A Simple Phase Unwrapping Algorithm and its Application to Phase-Based Frequency Estimation

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Abstract: The phase-based frequency estimation is investigated in this paper. We first discuss the impact of noise on the phase unwrapping and find that when the phase difference of the adjacent samples is π , namely, under the condition of two samples per cycle, the phase unwrapping has the best performance. An improved Kay estimator and a hybrid frequency estimator are then proposed according to this property. Their performance is improved by moving the frequency to a new one close to π . The phase noise and the phase-domain SNR are analyzed and the mean square error of the phase-based sinusoid frequency estimators is derived. We find that those frequency estimators, which use only the measurement phases, are suboptimal and do not attain the Cramer-Rao lower bound (CRLB).

Keywords: Cramer-Rao lower bound (CRLB), frequency estimation, phase noise, phase unwrapping, sinusoid.

1. INTRODUCTION

Estimating the frequency of a single sinusoid corrupted by additive, white, Gaussian noise (AWGN) is an important problem in communications, radar and sonar signal processing. There are two kinds of maximum likelihood (ML) frequency estimators, which are operated in the frequency domain [1-4] and the time domain [5-11] respectively.

ML estimation in frequency-domain was studied by Rife and Boorstyn in [2]; however, zero-padding is often required to obtain sufficient resolution. In such cases, the algorithm's complexity can be large. In order to reduce the computation complexity, spectral line interpolation method was proposed by Rife and Vincent [3, 4]. However, the accuracy of interpolation algorithm is insufficient and is depend on the relative ubiety of the true frequency and the discrete frequency.

The time-domain estimators in [5-11] are derived from the ML principle, but avoid the exhaustive search in the frequency domain that was used in [2]. Tretter proposed unwrapping the signal phase and performing linear regression to obtain a frequency estimate [5]. This approach was shown to approach the CRLB at high signal-to-noise ratio (SNR) and its computation load is low. However, the phase unwrapping algorithms [12-15] cited in [5] can only work well at high SNR. Kay addressed the phase unwrapping problem by only considering the phase differences and presented a simple frequency estimation algorithm, namely, Kay's estimator [6]. Like the ML estimator, this computationally simple estimator reaches the CRLB for SNRs above a threshold. However, at frequencies approaching π , the phase differences themselves are likely to wrap, causing large increases in mean-squared error. The estimator therefore performs poorly at frequencies near half of the sampling frequency. The work of [7] provides a computationally efficient estimation approach compared with ML estimation, via the fast Fourier transform algorithm. A class of smoothed central finite difference instantaneous frequency estimators was examined in [8]. In [9] an improved hybrid phase-based estimator, whose initial value was the Kay estimate, was proposed. Its threshold SNR is higher than that of Kay's. Hua Fu proposed a sample-by-sample iterative ML algorithm [11], which makes use of both the instantaneous signal phase and the magnitude of the received signal samples in the estimation process. However, when the frequency is close to 0 and π , its performance is poor, which is similar to Kay's estimator.

If the unwrapped phase can be obtained, the estimation of frequency and phase are straightforward. However, some of the available phase unwrapping algorithms need comparatively high SNR [12-16] and others suffer from a heavy computation load [17].

We investigate the phase-domain frequency estimation in this paper. First, the impact of noise on the phase unwrapping is discussed. We find that when the phase difference of the adjacent samples is π , namely, under the condition of two samples per cycle, the phase unwrapping has the best performance. An improved Kay estimator is then proposed according to this property. We first move the frequency to half of the sampling rate before unwrapping the phase and then estimate the phase difference of the adjacent samples using the unwrapped phase. Finally, substituting the phase difference values into Kay's estimator yields the frequency estimate. In practical applications, there may be no prior knowledge of the frequency and thus the phase can not be unwrapped under the condition of two samples per

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cycle. In this case, we propose a hybrid frequency estimator (HFE), whose performance is improved by moving the frequency to a new one close to π . The phase noise and the phase-domain SNR are analyzed and the variance of the phase-based sinusoid frequency estimators is derived. We find that those frequency estimators, which use only the measurement phases, are suboptimal and do not attain the CRLB. Finally, Monte Carlo simulations are performed to verify these conclusions.

