

# Analytical Solution for the Combined Solar Radiation Pressure and Luni- Solar Effects on the Orbits of High Altitude Satellites

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**Abstract:** A new analytical solution is obtained for the motion of high-altitude earth satellites. The basic idea is to study the joint effects of direct solar radiation pressure and the gravitation of the Earth, Moon and the Sun on the orbits of the satellites. The mathematical model includes the zonal harmonics of the geopotential effects up to  $J_4$ . The order of magnitude of each perturbing term is assessed. The formulae for the all perturbations forces are obtained up to the fourth order (where the mean motion of the moon ( $\sim 10^{-2}$ ) is considered a small quantity of first order). The short, intermediate and long-period terms are eliminated from the Hamiltonian using a perturbation technique based on the Lie-Deprit-Kamel transform through three canonical transformations. The solution is intended to be used for ephemeris predictions for orbiters whose orbital semimajor axes are in the range of 15000 to 70000 kms.

**Key Words:** Astrodynamics, artificial satellites theories, Hamiltonian system, Lie-Deprit-Kamel transform, solar radiation pressure, luni- solar effects.

## 1. INTRODUCTION

The study of the effect of direct solar radiation pressure on the orbits of the artificial satellites was discussed in a lot of literatures, starting from pioneering works of [1-4] and other of [5-9]. Also the theory of third body perturbations on an artificial satellite motion was developed by many authors in the past. The formulae for secular and periodic perturbations in orbital elements were obtained with the use of different methods [10-17]. But the joint effects of radiation pressure and the gravitational attraction of the Earth, Moon and Sun are rarely mentioned in the literature.

Musen (1960) [1] derived first order expressions for the rates of change in the osculating elements caused by the direct solar radiation pressure. He used the method of variation of vector elements.

Kaula (1962) [18] just developed the Lunar and Solar disturbing function for a close satellite and developed a quasi potential for the radiation pressure effects for use it in the equation of motion. He did not obtain the solution.

Sehna (1975) [19] discussed the direct solar radiation pressure, as one of the non-gravitational forces, from all its different aspects.

Anselmo *et al.* (1983) [20] had analyzed the perturbations due to solar radiation pressure, only, on the orbit of a high artificial satellite. The latter was modeled in a simplified way (axisymmetric body), which seems suitable to describe the main effects for existing telecommunication satellites. He used the regularized general perturbation equations,

by expressing the force in the moving Gauss' reference frame and by expanding the results in terms of some small parameters, referring both to the orbit (small eccentricity and inclination) and to the spacecraft's attitude. He used the general perturbation equations in Gauss' form and he neglected the oblateness of the Earth.

The present work considers the motion of an earth satellite orbiting at a high altitude in an orbit with semimajor axis in the range of 15000 to 70000 km at low inclination  $I \leq 10^\circ$ , with no restriction on the orbital eccentricity. The mathematical model includes the non sphericity of the earth's gravitational field up to  $J_4$  besides the effects of the lunar and solar gravitational attractions and solar radiation pressure. The disturbing function is developed completely apart from considerations of order [21]. Then the orders are assigned such that the terms are retained whenever their contributions are of fourth order, where the mean motion of the moon ( $\sim 10^{-2}$ ) is considered small quantity of the first order. The Hamiltonian has been formed in terms of a set of ten canonical elements representing the Delaunay variables augmented by the arguments of latitude of the moon and sun and their conjugate momenta. Then the short, intermediate and long-period terms are eliminated from the Hamiltonian with analytical perturbation technique based on the Lie-Deprit-Kamel transform [22, 23].

Finally a procedure for the computation of the position and velocity at any time is presented.

## 2. THE FORCE FUNCTION

### 2.1. The Force Functions Due to the Gravitational Effects

Firstly we derive the equations of motion of the satellite under the gravitation effects of the earth (as primary), and the moon and sun (as perturbations). Let the subscripts 0, 1, 2, 3 refer respectively to the earth, satellite, moon, and sun.

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The equations of motion of the four point masses  $m_i$  in an inertial frame are

$$m_i \ddot{\bar{\rho}}_i = \tilde{\nabla}_{\rho_i} U \quad (i = 0, 1, 2, 3) \quad (1)$$

where  $U$  is the force function given by

$$U = k^2 \sum_{j>i}^3 \frac{m_i m_j}{r_{ij}} \quad (i, j = 0, \dots, 3)$$

$\bar{\rho}_i = (\xi_i, \eta_i, \zeta_i)$  is the position vector of anyone of the four bodies with respect to the inertial frame and  $r_{ij}$  is the distance between the mass points .

It is convenient to refer the satellite and the moon to the earth and the sun to the centre of mass of the earth-moon system. Denoting the new position vectors by  $\bar{r}_i$ , we have

$$\begin{aligned} \bar{r}_1 &= \bar{\rho}_1 - \bar{\rho}_0 \\ \bar{r}_2 &= \bar{\rho}_2 - \bar{\rho}_0 \\ \bar{r}_3 &= \bar{\rho}_3 - \frac{m_0 \bar{\rho}_0 + m_2 \bar{\rho}_2}{m_0 + m_2} \end{aligned} \quad (2.1)$$

The equations of motion (1) must be transformed accordingly where the partials in Eq.(1) transform as:

$$\frac{\partial U}{\partial \bar{\rho}_j} = \sum_{k=1}^3 \frac{\partial U}{\partial \bar{r}_k} \frac{\partial \bar{r}_k}{\partial \bar{\rho}_j}, \quad (j = 0, 1, 2, 3)$$

Resulting in

$$\begin{aligned} \tilde{\nabla}_0 U &= -\nabla_1 U - \nabla_2 U - \frac{m_0}{m_0 + m_2} \nabla_3 U, \\ \tilde{\nabla}_1 U &= \nabla_1 U \\ \tilde{\nabla}_2 U &= \nabla_2 U - \frac{m_2}{m_0 + m_2} \nabla_3 U, \quad \tilde{\nabla}_3 U = \nabla_3 U \end{aligned} \quad (2.2)$$

where  $\nabla \equiv \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)^T$ ,  $\tilde{\nabla} \equiv \left( \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta} \right)^T$  Making use of

Eqs (1), (2.1), and (2.2)

the equations of motion of  $m_1$  can be written as:

$$\ddot{\bar{r}}_1 = \frac{m_0 + m_1}{m_0 m_1} \nabla_1 U + \frac{1}{m_0} \nabla_2 U + \frac{1}{m_0 + m_2} \nabla_3 U \quad (3)$$

In order to express U in terms of the new variables, we first choose the centre of mass of the entire system as origin of

the inertial frame, with this choice  $\sum_{j=0}^3 m_j \bar{\rho}_j = 0$  (4)

Combining Eq. (2.1) and Eq. (4) then:

$$\begin{aligned} \bar{\rho}_0 &= -\frac{m_2}{m_0 + m_2} \bar{r}_2 - \frac{m_3}{m_0 + m_2 + m_3} \bar{r}_3, \\ \bar{\rho}_1 &= \bar{r}_1 - \frac{m_2}{m_0 + m_2} \bar{r}_2 - \frac{m_3}{m_0 + m_2 + m_3} \bar{r}_3 \\ \bar{\rho}_2 &= \frac{m_0}{m_0 + m_2} \bar{r}_2 - \frac{m_3}{m_0 + m_2 + m_3} \bar{r}_3, \quad \bar{\rho}_3 = \frac{m_0 + m_2}{m_0 + m_2 + m_3} \bar{r}_3 \end{aligned}$$