2. SIGNAL MODEL

A mono-component sinusoid contaminated by AWGN can be modeled as

$$r(n) = A \exp\{j\phi_T(n)\} + z(n), n = 1, \cdots, N$$
(1)

where A is the unknown amplitude, and N is the number of samples. The noise z is a zero-mean complex white Gaussian process with $z(n) = z_R(n) + jz_I(n)$. Its components $z_R(n)$ and $z_I(n)$ are real, uncorrelated, zero-mean Gaussian random variables with variance $\sigma^2/2$. For a sinusoid signal, $\phi_T(n)$ can be expressed as

$$\phi_T(n) = \omega n + \theta \tag{2}$$

where $\omega(-\pi \le \omega < \pi)$ and θ $(-\pi \le \theta < \pi)$ are the frequency and the initial phase respectively.

The instantaneous phase $\phi(n)$ can be obtained by taking arctan to (1)

$$\phi(n) = \operatorname{tg}^{-1} \left[\frac{\operatorname{Im}[r(n)]}{\operatorname{Re}[r(n)]} \right]$$
(3)

where Im[x] and Re[x] denote the imaginary part and the

real part of x respectively, and tg⁻¹ denotes the arctangent function. Unfortunately, one is only able to measure a wrapped version of the phase, rather than the true phase. Under the noise-free situation, the measured phase at instant n, $\phi(n)$, is actually obtained from the true phase, $\phi_T(n)$, by a modulo operation as follows

$$\phi(n) = ((\phi_T(n)))_{2\pi}$$
(4)

where $(())_{2\pi}$ represents reduction modulo 2π onto the domain $(-\pi,\pi)$. The phase unwrapping problem is then to obtain an estimate for the true phase, $\phi_T(n)$, from the measured wrapped phase, $\phi(n)$. Thus, the wrapped value must be unwrapped through some method to estimate $\phi_T(n)$, which contains some physical quantity of interest. The phase unwrapping process is illustrated by a discrete time sinusoid next.

If the Nyquist theorem is met during sampling, the process of sampling a sinusoid with the sampling rate f_s can be illustrated in Fig. (1).

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Fig. (1). The sampling of a continuous sinusoid.

In Fig. (1), there are four samples whose phase measurement values, i.e. p_1 , p_2 , p_3 and p_4 , can be calculated from equation (4). Since p_1 , p_2 and p_3 are in the same period, the relationship

$$p_1 < p_2 < p_3 \tag{5}$$

holds. While the measurement phases of P3 and P4 satisfy

$$p_4 < p_3 \,. \tag{6}$$

Although the true phase of P4 is actually larger than that of P3, the modulo 2π operation during the argument calculation causes that p_4 is smaller than p_3 . The phase order relations of P3 and P4 can be recovered by adding multiple of 2π to p_4 . Assume that P1, P2 and P3 are in the (l+1)th period and P4 is in the (l+2)th period. Their true phases, respectively, are

$$2\pi l + p_1, 2\pi l + p_2, 2\pi l + p_3, 2\pi (l+1) + p_4.$$
⁽⁷⁾

The period number they are in can be determined by the phase measurement values of the adjacent samples. Thus the true phase, $\phi_T(t)$, can be recovered. In the noise-free cases, $\phi_T(n)$ can be recovered perfectly. However, when the noise is considered, the phase order relations of the adjacent samples may be reversed and then $\phi_T(n)$ can not be reconstructed correctly. In the following section, the impact of noise on the phase unwrapping process is discussed and an improved phase unwrapping algorithm is presented, which can work well at comparatively low SNR.

3. THE PHASE UNWRAPPING ON NOISE

Tretter [5] pointed out that when the SNR is higher than 5 dB, r(t) can be approximately expressed as

$$r(n) = A \exp\{j[\phi_T(n) + \varepsilon(n)]\}.$$
(8)

Furthermore if z(n) is a complex AGWN, $\varepsilon(n)$ will be a real Gaussian noise. The approximation of (8) requires high-enough SNR. In this paper, we discuss more general cases. Let $\psi = \theta - \phi$, then the probability density function (pdf) of ψ is given by [18 (Ch 4), 19, 20]

$$f(\psi / \theta) = \frac{1}{2\pi} e^{-\frac{\alpha^2}{2}} + \frac{\alpha}{2} \frac{\cos(\psi)}{\sqrt{2\pi}} e^{-\frac{\alpha^2 \sin^2 \psi}{2}} \cdot \left[1 + \operatorname{erf}\left(\frac{\alpha \cos(\psi)}{\sqrt{2}}\right)\right]$$
(9)

where $\alpha = A / \sigma$. The error function is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2/2} \, dz \,. \tag{10}$$

Equation (4) can be rewritten as

$$\phi(n) = \phi_T(n) - k_n 2\pi \tag{11}$$

where $k_n(k_n \in \mathbb{Z})$ is the period number of the sample at instant n. The phase difference between adjacent samples is

$$\Delta\phi(n) = \Delta\phi_T(n) - (k_n - k_{n-1})2\pi \tag{12}$$

where
$$\Delta \phi_T(n) = \phi_T(n) - \phi_T(n-1)$$
,
 $\Delta \phi(1) = \phi(1), \Delta \phi_T(1) = \phi_T(1), k_0 = 0$ and $k_1 = 1$. When
 $\phi_T(n)$ and $\phi_T(n-1)$ are in the same period, $\Delta \phi(n) > 0$.
However when they are in the adjacent period, $\Delta \phi_n < 0$
(Under the condition of uniform sampling and the Nyquist
theorem being met, $\Delta \phi_T(n) \le \pi$. Since $k_n - k_{n-1} = 1$, the
inequality $\Delta \phi(n) = \Delta \phi_T(n) - 2\pi < 0$ holds). Thus the true
phase can be recovered according to the phase difference
between adjacent samples. The true phase at the instant *n* is

0

$$\phi_T(n) = \phi(n) + k_n 2\pi, \begin{cases} k_n = k_{n-1}, & \text{if } \Delta\phi(n) \ge 0\\ k_n = k_{n-1} + 1, & \text{if } \Delta\phi(n) < 0 \end{cases}.$$
(13)

In the noise cases, (4) can be rewritten as

$$\phi(n) = \left(\left(\phi_T(n) + \varepsilon(n)\right)\right)_{2\pi} \tag{14a}$$

or

$$\phi(n) = [\phi_T(n) + \varepsilon(n)] - k_n 2\pi .$$
(14b)

The pdf of $\varepsilon(n)$ is (9). The phase difference of the adjacent samples is

$$\Delta\phi(n) = \Delta\phi_T(n) - (k_n - k_{n-1})2\pi + \Delta\varepsilon(n)$$
(15)

where $\Delta \varepsilon(n) = \varepsilon(n) - \varepsilon(n-1)$ and $\Delta \varepsilon(1) = \varepsilon(1)$.

When $\phi_T(n)$ and $\phi_T(n-1)$ are in the same cycle, $k_n - k_{n-1} = 0$ and $\Delta \phi_T(n) \in (0, \pi]$, then (15) is reduced to

$$\Delta \phi(n) = \Delta \phi_T(n) + \Delta \varepsilon(n) . \tag{16}$$

when $\Delta \varepsilon(n) > -\Delta \phi_T(n)$, $\Delta \phi(n) > 0$ and the correct $\phi_T(n)$ can be obtained from (13). When $\Delta \varepsilon(n) < -\Delta \phi_T(n)$, $\Delta \phi(n) < 0$. $\phi_T(n)$ obtained from (13) will be wrong. Since $\varepsilon(n)$ and $\varepsilon(n-1)$ are mutually independent, the pdf of $\Delta \varepsilon(n)$ can be written as

$$f_{\Delta\varepsilon}(y) = \int_{-\pi}^{\pi} f(y+x)f(x)dx$$
(17)

where $f(x) = f(\psi / \theta)$. Then the error probability of phase unwrapping can be defined as

$$P_{w} = \int_{-2\pi}^{-\Delta\phi_{T}(n)} f_{\Delta\varepsilon}(y) \, dy = \int_{-2\pi}^{-\Delta\phi_{T}(n)} \int_{\beta}^{\gamma} f(y+x) f(x) \, dx \, dy \quad (18)$$

where $\beta = \max(-\pi - y, -\pi)$ and $\gamma = \min(\pi - y, \pi)$. In order to attain the minimum error probability, $-\Delta \phi_T(n)$ should be as small as possible, namely, $\Delta \phi_T(n)$ should be as large as possible. On the other hand, $\Delta \phi_T(n) \in (0,\pi]$. Thus the phase unwrapping using (13) has the best performance when $\Delta \phi_T(n) = \pi$. When $\phi_T(n)$ and $\phi_T(n-1)$ are in different cycles, the same conclusion can be drawn. Fig. (2) shows the error probability of phase unwrapping when SNR is from 0 dB to 10 dB and the frequencies are π , $2\pi/3$ and $2\pi/5$ respectively. The error probabilities calculated from (18) are A, C and E, while B, D and F are the results of Monte-Carlo simulations. The number of simulation runs is set to 10 000 000 for each case. We see that the simulation results are identical with the theoretical values.



Fig. (2). The error probability of phase unwrapping.