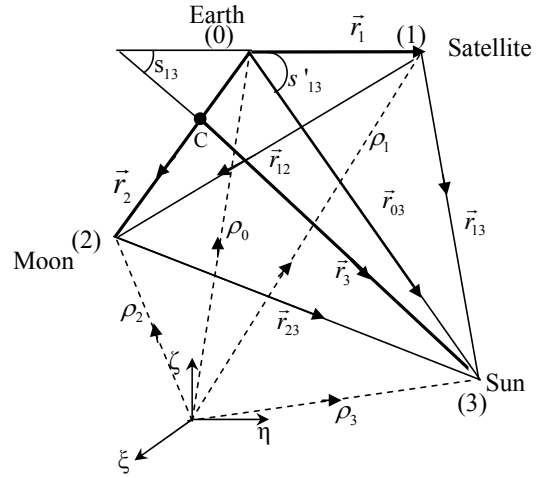


Fig. (1). Relative positions of the earth, satellite, moon and sun.

appealing now to the relations

$$r_{ij}^2 = (\xi_j - \xi_i)^2 + (\eta_j - \eta_i)^2 + (\zeta_j - \zeta_i)^2$$

and from Fig. (1),  $r_{01}^2 = r_1^2$ ,  $r_{02}^2 = r_2^2$

$$r_{03}^2 = r_3^2 + \frac{2m_2}{m_0 + m_2} \bar{r}_2 \cdot \bar{r}_3 + \left( \frac{m_2}{m_0 + m_2} \right)^2 r_2^2$$

$$r_2^2 = r^2 - 2\bar{r} \cdot \bar{r}_2 + r_2^2$$

$$r_{13}^2 = r_3^2 + \frac{2m_2}{m_0 + m_2} \bar{r}_2 \cdot \bar{r}_3 + \left( \frac{m_2}{m_0 + m_2} \right)^2 r_2^2 - 2\bar{r}_1 \cdot \bar{r}_3 - \frac{2m_2}{m_0 + m_2} \bar{r}_1 \cdot \bar{r}_2 + r_1^2$$

$$r_{23}^2 = r_3^2 - \frac{2m_0}{m_0 + m_2} \bar{r}_2 \cdot \bar{r}_3 + \left( \frac{m_0}{m_0 + m_2} \right)^2 r_2^2$$

with these relations we can express  $U$  in terms of the new coordinates.

Next, we have to express the partials in Eq. (3) explicitly in terms of the new coordinates. In doing so we make use of

$$\text{the relation: } \nabla \frac{1}{r} = -\frac{\bar{r}}{r^3} \nabla \bar{r}$$

which leads to the following, required partial derivatives

$$\nabla_1 \frac{1}{r_1} = -\frac{\bar{r}_1}{r_1^3},$$

$$\nabla_1 \frac{1}{r_{12}} = \frac{\bar{r}_2 - \bar{r}_1}{r_{12}^3}, \quad \nabla_1 \frac{1}{r_{13}} = \frac{\bar{r}_3}{r_{13}^3} - \frac{m_0}{m_0 + m_2} \frac{\bar{r}_0}{r_0^3} - \frac{m_3}{m_0 + m_2 + m_3} \frac{\bar{r}_3}{r_3^3},$$

$$\nabla_2 \frac{1}{r_2} = -\frac{\bar{r}_2}{r_2^3}$$

$$\nabla_2 \frac{1}{r_{03}} = -\frac{m_2}{m_0 + m_2} \frac{1}{r_{03}^3} (\bar{r}_3 + \frac{m_2}{m_0 + m_2} \bar{r}_2),$$

$$\nabla_2 \frac{1}{r_{23}} = \frac{m_0}{m_0 + m_2} \frac{1}{r_{23}^3} (\bar{r}_3 - \frac{m_0}{m_0 + m_2} \bar{r}_2)$$

$$\frac{1}{r_{03}} = \frac{1}{r_3} \bar{r}_3 + \frac{m_2}{m_0 + m_2} \bar{r}_2,$$

$$\frac{1}{r_{13}} = \frac{1}{r_3} (\bar{r}_3 - \frac{m_1}{m_1 + m_2} \bar{r}_1)$$

Substituting these derivatives into Eq. (3) it follows, after some simplifications, that:

$$\ddot{\bar{r}}_1 = k^2 \left[ \frac{m_0}{r_1^3} \bar{r}_1 + m_2 \frac{\bar{r}_2}{r_2^3} + m_2 \frac{\bar{r}_1 \bar{r}_2}{r_3^3} + \frac{m_3}{r_{13}^3} \bar{r}_1 - \frac{m_2}{m_0 + m_2} \bar{r}_2 \right] + \frac{m_3}{r_{03}^3} \left( \bar{r}_3 + \frac{m_2}{m_0 + m_2} \bar{r}_2 \right)$$

This may be, better, written as  $\ddot{\bar{r}}_1 + \frac{\mu}{r_1} \bar{r}_1 = U_{grav.} + \frac{\mu}{r_1} \bar{r}_1$ ,

where

$$U_{grav.} = k^2 \left[ \frac{m_2}{r_{12}} + \frac{m_3}{r_{13}} - \frac{m_2}{r_2^3} \bar{r}_1 \cdot \bar{r}_2 - \frac{m_3}{r_{03}^3} \left( \frac{m_2}{m_0 + m_2} \bar{r}_1 \cdot \bar{r}_2 + \bar{r}_1 \cdot \bar{r}_3 \right) \right] \quad (5)$$

$\mu = k^2 m_0$  and  $U_{grav.}$  represents the disturbing function due to the attractions of the moon and of the sun. In the following subsections we develop  $U_{grav.}$  in forms suitable to any subsequent analysis.

### 2.2. Development of the Gravitational Disturbing Function

The first step in the development of the disturbing function requires the expansion of  $1/r_{12}$ ,  $1/r_{13}$ , and  $1/r_{03}$  in series in the ratios  $r_1/r_2$ ,  $r_1/r_3$  and  $r_2/r_3$  and in terms of three angles  $s_{12}$ ,  $s_{13}$  and  $s_{23}$  where  $\cos S_{ij} = \bar{r}_i \cdot \bar{r}_j / r_i r_j$ . Then

$$\frac{1}{r_{12}} = \frac{1}{r_2} \left[ 1 + 2 \frac{r_1}{r_2} \cos s_{12} + \left( \frac{r_1}{r_2} \right)^2 \right]^{-1/2}$$

$$\frac{1}{r_{13}} = \frac{1}{r_3} \left[ 1 + \frac{m_1}{m_1 + m_2} \frac{r_1}{r_3} 2 \cos s_{13} + \left( \frac{m_1}{m_1 + m_2} \right) \left( \frac{r_1}{r_3} \right)^2 - \frac{m_1}{m_1 + m_2} \frac{r_1 r_2}{r_3 r_3} 2 \cos s_{13} + \left( \frac{r_1}{r_3} \right)^3 \right]^{-1/2}$$

$$\frac{1}{r_{03}} = \frac{1}{r_3} \left[ 1 + 2 \frac{m_2}{m_0 + m_2} \frac{r_2}{r_3} \cos s_{23} + \frac{m_2}{m_0 + m_2} \left( \frac{r_2}{r_3} \right)^2 \right]^{-1/2}$$

Clearly the first and third of these may be expanded with the aid of Legendre polynomials, while for the second a binomial expansion will serve, so that:

$$\frac{1}{r_{12}} = \frac{1}{r_2} \sum_{n=0}^{\infty} \left( \frac{r_1}{r_2} \right)^n P_n(\cos s_{12})$$

$$\frac{1}{r_{03}} = \frac{1}{r_3} \sum_{n=0}^{\infty} \left( \frac{m_2}{m_0 + m_2} \frac{r_2}{r_3} \right)^n P_n(\cos s_{23})$$

$$\frac{1}{r_{13}} = \frac{1}{r_3} \left[ 1 + \frac{3}{8} X^2 + \frac{5}{16} X^3 + \frac{35}{128} X^4 + \dots \right]$$

where

$$X = 2 \frac{m_2}{m_0 + m_2} \frac{r_2}{r_3} \cos s_{23} + \frac{m_2}{m_0 + m_2} \frac{r_2^2}{r_3^2} \left[ 2 \frac{r_1}{r_3} \cos s_{13} - 2 \frac{m_2}{m_0 + m_2} \frac{r_1 r_2}{r_3 r_3} \cos s_{12} + \frac{r_1^2}{r_3^2} \right]$$

In order to determine the truncation point for each series we evaluate the coefficients and retaining terms with numerical values up to  $10^{-8}$  then the above expressions yield

$$\frac{1}{r_{12}} = \frac{1}{r_2} + \frac{r_1}{r_2^2} \cos s_{12} + \frac{r_1^2}{2r_2^3} (3 \cos^2 s_{12} - 1) + \frac{r_1^3}{2r_2^4} (5 \cos^3 s_{12} - 3 \cos s_{12}) + \frac{r_1^4}{8r_2^5} (35 \cos^4 s_{12} - 30 \cos^2 s_{12} + 3) \quad (6.1)$$

$$\frac{1}{r_{03}} = \frac{1}{r_3} \left[ 1 + 3M \frac{r_2}{r_3} \cos s_{23} \right] \quad (6.2)$$

$$\frac{1}{r_{13}} = \frac{1}{r_3} - M \frac{r_2}{r_3^2} \cos s_{23} + \frac{r_1}{r_3} \cos s_{13} + M \frac{r_1 r_2}{r_3^3} \cos s_{12}$$

$$- \frac{1}{2} M^2 \frac{r_2^2}{r_3^3} - \frac{r_1^2}{2r_3^3} + \frac{3}{2} M^2 \frac{r_2^2}{r_3^3} \cos^2 s_{23} \quad (6.3)$$

$$+ \frac{3r_1^2}{2r_3^3} \cos^2 s_{13} - 3M \frac{r_1 r_2}{r_3^3} \cos s_{13} \cos s_{23} + 3M \frac{r_1^2 r_2}{r_3^4} \cos s_{12} \cos s_{13}$$

where  $M = m_2 / (m_0 + m_2)$ . Substituting into Eq. (5), retaining terms up to  $10^{08}$  we obtain

$$U_{grav.} = k^2 \left[ \frac{m_2}{2} \frac{r_1^2}{r_2^3} (3 \cos^2 s_{12} - 1) + \frac{m_2}{2} \frac{r_1^3}{r_2^4} (5 \cos^3 s_{12} - 3 \cos s_{12}) + \frac{m_3}{2} \frac{r_1^2}{r_3^3} (3 \cos^2 s_{13} - 1) \right] \quad (7)$$

The next step is to express  $\cos s_{12}$  and  $\cos s_{13}$  in terms of the orbital elements of the orbiter, the moon, the earth and the sun. It is readily clear from Fig. (2) that the base vectors along  $\bar{r}_A$  and  $\bar{r}_B$  may be written as:

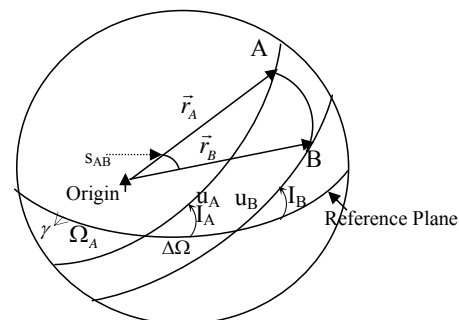


Fig. (2). Geometric relation between the angle  $S_{AB}$  and the orbital elements of any two orbits A and B.

$$\hat{r}_A = \begin{pmatrix} \cos u_A \\ \cos s_A \sin u_A \\ \sin s_A \sin u_A \end{pmatrix}$$

$$\hat{r}_B = \begin{pmatrix} \cos \Delta\Omega \cos u_B - \sin \Delta\Omega \sin u_B \\ \sin \Delta\Omega \cos u_B + \cos \Delta\Omega \sin u_B \\ \sin s_B \sin u_B \end{pmatrix}$$

where  $\hat{r}_i$  is a unit vector,  $\Delta\Omega = \Omega_B - \Omega_A$  and  $u_i = f_i + \varpi_i$  from which

$$\cos S_{AB} = \cos \Delta\Omega \cos u_A \cos u_B - \sin \Delta\Omega \cos u_A \sin u_B + \sin \Delta\Omega \sin u_A \cos u_B + \cos \Delta\Omega \sin u_A \sin u_B + \sin s_A \sin s_B \sin u_A \sin u_B$$

Adopting reference frame is an equatorial system with the positive X-axis toward the vernal equinox, Z- axis toward the north pole of the equator, and the Y-axis completing a right handed system. We note in the development of  $\cos s_{12}$  and  $\cos s_{cs}$  the appearance of  $I_1, I_2$  and  $I_3$  the inclinations, to the equator, of the orbits of the satellite, the moon and the sun respectively. The final expressions for  $\cos s_{12}$  and  $\cos s_{13}$  become after some lengthy algebra:

$$\begin{aligned} \cos s_{12} &= \frac{1}{2} [(1 - c_1 c_2) \cos \Delta\Omega_2 \cos(u_1 + u_2) + \\ & (1 + c_1 c_2) \cos \Delta\Omega_2 \cos(u_1 - u_2) \\ & + (c_1 - c_2) \sin \Delta\Omega_2 \sin(u_1 + u_2) + (c_1 + c_2) \sin \Delta\Omega_2 \sin(u_1 - u_2)] \quad (8) \\ \cos s_{13} &+ \frac{1}{2} [(1 - c_1 c_3) \cos \Delta\Omega_3 \cos(u_1 + u_3) + \\ & (1 + c_1 c_3) \cos \Delta\Omega_3 \cos(u_1 - u_3) \\ & + (c_1 - c_3) \sin \Delta\Omega_3 \sin(u_1 + u_3) + (c_1 + c_3) \sin \Delta\Omega_3 \sin(u_1 - u_3)] \end{aligned}$$

where  $c_i = \cos I_i, s_i = \sin I_i$  and  $\Delta\Omega_j = \Omega_j - \Omega_i$ .

Substituting Eqs. (8) into Eq. (7) and dropping the subscript '1' designating the satellite, we obtain the force function  $U_{grav}$ .

### 2.3. The Force Functions Due to the Oblateness

We have so far considered the earth as a point mass introducing the effect of earth oblateness we can write the disturbing function in the form:

$$U = \frac{\mu}{r_1} + U_{grav.} + U_{ob.} \quad (9)$$

where  $U_{ob}$  represents the contribution of the oblateness of the earth to the disturbing function. Assuming an axially symmetric geopotential truncated beyond the fourth zonal

harmonic, the disturbing function due to earth oblateness acquires the form (e. g. Fitzpatrick (1970) [24])

$$U_{oo.} = \sum_{r=2}^4 \frac{\mu}{r} \sum_{n=0}^r J_n \left(\frac{R}{r}\right)^n P_n(\sin \delta) \quad (10)$$

where  $r$  and  $\delta$  are the geocentric distance and declination of the satellite,  $J_n$  are harmonic coefficients with  $J_2 \approx 10^{-6}$  and  $J_n (n > 4) \approx 0^{-6}$ ,  $R$  is the equatorial radius of the earth,  $P_n(\sin \delta)$  are Legendre polynomials.

### 2.4. The Force Functions Due to Solar Radiation Pressure

Consider  $\sigma$  to be the absolute value of the acceleration of the satellite arising from the solar radiation pressure. Then the disturbing function for the radiation pressure will be given by (see the earlier section, 'the Angles  $S_{13}$  and  $S'_{13}$ ).

$$R_{\sigma} = \sigma r \cos S'_{13} \quad (11)$$

The order of magnitude of the radiation pressure acceleration depends on the area to the mass ratio of the satellite. If the area to the mass ratio of the satellite is of  $O(10^{\square})$  then the disturbing acceleration due to the radiation pressure is of fourth order.