From just analysis, we can conclude that when the true phase is recovered through phase unwrapping and the phase difference $\Delta \phi_T(n)$ is in interval $[0,\pi]$, the larger

the $\Delta \phi_T(n)$, the better the noise-immune performance will be. When $\Delta \phi_T(n) = \pi$, namely, two samples per cycle, the best performance is achieved.

4. THE APPLICATION OF PHASE UNWRAPPING TO FREQUENCY ESTIMATION

The estimation of the frequency of a single complex sinusoid in AWGN is a classical problem in signal processing. Kay's estimator, which was proposed by Steven Kay [6], can approach the CRLB for moderately high SNR's and has been widely applied in many fields. Kay's estimator is given by

$$f_{c}^{n} = \sum_{i=1}^{N-1} w_{i} \angle r^{*}(i)r(i+1)$$
(19)

where $\angle x$ denotes the phase of x, and w_i are the weights of Kay's window defined by

$$w_{i} = \frac{3N/2}{N^{2} - 1} \left\{ 1 - \left[\frac{i - (N/2 - 1)}{N/2} \right]^{2} \right\}.$$
 (20)

When the SNR is below 8 dB, Kay's estimator degrades rapidly. This is because the later three items of the first-order autocorrelation of signal, which appears as

$$r^{*}(i)r(i+1) = s^{*}(i)s(i+1) + s^{*}(i)z(i+1) + s(i+1)z^{*}(i) + z^{*}(i)z(i+1)$$
(21)

are noise terms, thus the 2π ambiguity is likely to appear when calculating the argument of $r^*(i)r(i+1)$ and the frequency estimate $\int_c^{\mathbf{R}}$ obtained from (19) will have a comparatively large error. If the phase difference is calculated from the unwrapped phase, Kay's estimator can be improved. The improved frequency estimator has the form

$$f_{c}^{\mathbf{I}} = \sum_{i=1}^{N-1} w_{i} \Big[\phi_{I}(i+1) - \phi_{I}(i) \Big].$$
(22)

In some applications, such as frequency tracking, a prior knowledge of frequency is available. In this case, the frequency of the signal can be shifted to half of the sampling rate first and then the phase unwrapping is performed. Assuming $\tilde{\omega}$ is the initial frequency, the frequency shift value is

$$\omega_{\rm shift} = \pi - \tilde{\omega} \,. \tag{23}$$

In phase domain, frequency shift can be replaced by phase shift. According to the principal value $\phi(n)$, the wrapped phase after frequency shifting is given by

$$\phi'(n) = ((\phi(n) + n\omega_{shift}))_{2\pi}$$
 (24)

Appling (13) to $\phi'(n)$ yields the instantaneous phase $\phi'_I(n)$ (In noise cases, we can only obtain the noised phase rather than the true phase $\phi_T(n)$ so that the subscript *T* is replaced by *I*.), the final frequency estimate is given by

$$\boldsymbol{\alpha} = \sum_{i=1}^{N-1} w_i \Big[\phi_i'(i+1) - \phi_i'(i) \Big] - \omega_{shift} \,.$$
(25)

For convenience we call (25) as IKay.

In practical applications, the frequency is commonly unknown. In this case, the phase can not be unwrapped under the condition of two samples per cycle. Next, we present a hybrid frequency estimator, whose performance is improved by moving the frequency to a new one close to π .

Firstly, frequency shift is performed to the signal received, and the shift values are $\pi/2$, π and $3\pi/2$ respectively. The wrapped phases after frequency shifting are given by

$$\phi_1(n) = ((\phi(n) + n\pi / 2))_{2\pi}$$
(26)