### 2.5. The Force Functions in Terms of the Delaunay Elements

Reproducing Eq. (9), the force function can now be written as:

$$U = \frac{\mu}{r} + U_{grav.} + U_{ob.} + R_{\sigma}$$

where  $U_{grav.}, U_{ob.}$  and  $R_{\sigma}$  are given by Eq.(7), Eq.(10) and Eq.(11) respectively.

We can assume, with sufficient accuracy, that the moon and the sun move in circular orbits so that  $u_2 = n_2 t + \text{constant}$  and  $u_3 = n_3 t + \text{constant}$ ,

where  $n_2$  and  $n_3$  are the respective mean motions. In what follows we form the Hamiltonian of the problem in terms of the Delaunay elements. We have

$$F = \mu \frac{\mu^2}{2L^2} + U_{grav.} + U_{of.} + R_{\sigma} \quad (12)$$

Before proceeding to write  $U_{grav.}, U_{ob.}$  and  $R_{\sigma}$  in terms of Delaunay's elements we note that the time  $t$  appears explicitly in  $F'$  through its explicit appearance in  $u_2, u_3$  and  $R_{\sigma}$ . Hence we augment the set by the pair  $k_{\circ} = n_2 t + \text{constant}, k_{\ominus} = n_3 t + \text{constant}$ , and their conjugates  $K_{\circ}$  and  $K_{\ominus}$  Hori (1966) [25]

Our set then consists of the ten elements

$$l = \text{mean anomaly}, \quad L = (\mu a)^{1/2}$$

$$g = \text{argument of perigee, } G = L(1 - e^2)^{1/2} \quad (13)$$

$$h = \Omega - \Omega_2 \quad H = Gc$$

= longitude of the node measured, from the node of the moon's orbit

$$k_o = u_2, \quad K_o$$

$$k_\theta = u_3, \quad K_\theta$$

In terms of this set the Hamiltonian becomes

$$F = \frac{\mu^2}{2L^2} - n_2 k_o - n_3 k_\theta + U_{grav.} + U_{ob.} + R_\sigma \quad (14)$$

### 3. THE HAMILTONIAN

Regarding  $n_2$  (the mean motion of the moon) as the small parameter of the problem, therefore the perturbing terms in the Hamiltonian can be grouped and expressed in expandable form as:

$$F = \sum_{n=0}^5 \frac{\epsilon^n}{n!} F_n \quad (\epsilon = n_2) \quad (15)$$

with

$$F_o = \frac{\mu^2}{2L^2} \quad (16.1)$$

$$F_1 = -K_o \quad (16.2)$$

$$F_2 = \gamma_2 \phi^3 z_2 + AK_\theta \quad (16.3)$$

$$F_3 = 0 \quad (16.4)$$

$$F_4 = \sum_{m=1}^5 F_{4m} \quad (16.5)$$

$$F_{41} = A_1 \frac{L^4 x_1}{\phi^2}, \quad F_{42} = A_3 \frac{L^4 x_3}{\phi^2}, \quad F_{43} = \gamma_3 \phi^4 z_3, \quad F_{44} = \gamma_4 \phi^5 z_4, \quad F_{45} = R_\sigma.$$

$$F_5 = A_2 \frac{L^6 x_2}{\phi^3} \quad (16.6)$$

where

$$\begin{aligned} x_1 &= \theta_o + \theta_1 \cos h + (\theta_2 - \theta_1 \cos h + \theta_3 \cos 2h) \cos 2u \\ &\quad + (\theta_3 - \theta_1 \cos h + \theta_2 \cos 2h) \cos 2k_o - (\theta_4 \sin 2h + \theta_5 \sin h) \sin 2u \\ &\quad - (\theta_6 \sin h + \theta_7 \sin 2h) \sin 2k_o + \theta_8 \cos 2h \cos 2(u + k_o) \\ &\quad + [\theta_9 \cos 2h + (4\theta_{10} + \theta_1) \cos h] \cos(u - k_o) + 2\theta_{10} \sin h \sin 2(u + k_o) \\ &\quad - [(\theta_6 - \theta_5) \sin h + \theta_{11} \sin 2h] \sin 2(u - k_o) \\ x_2 &= \theta_{12} \cos h \cos(u + k_o) + \theta_{14} \cos h \cos(u - k_o) + \theta_{16} \cos 3h \cos 3(u - k_o) \\ &\quad + (\theta_{17} \cos h + \theta_{18} \cos 3h) [\cos(3u - k_o) + \cos(u - 3k_o)] \\ &\quad - \theta_{20} \sin 3h \sin 3(u - k_o) - \theta_{21} \sin h \sin(u + k_o) - \theta_{22} \sin h \sin(u - k_o) \\ &\quad - (\theta_{23} \sin h + \theta_{18} \sin 3h) [\sin(3u - k_o) - \sin(u - 3k_o)] \\ x_3 &= \theta_{24} + \theta_{25} \cos 2(h + \Gamma) \cos 2u + \theta_{25} \cos 2k_\theta - \theta_{26} \sin 2(h + \Gamma) \sin 2u \\ &\quad + \theta_{27} \cos 2(h + \Gamma) \cos 2(u + k_\theta) + \theta_{28} \cos 2(h + \Gamma) \cos 2(u - k_\theta) \\ &\quad - \theta_{29} \sin 2(h + \Gamma) \sin 2(u - k_\theta) \end{aligned}$$

$$z_2 = \frac{1}{L^6} (\theta_{30} + \frac{3}{2} \theta_2 \cos 2u),$$

$$z_3 = \frac{1}{L^8} (\theta_{31} \sin u + \theta_{32} \sin 3u),$$

$$z_4 = \frac{1}{L^{10}} (\theta_{33} + \theta_{34} \cos 2u + \theta_{35} \cos 4u)$$

$$\begin{aligned} R_\sigma &= -\frac{1}{2} \sigma r [(1 - cc_3) \cos(h + \Gamma) \cos(u + k_\theta) + \\ &\quad (1 + cc_3) \cos(h + \Gamma) \cos(u - k_\theta) - (c - c_3) \sin(h + \Gamma) \sin(u + k_\theta) \\ &\quad - (c + c_3) \sin(h + \Gamma) \sin(u - k_\theta)] \\ &= -\frac{1}{4} \sigma r \sum_{i=-1}^1 \sum_{j=-1}^1 \vartheta_{ij} \{ \cos[g + i(h + \Gamma) + jk_\theta] \cos f \\ &\quad - \sin[g + i(h + \Gamma) + jk_\theta] \sin f \} \end{aligned}$$

the  $A$ 's and  $\gamma$ 's are zero order quantities defined by:

$$A = -2n_3 / \epsilon^2 \quad A_1 = \frac{9}{2\epsilon^4} \frac{k^2 m_2}{\mu^2 a_2^3}, \quad A_2 = \frac{75}{32\epsilon^5} \frac{k^2 m_2}{\mu^3 a_2^4},$$

$$A_3 = \frac{9}{2\epsilon^4} \frac{k^2 m_3}{\mu^2 a_3^3},$$

$$\gamma_2 = \frac{1}{2\epsilon^2} R^2 \mu^4 J_2, \quad \gamma_3 = \frac{3}{\epsilon^4} R^3 \mu^5 J_3, \quad \gamma_4 = -\frac{3}{8\epsilon^4} R^4 \mu^6 J_4,$$

$$\Gamma = \Omega_2 - \Omega_3,$$

and  $\theta_i (i = 0, 1, 2, \dots, 30)$  are functions of  $c_i$  and  $s_i$ , given

by:

$$\theta_0 = (4/3) - 2(s^2 + s_2^2), \quad \theta_1 = 4ss_2cc_2, \quad \theta_2 = 2s^2,$$

$$\theta_3 = 2s_2^2, \quad \theta_4 = cs_2^2, \quad \theta_5 = -2ss_2c_2,$$

$$\theta_6 = 2ss_2c, \quad \theta_7 = -c_2s^2, \quad \theta_8 = 2 - s^2 - s_2^2 - 2cc_2,$$

$$\theta_9 = 2 - s^2 - s_2^2 + 2cc_2,$$

$$\theta_{10} = ss_2, \quad \theta_{11} = c + cc_2^2 + c^2c_2 + c_2$$

$$\theta_{12} = -(102/5) + (347/5)cc_2 - 6cc_2^3 + 6c^2 - 30c_2^2c^2 - 6c^3c_2 - 18c^3c_2^3 + 6c_2^2,$$

$$\theta_{13} = 1 - c^2 + c^2c_2^2 + c_2^2,$$

$$\theta_{14} = -(147/5) - (27/5)cc_2 + 3cc_2^3 + 3c^2 + 33c^2c_2^2 + 3c^3c_2 + 9c^3c_2^3 + 3c_2^2,$$

$$\theta_{15} = 3(1 + cc_2 - cc_2^3 - c^2 + c^2c_2^2 - c^3c_2 - c^3c_2^3 - c_2^2),$$

$$\theta_{16} = 1 + 8cc_2 + 3c^2 + c^2c_2^2 + 3c_2^2,$$

$$\theta_{17} = 3(3 + cc_2 + cc_2^3 - c^2 - c^2c_2^2 + c^3c_2 - 3c^3c_2^3 - c_2^2),$$

$$\theta_{18} = 1 + 4cc_2 + c^2 + c^2c_2^2 + c_2^2,$$

$$\theta_{19} = 3(c - cc_2^2 - c^2c_2 + c^2c_2^3 - c^3 + c^3c_2^2 + c_2 - c_2^3),$$

$$\theta_{20} = 3 + 8cc_2 + c^2 + 3c^2c_2^2 + c_2^2,$$

$$\begin{aligned} \theta_{21} &= -(162/5)c - 30cc_2^2 + 30c^2c_2 - 6c^2c_2^3 + 18c^3 + 6c^3c_2^2 + (222/5)c_2 - 18c_2^3, \\ \theta_{22} &= -(177/5)c + 33cc_2^2 + 33c^2c_2 + 3c^2c_2^3 + 9c^3 + 3c^3c_2^2 - (177/5)c_2 + 9c_2^3, \\ \theta_{23} &= 3(c + cc_2^2 - c^2c_2 - c^2c_2^3 - 3c^3 + c^3c_2^2 - c_2 + 3c_2^3), \\ \theta_{24} &= (4/3) - 2s_3^2, \quad \theta_{25} = 2s_3^2, \\ \theta_{26} &= cs_3^2, \quad \theta_{27} = 2 - s_3^2 - 2cc_3, \quad \theta_{28} = 2 - s_3^2 + 2cc_3, \\ \theta_{29} &= c + cc_3^2 + c^2c_3 + c_3, \quad \theta_{30} = 2 - 3s_3^2, \\ \vartheta_{-1,-1} &= 1 - c - c_3 + cc_3, \quad \vartheta_{1,-1} = 1 + c + c_3 + cc_3 \\ \vartheta_{-1,1} &= 1 - c + c_3 - cc_3 \quad \text{and} \quad \vartheta_{1,1} = 1 + c - c_3 - cc_3. \end{aligned}$$

4. THE PROCEDURE FOR SOLUTION

The Hamiltonian  $F$  of the problem defined by Equations

(15) and (16) can be represented by  $F = F_0 + \sum_{n=1}^5 \frac{\epsilon^n}{n!} F_n$

where  $F_0$  is the unperturbed Hamiltonian. It represents the integrable part of the problem. Then the equations of motion can be written in the form

$$\dot{u} = \frac{\delta F}{\delta U}, \quad \dot{U} = -\frac{\delta F}{\delta u} \tag{17}$$

where  $(u, U)$  is the ten-vector of adopted canonical variables  $(l, g, h, k_\theta, k_\Theta; L, G, H, K_\theta, K_\Theta)$ .

To develop the solution including the effects of the perturbing Hamiltonian  $F_n$ , it is required now to construct three canonical transformations  $(u, U; \epsilon) \rightarrow (u', U')$  and  $(u', U'; \epsilon) \rightarrow (u'', U'')$  and  $(u'', U''; \epsilon) \rightarrow (u''', U''')$  analytic in  $\epsilon$  at  $\epsilon = 0$ , to remove in succession the fast and slow angles from  $F_n$ . This process reduces  $U'''$  to constants and  $u'''$  to linear functions of time.

A theorem by Lie has been applied to construct explicit transformations. Deprit (1969) [22] constructed an algorithm to generate the new Hamiltonian recursively using the Lie transform, while Kamel (1969) [23] simplified Deprit's algorithm. We now outline the perturbation technique based on Kamel [23] to evaluate the secular and the periodic perturbations.

The transformed Hamiltonians and the corresponding generators will be assumed expandable as

$$F^*(-, u'_2, u'_3, u'_4, u'_5; U'; \epsilon) = F_0^*(U'_1) + \sum_{n=1}^6 \frac{\epsilon^n}{n!} F_n^*(-, u'_2, u'_3, u'_4, u'_5; U') \tag{18.1}$$

$$w(u', U'; \epsilon) = \sum_{n=0}^4 \frac{\epsilon^n}{n!} w_{n+1}(u'; U') \tag{18.2}$$

$$F^{***}(-, -, u''_3, u''_4, u''_5; U''; \epsilon) = F_0^{***}(U''_1) + F_1^{***}(U''_2) + \sum_{n=2}^6 \frac{\epsilon^n}{n!} F_n^{***}(-, -, u''_3, u''_4, u''_5; U'') \tag{19.1}$$

$$w^*(-, u''_2, u''_3, u''_4, u''_5; U''; \epsilon) = \sum_{n=0}^3 \frac{\epsilon^n}{n!} w_{n+1}^*(-, u''_2, u''_3, u''_4, u''_5; U'') \tag{19.2}$$

$$F^{****}(-; U'''; \epsilon) = F_0^{****}(U'''_1) + F_1^{****}(U'''_2) + \sum_{n=2}^6 \frac{\epsilon^n}{n!} F_n^{****}(-; U''') \tag{20.1}$$

$$w^{**}(-, -, u'''_3, u'''_4, u'''_5; U'''; \epsilon) = \sum_{n=0}^3 \frac{\epsilon^n}{n!} w_{n+1}^{**}(-, -, u'''_3, u'''_4, u'''_5; U''') \tag{20.2}$$

5. THE NORMALIZED HAMILTONIAN

As is clear from the process we have five angles of different rates of change. These angles are: the mean anomaly  $l$  is the fast variable, the Moon's mean longitude  $k_\theta$  is the intermediate variable, while the argument of perigee  $g$ , the longitude of the ascending node  $h$  and the Sun's mean longitude  $k_\Theta$  are the slow variables. We thus need three canonical transformations to eliminate in succession the short, the intermediate and the long period terms from the Hamiltonian using a perturbation technique based on Lie series and Lie transform Kamel [23].

5.1. Elimination of the Short Period Terms

Since the integrable part of the Hamiltonian  $F_0$  depends on  $L$ , then the mean anomaly  $l$  is considered as the fast variables of the problem. We thus use the Lie transform to eliminate the short periodic terms from the Hamiltonian with the averages being taken over  $l$ . This step effected by a canonical transformation from the old Hamiltonian Eqns. (15) to a new one  $F^*(-, g', h', k'_\theta, k'_\Theta; L', G', H', K'_\theta, K'_\Theta)$  that is to be performed via a generating function  $W(l', g', h', k'_\theta, k'_\Theta; L', G', H', K'_\theta, K'_\Theta)$ .