$$\phi_2(n) = ((\phi(n) + n\pi))_{2\pi} \tag{27}$$

$$\phi_3(n) = ((\phi(n) + 3n\pi/2))_{2\pi} . \tag{28}$$

The corresponding frequencies to $\phi_1(n)$, $\phi_2(n)$ and $\phi_3(n)$ are $\omega_1(\omega_1 = ((\omega + \pi/2))_{2\pi}), \quad \omega_2(\omega_2 = ((\omega + \pi))_{2\pi})$ and $\omega_3 (\omega_3 = ((\omega + 3\pi/2))_{2\pi})$ respectively. Unwrapping $\phi(n)$, $\phi_1(n)$, $\phi_2(n)$ and $\phi_3(n)$ yield $\phi_1(n)$, $\phi_{11}(n)$, $\phi_{12}(n)$ and $\phi_{I3}(n)$. If there is no error during phase unwrapping, $\phi_{Ii}(n)$ ($i = 0, 1, 2, 3, \phi_{I0}(n) = \phi_{I}(n)$) will be noisecontaminated straight lines, i.e. $\omega_i n + \theta (\omega_0 = \omega)$. Otherwise, the unwrapped phase $\phi_{li}(n)$ will have a phase jump, namely, the line $\phi_{li}(n)$ will have an inflection point at the error position. In a word, the linearity of $\phi_{I_i}(n)$ is related to the performance of phase unwrapping. As we know, the linearity can be measured by the variance of linear regression. Consequently, a suboptimal frequency estimator can be obtained: the frequency a_{i} is estimated by performing Tretter's estimator (linear regression) to $\phi_{li}(n)$ and the variances of fitting are calculated at the same time. The optimum frequency estimate is the one whose corresponding fitting variance is minimal. Since $\omega \in [-\pi, \pi)$, the frequency estimates α_{i} should be wrapped into $[-\pi,\pi)$ after subtracting the frequency shift value from $\boldsymbol{\omega}_{i}$. Thus the relationships between the estimate $\hat{\omega}$ and ω , are

$$\widehat{\boldsymbol{\omega}} = \left((\boldsymbol{\alpha}_{0}) \right)_{2\pi} \tag{29}$$

$$\widehat{\omega} = \begin{cases} ((\mathbf{a}_{1}))_{2\pi} - \pi / 2, \text{ if } \mathbf{a}_{1} > 3\pi / 2\\ \mathbf{a}_{1} - \pi / 2, \text{ if } \mathbf{a}_{1} \le 3\pi / 2 \end{cases}$$
(30)

$$\hat{\omega} = \boldsymbol{\alpha}_2 - \boldsymbol{\pi} \tag{31}$$

$$\widehat{\omega} = \begin{cases} \alpha_3 + \pi / 2, & \text{if } \alpha_3 < \pi / 2 \\ \alpha_3 - 3\pi / 2, & \text{if } \alpha_3 \ge \pi / 2 \end{cases}$$
(32)

Next we demonstrate this process with an example for clarity. In this example, the frequency ω is set to 0.5 and N = 11. The estimates are $\omega_0 = 1.8651$, $\omega_1 = 2.1222$,

 $\mathbf{a}_2 = 3.6929$ and $\mathbf{a}_3 = 4.2927$, and the fitting variances are $\operatorname{Var}_0 = 1.5795$, $\operatorname{Var}_1 = 0.4773$, $\operatorname{Var}_2 = 0.4773$ and $\operatorname{Var}_3 = 2.7948$ respectively. The minimum variance is 0.4773 ($\operatorname{Var}_1 = \operatorname{Var}_2 = 0.4773$). Thus substituting \mathbf{a}_1 into (30) or substituting \mathbf{a}_2 into (31) yields the desired estimate, i.e. $\hat{\omega}$. Var_1 being equal to Var_2 implies that no error occurs during the unwrapping of $\phi_1(n)$ and $\phi_2(n)$ so that the frequency estimates obtained from them will be the same.

If ω is close to $\pm \pi$, the estimate obtained from the suboptimum estimator may still be ambiguous, e.g. when ω is close to π , $\hat{\omega}$ may be a value close to $-\pi$. While for ω close to $-\pi$, $\hat{\omega}$ may be close to π . To solve frequency ambiguity, we need to know the rough range of the true frequency, which can be determined by the spectral interpolation scheme [21] given by

$$\tilde{\omega} = \left(\left(\frac{2\pi}{N} \left(k_0 + r \frac{|R(k_0 + r)|}{|R(k_0)| + |R(k_0 + r)|} \right) \right) \right)_{2\pi}$$
(33)

where R(K) is the power spectrum of r(n) and K_0 is the index of the largest peak. When $|X(k_0+1)| \le |X(k_0-1)|$, r = -1, while for $|X(k_0+1)| > |X(k_0-1)|$, r = 1. The final frequency estimate of HFE is given by

$$\boldsymbol{\omega} = \begin{cases} \hat{\omega} + 2\pi, & \text{if } \hat{\omega} < -0.95\pi \text{ and } \tilde{\omega} > 0\\ \hat{\omega} - 2\pi, & \text{if } \hat{\omega} > 0.95\pi \text{ and } \tilde{\omega} < 0\\ \hat{\omega}, & \text{else} \end{cases}$$
(34)

In Fig. (3) the block diagram shows the scheme of the HFE estimator.

where PU denotes the phase unwrapping unit, and the decision conditions, i.e. con1 - con6 respectively denote if $\omega_1 > 3\pi/2$, if $\omega_3 < \pi/2$, if $|\hat{\omega}| > \frac{3}{4}\pi$, if $\hat{\omega} > 0.95\pi$, if $\tilde{\omega} < 0$, and if $\tilde{\omega} > 0$.