5.1.1. Results of the Computations

Applying the recursive process developed by Deprit [22] and Kamel [23] to the Hamiltonian defined by Eqs. (15) and (16) the following results for  $F^*$  and  $W$  are obtained at different orders after some lengthy manipulations. All variables are understood to be single - primed, but the primes are dropped for the sake of simplicity of writing.

$$F_0^* = \frac{\mu^2}{2L^2} \tag{21}$$

$$F_1^* = -K_\Theta \tag{22}$$

$$F_2^* = \gamma_2 \theta_{30} \eta_{33} + AK_\Theta, (\eta_{ij} = L^{-i}G^{-j}, \eta = L/G) \tag{23}$$

$$F_3^* = 0 \tag{24}$$

$$F_4^* = L^4 [a_0(1 + \frac{3}{2}e^2) + \frac{5}{2}e^2(a_2 \cos 2g + b_2 \sin 2g)] + \sum_{i=0}^1 (\chi_i^c \cos 2ig) + \chi_i^s \sin ig + \frac{3\sigma ae}{8} \sum_{i=-1}^1 \sum_{j=-1}^1 \vartheta_{ij} \cos(g + i(h + \Gamma) + jk_\Theta) \tag{25}$$

$$F_5^* = A_2 L^6 [(-\frac{5}{2}e - \frac{15}{8}e^3)(a_1 \cos g + b_1 \sin g) - \frac{35}{8}e^3(a_3 \cos 3g + b_3 \sin 3g)] - \frac{25}{2} \frac{A_1}{\mu^2} (\eta_{-6,-1} - \eta_{-4,-3}) [(\tilde{v}_3^c \cos 2g - \tilde{v}_2^c \sin 2g) \sin 2k_\theta + (\tilde{v}_3^c \cos 2g - \tilde{v}_2^c \sin 2g) \cos 2k_\theta] \tag{26}$$

$$F_6^* = \sum_{i=1}^6 F_{6i}^* \quad (27)$$

$$F_{61}^* = -15 \sum_{j=0}^2 d_j \cos 2jg$$

$$F_{62}^* = \sum_{j=1}^3 (X_j^c \cos jk_o + X_j^s \sin jk_o)$$

$$F_{63}^* = 10 \frac{\gamma_2}{\mu^2} \sum_{j=0}^1 (M_j^c \cos 2jk_o + M_j^s \sin 2jk_o)$$

$$F_{64}^* = -5\gamma_2 \sum_{j=0}^1 (N_j^c \cos 2jk_o + N_j^s \sin 2jk_o)$$

$$F_{65}^* = 5 \sum_{j=0}^1 (m_j^c \cos 2jk_o + m_j^s \sin 2jk_o)$$

$$F_{66}^* = 10 \sum_{j=0}^1 (n_j^c \cos 2jk_o + n_j^s \sin 2jk_o)$$

Where the quantities  $a_i, b_i, \chi_i^c, \chi_i^s, \tilde{v}_i^c, \tilde{v}_i^s, X_j^c, X_j^s, d_j, M_j^c, M_j^s, N_j^c, N_j^s, m_j^c, m_j^s, n_j^c$  and  $n_j^s$  are functions of the action and angular variables.

$$W_1 = 0 \quad (28)$$

$$W_2 = \frac{-\gamma_2}{\mu^2 G^3} \left\{ \theta_{30} (f - l + e \sin f) + \frac{3}{2} \theta_2 \left[ \frac{e}{2} \sin F_{12} + \frac{1}{2} \sin F_{22} + \frac{e}{6} \sin F_{32} \right] \right\} \quad (29)$$

$$W_3 = 0 \quad (30)$$

$$W_4 = \sum_{i=1}^4 W_{4i} \quad (31)$$

with

$$W_{41} = -\frac{L^7}{\mu^2} \left\{ \sum_{i=1}^3 \sum_{j=1}^1 [B_{ij}^c \cos(iE + 2jg) + B_{ij}^s \sin(iE + 2jg)] + B_{\infty}(E - l) \right\}$$

$$W_{42} = -\frac{L^3}{\mu^2} \left\{ \sum_{i=1}^7 \sum_{j=1}^3 [D_{ij}^c \cos F_{ij} + D_{ij}^s \sin F_{i,2j}] + D_{\infty}(f - l) \right\}$$

$$W_{43} = -\frac{9\gamma_2^2}{\mu^4} \eta_{52} \left\{ \sum_{i=1}^6 \sum_{j=1}^2 [\psi_{ij} \sin F_{i,2j} + \sum_{j=0}^1 \tilde{\psi}_j (f - l) \cos 2jg] + \sum_{i=1}^4 \psi_i + \frac{1}{e^2} (5s^4 - 4s^2)(-\eta^5 + 3\eta^3 - 2\eta^2)(E - l) \cos 2g \right\}$$

$$W_{44} = \frac{L^3 \sigma a}{16\mu^2} \sum_{i=1}^1 \sum_{j=1}^1 \vartheta_{ij} \{ [4(1 + e^2) \sin E - e \sin 2E - 6e(E - l)] \times \cos(g + i(h + \Gamma) + jk_o) + \frac{1}{\eta} (4 \cos E - e \cos 2E) \times \sin(g + i(h + \Gamma) + jk_o) \}$$

$$W_5 = W_{51} + W_{52} \quad (32)$$

with

$$W_{51} = -\frac{A_j}{\mu^2} L^9 \left\{ \sum_{i=0}^4 \sum_{j=-3}^3 [\xi^{sy} \sin(iE + jg) + \xi^{cy} \cos(iE + jg)] + \xi(E - l) \right\}$$

$$W_{52} = \frac{5L^{10}}{\mu^2} \left\{ \sum_{i=1}^4 \sum_{j=-1}^1 [\xi_{ij}^c \cos(iE + 2jg) + \xi_{ij}^s \sin(iE + 2jg)] + \xi_{\infty}(E - l) \right\}$$

where  $F_{ij} = if + ig$

### 5.2. Intermediate - Period Transformation

In this section the intermediate - period terms (those periodic in  $k_o$ ) are eliminated via a second canonical transformation, leaving the Hamiltonian containing only the slow angle variables. The procedure is essentially similar to the short periodic terms but with averages being taken over  $k_o$

#### 5.2.1. Results of Computation

Application of the above procedure to the Hamiltonian  $F^*$  given by (21) to (27) yields: (all variables on the right are now double primed)

$$F_0^{**} = \frac{\mu^2}{2L^2} \quad (33)$$

$$F_1^{**} = F_1^* = -K_o, \quad F_2^{**} = F_2^* = \gamma_2 \theta_{30} \eta_{33} + AK_o, \quad F_3^{**} = 0 \quad (34)$$

$$F_4^{**} = L^4 \left[ \left(1 + \frac{3}{2} e^2\right) \lambda_{36} + \frac{5}{2} e^2 (\lambda_{39} \cos 2g + \lambda_{42} \sin 2g) \right] + \sum_{i=0}^1 (x_i^c \cos 2ig + x_i^s \sin 2ig) + \frac{3\sigma a e}{8} \sum_{i=1}^1 \sum_{j=1}^1 \vartheta_{ij} \cos(g + i(h + \Gamma) + jk_o) \quad (35)$$

$$F_5^{**} = 0 \quad (36)$$

$$F_6^{**} = -15 \sum_{i=0}^2 d_i \cos 2ig + \frac{10\gamma_2}{\mu^2} M_0^c + 5m_0^c + 10n_0^c - 5\gamma_2 N_0^c \quad (37)$$

$$W_1^* = W_2^* = 0 \quad (38)$$

$$W_3^* = \frac{L^4}{8} \left\{ \left[ \left(1 + \frac{3}{2} e^2\right) \lambda_{37} + \frac{5}{2} e^2 (\lambda_{40} \cos 2g + \lambda_{43} \sin 2g) \right] \cos 2k_o - \left[ \left(1 + \frac{3}{2} e^2\right) \lambda_{38} + \frac{5}{2} e^2 (\lambda_{41} \cos 2g + \lambda_{44} \sin 2g) \right] \sin 2k_o \right\} \quad (39)$$

$$W_4^* = -\frac{1}{5} \sum_{i=1}^3 \frac{1}{i} (v_i^c \sin ik_o - v_i^s \cos ik_o) \quad (40)$$

### 5.3. Elimination of Long-Period Terms

The long-period terms, those periodic in  $g, h$  and  $k_o$  will be eliminated and the elements of long period transformation will be obtained. The transformation is being made via a generator  $W^{**}$  where the old and new Hamiltonians are related through

$$F^{**}(-, g'', h'', -, k_o''; L'', G'', H'', K_o'', K_{\theta}'') = F^{***}(L''', G''', H''', K_o''', K_{\theta}''')$$

The transformation equations are essentially the same as those of last section with relevant changes of primes and asterisks and the averages being taken over  $g, h$  and  $k_o$ .