5. PERFORMANCE ANALYSIS

Kay [6] and Tretter [5] pointed out that at moderately high SNR Kay's estimator and Tretter's estimator can attain the CRLB on variance, which is deduced using approximation. Then we have a question herein: can these two estimators indeed attain the CRLB at high-enough SNR? Hua Fu [11] indicated that those estimators using only the measurement phases would be suboptimal. He presented a recursive ML estimator which makes use of both the measurement phases and the measurement magnitudes. For a sine wave with $\omega = 0.5$, and N = 11, and SNR=12 dB, the root mean square error (RMSE) of Kay's estimator is 0.01724, while for HuaFu's estimator, the RMSE is 0.01699. The results are the product of a Monte-Carlo simulation of 100 000 runs and the CRLB is 0.01687. From the results we can see that Kay's estimator does not attain the CRLB, while HuaFu's estimator is more close to it. Since both Kay's estimator and Tretter's estimator are not maximum likelihood in a real sense, what are the true MSEs of these two estimators? Next we analyze the phase noise and the phase SNR, and derive the MSE of frequency estimators which only utilize the phase measurements.

The difference of the unwrapped phase is given by

$$\Delta \phi_i = \phi_I(i+1) - \phi_I(i) = \omega + \varepsilon(i+1) - \varepsilon(i)$$
(35)

where $i = 1, \dots, N-1$. Equation (35) indicates that the problem now is to estimate the mean, ω , in noise. The process is actually a moving average with coefficients 1 and - 1. The minimum variance unbiased estimator for the linear model of (35) is found by minimizing

$$\mathbf{J} = (\Delta - \omega \mathbf{I})^T \mathbf{C}^{-1} (\Delta - \omega \mathbf{I})$$
(36)

where $\Delta = [\Delta \phi_1, \Delta \phi_2, \dots, \Delta \phi_{N-1}]^T$, and $\mathbf{I} = [1, 1, \dots, 1]^T$, and \mathbf{C} is the $(N-1) \times (N-1)$ covariance matrix of Δ . The solution to the problem is

$$\boldsymbol{\alpha} = \frac{\mathbf{I}^T \mathbf{C}^{-1} \boldsymbol{\Delta}}{\mathbf{I}^T \mathbf{C}^{-1} \mathbf{I}} \,. \tag{37}$$

The variance of this estimator is

$$\operatorname{Var}(\boldsymbol{\alpha}) = \frac{1}{\mathbf{I}^T \mathbf{C}^{-1} \mathbf{I}} \,. \tag{38}$$



Fig. (3). Scheme for HFE estimator.

Since $\Delta \phi_i$ is a real moving average process with coefficients $b_0 = 1$ and $b_1 = -1$, the covariance matrix has the form [6]

$$\mathbf{C} = \frac{\sigma_{\varepsilon}^2}{2} \begin{pmatrix} 2 - 1 & 0 & 0 & \cdots & 0 \\ -1 & 2 - 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 - 1 & 2 \end{pmatrix}$$
(39)

where σ_{ε}^2 is the variance of phase noise.

In the following section, we investigate the variance of phase noise and the phase-domain SNR. In phase domain, the signal component of interest is the true phase $\phi_T(n)$, which is contaminated by noise. Tretter [5] proved that for large SNR, the additive noise can be converted into an equivalent additive phase noise. We do not use approximate processing herein, but consider more general cases. The phase-domain signal can be modeled as

$$p(n) = \phi_{\tau}(n) + \varepsilon(n) \tag{40}$$

where p(n) and $\phi_T(n)$ are equivalent to φ and θ in (9) respectively. Thus $\sigma_{\varepsilon}^2 = \operatorname{Var}[\varphi] = \operatorname{Var}[\psi]$. The phase-domain SNR is then defined by

$$SNR_{p} = 1/\sigma_{\varepsilon}^{2}.$$
 (41)

Since $E[\psi] = 0$, we have

$$\operatorname{Var}[\psi] = \int_{-\pi}^{\pi} \psi^2 f(\psi / \theta) d\psi \,. \tag{42}$$

Because the close form of $Var[\psi]$ is difficult to obtain, the numerical calculation function of MATLAB is utilized to evaluate $Var[\psi]$, namely, σ_{e}^{2} . Fig. (4) indicates the comparison of SNR_p (dashed line) and the time-domain SNR (solid line). We can see that SNR_p is larger than SNR and the difference between them is varying with SNR. Fig. (5) shows this difference per signal-to-noise ratio, starting from -10 to 20 dB in steps of 0.1 dB. As can be observed from the plot, when SNR<1 dB, SNR_p-SNR increases gradually. While the difference approaches 3 dB when SNR \gg 1 dB, which is identical with the Tretter's approximation. He proved the phase noise variance is $\sigma_{e}^{2} = 1/(2SNR)$ [5] at high SNR, namely, SNR_p is approximately equal to 2SNR, or 3 dB higher than SNR.