5.3.1. Results of Computation

All variables on the right are now triple primed

$$F_0^{***} = \frac{\mu^2}{2L^2} \tag{41}$$

$$F_1^{***} = -k_\theta \tag{42}$$

$$F_2^{***} = F_2^{**} = \gamma_2 \theta_{30} \eta_{33} + AK_\theta \tag{43}$$

$$F_3^{***} = 0 \tag{44}$$

$$F_4^{***} = L^4 \left[ \left(1 + \frac{3}{2} e^2\right) (A_1 \theta_0 + A_3 \theta_{24}) \right] \tag{45}$$

$$F_5^{***} = 0 \tag{46}$$

$$F_6^{***} = -15d_0 + \frac{\gamma_2}{\mu^2} \{10(A_1 L^4 \tilde{\sigma}_{00} + \tilde{\chi}_0^c) + 5[\gamma_4 \tilde{M}_0^c + L^4 (A_1 \tilde{\rho}_{00}^c + A_3 \tilde{\rho}_{000}^c)]\} \\ + 10[\tilde{W}_{z=00}^c + \tilde{W}_{z=2,-1}^c + \tilde{W}_{z=-2,1}^c + A_1(\tilde{\lambda}_{z=000}^c + \tilde{\lambda}_{z=01,-1}^c + \tilde{\lambda}_{z=0,-11}^c) + A_3 \tilde{\omega}_{z=00}^c] \tag{47} \\ + \frac{1}{2} \sum_{i=1}^4 [\tilde{J}_0^i + \tilde{\rho}_0^i + A_1(\tilde{J}_1^i + \tilde{\rho}_1^i) + A_1 A_3(\tilde{J}_{13}^i + \tilde{\rho}_{13}^i) + A_1^2(\tilde{J}_{11}^i + \tilde{\rho}_{11}^i) \\ + A_3(\tilde{J}_3^i + \tilde{\rho}_3^i) + A_3^2(\tilde{J}_{33}^i + \tilde{\rho}_{33}^i)]$$

$$W_1^{**} = 0 \tag{48}$$

$$W_2^{**} = \frac{1}{6} A_1 \sum_{i=0}^{2'} \sum_{j=-1}^1 \Pi_{ij} \frac{\sin(ih + 2jg)}{i\tilde{\alpha}_1 + 2j\tilde{\alpha}_3} + \frac{1}{12\tilde{\alpha}_3} \sum_{i=0}^1 \frac{1}{i} (\chi_i^c \sin 2ig - 2\chi_i^s \cos ig) \\ + \frac{1}{12} A_3 \sum_{m=0}^1 \sum_{i=-1}^1 \sum_{j=-1}^1 \tilde{\Pi}_{mij} \frac{\sin 2[m(h+\Gamma) + ik_\theta + jg]}{(m\tilde{\alpha}_1 + i\tilde{\alpha}_2 + j\tilde{\alpha}_3)} \\ + \frac{\sigma a e}{16} \sum_{i=-1}^1 \sum_{j=-1}^1 \frac{\vartheta_{ij}}{(i\tilde{\alpha}_1 + j\tilde{\alpha}_2 + \tilde{\alpha}_3)} \sin(g + i(h+\Gamma) + jk_\theta) \tag{49}$$

where the prime over the summation signs indicates that the sine terms are to be replaced in succession by  $\frac{1}{2} \Pi_{00}(\frac{h}{\tilde{\alpha}_1} + \frac{g}{\tilde{\alpha}_3})$ ,  $\frac{1}{3} \tilde{\Pi}_{000}(\frac{g}{\tilde{\alpha}_3} + \frac{h}{\tilde{\alpha}_1} + \frac{k_\theta}{\tilde{\alpha}_2})$  and  $\chi_o^c \frac{g}{\tilde{\alpha}_3}$  respectively.

$$W_3^{**} = 0 \tag{50}$$

and

The z over the summation signs indicates that terms with zero divisors are to be excluded.

$$W_4^{**} = -\sum_{i=1}^{2'} \frac{d_i}{2i\tilde{\alpha}_3} \sin 2ig + \frac{2\gamma_2}{3\mu^2} \{L^4 [A_1 \sum_{i=0}^{2'} \sum_{j=-2}^{2'} \frac{\tilde{\sigma}_{ij}}{i\tilde{\alpha}_1 + 2j\tilde{\alpha}_3} \sin(ih + 2jg) \\ + A_3 \sum_{m=0}^2 \sum_{i=-1}^1 \sum_{j=-2}^2 \frac{\tilde{\sigma}_{mij}}{2(m\tilde{\alpha}_1 + i\tilde{\alpha}_2 + j\tilde{\alpha}_3)} \sin 2[m(h+\Gamma) \\ + ik_\theta + jg] + \sum_{i=0}^{2'} \frac{1}{\tilde{\alpha}_3} (\frac{1}{2i} \tilde{\chi}_i^c \sin 2ig - \frac{1}{1+i} \tilde{\chi}_i^s \cos(i+1)g)\} \\ - \frac{\gamma_2}{3\mu^2} \{A_1 \sum_{i=0}^2 \sum_{j=-1}^1 \frac{\tilde{N}_{ij}}{i\tilde{\alpha}_1 + 2j\tilde{\alpha}_3} \sin(2jg + ih) \\ + A_3 \sum_{i=-1}^1 \sum_{j=-1}^1 \frac{\tilde{N}_{ij}}{2(\tilde{\alpha}_1 + i\tilde{\alpha}_2 + j\tilde{\alpha}_3)} \sin 2(jg + h + \Gamma + ik_\theta) - 27 \frac{\gamma_2^2}{\mu^2} (1 - 5C^2) \times \\ \eta_{86} (\sum_{i=1}^{6z} \sum_{j=-1}^{2'} \frac{j}{i\tilde{\alpha}_3} \psi_{ij} \tilde{g}_{01i}^c \sin 2ig + \frac{1}{2\tilde{\alpha}_3} \psi \sin 2g) - 3(1 - 5C^2) \eta_{04} \sum_{i=-1}^3 \sum_{j=1}^7 \frac{1}{\tilde{\alpha}_3} \times \\ (D_{ij}^s \sin 2jg + D_{ij}^c \cos jg) \tilde{g}_{01i}^c \} + \frac{\gamma_2}{3\mu^2} \{ \sum_{i=0}^3 (\gamma_4 \tilde{M}_i^c \frac{1}{2i\tilde{\alpha}_3} \sin 2ig - \gamma_3 \tilde{M}_i^s \times \\ \frac{1}{(2i+1)\tilde{\alpha}_3} \cos(2i+1)g) + L^4 [A_1 \sum_{i=-2}^{2'} \sum_{j=0}^{2'} \frac{1}{j\tilde{\alpha}_1 + 2i\tilde{\alpha}_3} (\tilde{\rho}_{ij}^c \sin(2ig + jh) - \tilde{\rho}_{ij}^s \cos(2ig + jh)) \\ + A_3 \sum_{i=-2}^{2'} \sum_{j=-1}^1 \sum_{m=0}^{2'} \frac{1}{2(m\tilde{\alpha}_1 + j\tilde{\alpha}_2 + i\tilde{\alpha}_3)} (\tilde{\rho}_{mij}^c \sin 2(ig + m(h+\Gamma) + jk_\theta) \\ - \tilde{\rho}_{mij}^s \cos 2(ig + m(h+\Gamma) + jk_\theta)) \} + \frac{2}{3} \{ \sum_{z=0}^i \sum_{j=-2}^{2'} \sum_{i=-1}^1 \frac{1}{(j+2i)\tilde{\alpha}_3} (\tilde{W}_{zji}^c \sin(j+2i)g \\ - \tilde{W}_{zji}^s \cos(j+2i)g) + A_1 \sum_{z=0}^3 \sum_{j=-2}^{2'} \sum_{i=0}^{2'} \sum_{n=-1}^1 \frac{1}{i\tilde{\alpha}_1 + (2j+2n)\tilde{\alpha}_3} (\tilde{\lambda}_{zijn}^c \sin(2(j+n)g + ih) \\ - \tilde{\lambda}_{zijn}^s \cos(2(j+n)g + ih)) + A_3 \sum_{z=0}^3 \sum_{j=-2}^{2'} \sum_{i=-1}^1 \sum_{n=-1}^1 \frac{1}{2(\tilde{\alpha}_1 + i\tilde{\alpha}_2 + (j+n)\tilde{\alpha}_3)} (\tilde{\lambda}_{zijn}^c \sin 2((j+n)g \\ + h + \Gamma + ik_\theta) - \tilde{\lambda}_{zijn}^s \cos 2((j+n)g + h + \Gamma + ik_\theta)) + A_3 \sum_{z=0}^3 \sum_{i=0}^1 \sum_{j=-2}^{2'} \frac{1}{2(i\tilde{\alpha}_2 + j\tilde{\alpha}_3)} (\tilde{\omega}_{zij}^c \times$$



$$\begin{aligned}
 & \sin 2(jg + ik_{\theta}) - \tilde{\omega}_{zj}^s \cos 2(jg + ik_{\theta}) - AA_3 \sum_{i=1}^{l'} \sum_{j=1}^{l'} \frac{1}{2(\tilde{\alpha}_1 + i\tilde{\alpha}_2 + j\tilde{\alpha}_3)} \tilde{\omega}_{ij} \sin 2(jg + h \\
 & + \Gamma + ik_{\theta}) + \frac{1}{15} \left\{ - \sum_{i=0}^{2'} \sum_{j=1}^{l'} \frac{\tilde{\alpha}_{4ij}}{i\tilde{\alpha}_1 + 2j\tilde{\alpha}_3} [(2A_1^2 \tilde{\alpha}_{82} + 2A_1 A_3 \tilde{\alpha}_{83})j + A_1^2 \tilde{\alpha}_{84} i] \sin(2jg + ih) \right. \\
 & - 2 \sum_{m=0}^{l'} \sum_{i=1}^{l'} \sum_{j=1}^{l'} \frac{\tilde{\alpha}_{5mij}}{2(m\tilde{\alpha}_1 + i\tilde{\alpha}_2 + j\tilde{\alpha}_3)} [(A_1 A_3 \tilde{\alpha}_{82} + A_3^2 \tilde{\alpha}_{83})j + A_1 A_3 \tilde{\alpha}_{84} m] \sin 2[jg + m(h \\
 & + \Gamma) + ik_{\theta}] - \frac{1}{\tilde{\alpha}_3} (A_1 \tilde{\alpha}_{82} + A_3 \tilde{\alpha}_{83})(\tilde{\alpha}_{61}^s \sin 2g + \tilde{\alpha}_{61}^c \cos g) \left. \right\} + \frac{1}{30} \sum_{i=1}^{4'} [\tilde{J}_0^i + \tilde{\rho}_0^i \\
 & + A_1 (\tilde{J}_1^i + \tilde{\rho}_1^i) + A_1 A_3 (\tilde{J}_{13}^i + \tilde{\rho}_{13}^i) + A_1^2 (\tilde{J}_1^i + \tilde{\rho}_1^i) + A_3 (\tilde{J}_3^i + \tilde{\rho}_3^i) + A_3^2 (\tilde{J}_{33}^i + \tilde{\rho}_{33}^i)]
 \end{aligned} \tag{51}$$

**6. THE COMPUTATION OF POSITION AND VELOCITY**

The equations of motion are now reduced to

$$\frac{dU'''}{dt} = - \frac{\partial F''''}{\partial u'''} = 0 \quad \frac{du'''}{dt} = \frac{\partial F''''}{\partial U'''} = \square \tag{52}$$

where  $\square$  's are arbitrary constant. These equations can be integrated to

$$U''' = U_0''' \quad u''' = u_0''' + t \square \tag{53}$$

where the constant  $(u_0''', U_0''')$  are to be determined from the initial conditions.

Let the elements  $(u_0, U_0)$  be known at a given initial epoch  $t_0$  then we can obtain the constants  $(u_0''', U_0''')$  as follows:

- 1) One can compute initial values  $(u_0', U_0')$  from

$$u_0' = u_0 + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} u_0^{(n)} \quad U_0' = U_0 + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} U_0^{(n)} \tag{54}$$

- 2) Similarly we can compute the initial values  $(u_0'', U_0'')$  from

$$u_0'' = u_0' + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} u_0'^{(n)} \quad U_0'' = U_0' + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} U_0'{}^{(n)} \tag{55}$$

- 3) Also we can compute the constants  $(u_0''', U_0''')$  from

$$u_0''' = u_0'' + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} u_0''^{(n)} \quad U_0''' = U_0'' + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} U_0''^{(n)} \tag{56}$$

Now having determined  $U_0'''$  we can evaluate  $F'''' = F''''(U''')$  and in turn the constant  $\square$  s are now known. Then the position and velocity at any time  $t$  can now be computed as follows:

We compute  $u'' = u_0'' + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} u_0''^{(n)}$   $U'' = U_0'' + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} U_0''^{(n)}$   $(57)$

and  $u' = u_0' + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} u_0'^{(n)}$   $U' = U_0' + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} U_0'{}^{(n)}$   $(58)$

finally  $u = u_0 + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} u_0^{(n)}$   $U = U_0 + \sum_{n=1}^5 \frac{\varepsilon^n}{n!} U_0^{(n)}$   $(59)$

Having determined  $u = (l, g, h, k_o, k_{\theta})$  and  $U = (L, G, H, K_o, K_{\theta})$  at time  $t$ , one can easily compute the position and the velocity.

**CONCLUSION**

In this work we developed analytical solution for the combined effects of solar radiation pressure and the gravitation of the Earth, Moon and the Sun on the orbits of high altitude satellites. The mathematical model included the zonal harmonics of the geopotential effects up to  $J_4$ . The equations of motion are derived in terms of a set of ten canonical elements representing the Delaunay variables augmented by the arguments of latitude of the moon and sun and their conjugate momenta to account for the explicit appearance of the time through the motions of the sun and of the moon. The resulting disturbing function is developed in a form suitable to facilitate the subsequent developments to solve the equations of motion. The usefulness of the theory appears in using perturbation techniques based on the Lie-Deprit- Kamel transform for elimination of the short, intermediate and long-period terms from the Hamiltonian through three canonical transformations.

This technique has many advantages: the perturbation theory is based on explicit transformations; the main part of the development of perturbations is reduced to the evaluation of Poisson brackets, which facilitated the construction of recursive algorithms and made it more access able to be implemented by computers. As a result of the invariance of Poisson brackets under canonical transformations, the generators as well as the Hamiltonian are also invariant, the transformation and its inverse are usually obtained along the same lines and it is possible to give a direct expression for any function of the old variables in terms of the new variables.

Finally a procedure for the computation of the position and velocity at any time is presented.

In conclusion the analytical works are so important for the evaluation of the problems of orbital motions of different

bodies under the effects of external forces such as drag, radiation pressure...etc.

We believe that the treatments in the analytical models describing the forces and motion can improve the accuracy of the computations and satellite life time.

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