As we known, the autocorrelation calculation of the noise signal will incur SNR loss. Next we analyze the SNR after autocorrelation. Recall the first-order autocorrelation of signal

$$r^{*}(i)r(i+1) = s^{*}(i)s(i+1) + s^{*}(i)w(i+1) + s(i+1)w^{*}(i) + w^{*}(i)w(i+1)$$
(43)

The signal terms is

$$s^{*}(i)s(i+1) = A^{2}e^{j2\pi f_{c}T}$$
(44)

and the noise terms are

$$w = s^{*}(i)w(i+1) + s(i+1)w^{*}(i) + w^{*}(i)w(i+1).$$
(45)



Fig. (4). The relationship between the phase-domain SNR and the time-domain SNR.



Fig. (5). The difference between the phase-domain SNR and the time-domain SNR.

The mean and variance of w are 0 and $2A^2\sigma^2 + \sigma^4$ respectively. Thus the SNR after autocorrelation will be

$$\text{SNR}_{c} = \frac{A^{4}}{2A^{2}\sigma^{2} + \sigma^{4}} = \text{SNR}\frac{1}{2 + 1/\text{SNR}}$$
 (46)

As can be seen from (46), the SNR decreases after autocorrelation. At high SNR (SNR $\gg 1$), the SNR loss is about 3 dB. The loss is increasing gradually with the decreasing of SNR. For example, SNR_c is -4.7 dB when SNR is 0 dB. Recall Kay's estimator, the first-order autocorrelation in (19) will cause at least 3 dB loss in time domain. However, its performance approaches the CRLB when the SNR is higher than the threshold SNR and does not decrease 3 dB. Now this phenomenon can easily be explained according to the analysis of phase SNR.

Substituting σ_{ε}^2 into the covariance matrix (39) and substituting (39) into (38) yield the variance of IKay and HFE

$$\operatorname{Var}(\boldsymbol{a}) = \frac{6\sigma_{\varepsilon}^{2}}{N(N^{2} - 1)}.$$
(47)

Note that the variance is obtained under the assumption that no error occurs duration phase unwrapping. At low SNR, the error probability of phase unwrapping is comparatively high so that the actual MSEs are worse than (47). However, Ikay and HFE can attain (47) at high SNR.

Kay pointed out that at moderately SNR Kay's estimator attains the CRLB. The simulation shows, however, even when the SNR is up to 20 dB, it can still not attain the CRLB. The MSE of Kay's estimator was also given by [22,23], yet these two bounds are only valid at high SNR. As a matter of fact, the MSE of Kay's estimator are derived from (4) in [6], which is identical with (22) of our paper, so that the bound for Kay's estimator should be (47). The simulations in the next section verify this conclusion.

6. SIMULATION RESULTS

This section presents the simulation results to illustrate the behavior of the variances of the classical ML estimator given in [2] for locating the peak of a periodogram, the IKay estimator (25), the HFE estimator, Kay's estimator [6], the bound given by (47), the CRLB as a function of the various parameters. To ensure the accuracy, the number of Monte-Carlo simulation runs is set to 10^6 . For convenience, the results are provided with the inverse mean square error (IMSE), which is defined by $-10 \times \log 10$ (MSE) dB. The variances of IKay and HFE and the CRLB are respectively given by the inverse variance (IVar, IVar = $-10 \times \log 10$ (Var) dB) and the inverse CRLB (ICRLB, ICRLB = $-10 \times \log 10$ (CRLB) dB) as well.

Figs. (6-8) gives the performance comparison of four estimators for the frequency. The IVar and the ICRLB are also plotted for the sake of comparison. The actual frequency values to be estimated are 0.5π , 0 and -0.25π respectively; θ is assumed to be a uniformly distributed random phase; the number of samples N is equal to 11. Notice that when $\omega = 0$, the curves of IKay and Kay are overlapped. We define the threshold SNR of an estimator as the value of SNR at which its inverse variance curve dips by 1 dB from the ICRLB curve, as is common in the literature. The simulation results show that the threshold SNR for ML estimator is about 3 dB, while those for IKay's estimator and HFE estimator are 7 dB and 8dB respectively. For Kay's estimator, the threshold SNRs are 7 dB, 9.5 dB and 8.5 dB for $\omega = 0.5\pi$, $\omega = 0$ and $\omega = -0.25\pi$. At SNRs lower than the threshold SNR, the performance of HFE decreases faster than that of IKay. From the simulation results we can see that IKay has a better performance than Kay's estimator and HFE, but is outperformed by ML estimator.



Fig. (6). Performance comparison of four estimators for the frequency ω , with $\omega = 0.5\pi$, N = 11.



Fig. (7). Performance comparison of four estimators for the frequency ω , with $\omega = 0$, N = 11.

The performance of Kay's estimator is dependent on the frequency to be estimated. When $\omega = 0$, the performance is the best, and worse for $\omega = -0.25\pi$, and the worst for $\omega = 0.5\pi$. Fig. (9) shows the variation of the MSEs of HFE and Kay's estimator when the frequency is changing starting from $-\pi$ to π . We can see that HFE is stable over the whole frequency range, while for Kay's estimator, it suffers from significant performance degradation when the frequency is close to $\pm \pi$.

Figs. (10-12) are detailed zooms of Figs. (6-8) over $8dB \le SNR \le 10dB$. We see that when the SNR is increasing, Kay's estimator, IKay and HFE are close to the IVar gradually, yet keep a certain distance between them and

the ICRLB. Kay said that when the SNR is high-enough, the variance of Kay's estimator is equal to the CRLB. Some other literatures also presented that Kay's estimator can attain the CRLB when the SNR is higher than the threshold SNR. Well then can it indeed attain the CRLB at highenough SNR? Table 1 shows the performance of Kay's estimator and HFE over $11dB \le SNR \le 20dB$. The IVar and the ICRLB is also listed for comparison. The results show that Kay's estimator can not attain the CRLB. Hua Fu [11] indicated that Kay's estimator is not a maximum likelihood method in a real sense, since it only makes use of the measurement phases, yet discards the measurement magnitudes information. From the simulation results, his conclusion is verified. As a matter of fact, the MSE of Kay's estimator are derived from (4) in [6], which is identical with (22) in this paper, so that the bound for Kay's estimator should be (47). Furthermore, we can conclude that the lower bound of those estimators using only the measurement phases is (47), not the CRLB.



Fig. (8). Performance comparison of four estimators for the frequency ω , with $\omega = -0.25\pi$, N = 11.



Fig. (9). Impact of the actual frequency on Kay's estimator and HFE.



Fig. (10). Zoom of Fig. (6) over $8dB \le SNR \le 10dB$.



Fig. (11). Zoom of Fig. (7) over $8dB \le SNR \le 10dB$.

7. CONCLUSION

The instantaneous phase of the signal received can be obtained by a simple phase unwrapping algorithm and on this basis we propose two frequency estimators, i.e. IKay and HFE, whose performances are better than that of Kay's estimator. The phase noise variance and the phase-domain SNR are analyzed and the theoretical analysis shows that the phase-domain SNR is larger than the time-domain SNR. When SNR < 1 dB, SNR_p-SNR increases gradually. While the difference approaches 3 dB when SNR \gg 1 dB. The mean square error of the frequency estimators based on phase measurements is derived. In general, we can conclude that IKay and HFE is better than Kay's estimator and their threshold SNRs decrease for about 1-2 dB; HFE is stable over the whole frequency range, while Kay's estimator

	11 dB	12 dB	13 dB	14 dB	15 dB	16 dB	17 dB	18 dB	19 dB	20 dB
Kay	34.2184	35.2738	36.3135	37.2771	38.3314	39.3286	40.3922	41.3902	42.3956	43.4002
HFE	34.2184	35.2738	36.3135	37.2771	38.3314	39.3286	40.3922	41.3902	42.3956	43.4002
IVar	34.2331	35.2761	36.3087	37.3337	38.3530	39.3681	40.3799	41.3892	42.3965	43.4023
ICRLB	34.4242	35.4242	36.4242	37.4242	38.4242	39.4242	40.4242	41.4242	42.4242	43.4242

Table 1. Performance Comparison of Kay's Estimator and HFE, with $\omega = 0$ and N = 11

suffers from a significant performance degradation when the frequency is close to $\pm \pi$; the lower bound of those estimators using only the measurement phases is (47), yet can not attain the CRLB.



Fig. (12). Zoom of Fig. (8) over $8dB \le SNR \le 10dB$.

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