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Dynamical Behavior in the Vicinity of a Circular Anisotropic Ring

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Abstract: Motivated by the variety of rings around planets in our solar system, and arguably in other planetary systems as well, we will analytically treat the venerable problem of investigating the potential of a massive annular distribution. In a first approximation, we can consider the mass distribution as homogenous. Actually, a more realistic model should take into account the non-isotropy of density.In this work, we establish an analytical elliptic form of the potential generated by an anisotropic matter distribution. The study of the dynamical behavior is performed within the Hamiltonian formulation, which allows us to derive some orbits of the test particle.

Keywords: Potential - Anisotropic Distribution-Rings- Elliptic Integrals.

I. INTRODUCTION

The solar system is a large area with various kinds of celestial bodies; elongated, circular, elliptical or strongly irregular ones.

Their gravitational behaviour gives us much information. The relativistic studies allowed us to valid some aspect of this theory [1] or to detect relativistic effect about the element of orbits [2], or the increase of accuracy of the two body problem in the frame of general relativity [3]. The extra solar planets are studied too in [4-6] and [7]. The effect of atmosphere, potential type or gravitational radiation about orbits are studied in [8, 9], and [10]. Some other publications [11-14] are devoted to studies of rings around planets or even sun.

The discoveries of binary asteroids, the mission to Saturn rings gave a new interest in potential calculation. This subject is a new/old field of research, [15, 16]. In the literature we find studies like in Riaguas *et al.* [17] in which they estimated the potential generated by a homogeneous straight segment. Elipe and Lara [18] described the motion around Eros 433 with the same homogeneous model.

A harmonic polyhedron was used by Werner and Scheeres for 4769 Castalia [19, 20]. Ellipsoids, material points and a segment of double material were used by Bartczak and Breiter [21] and Bartczak *et al.* [22]. In our laboratory we used a new idea in a previous work [23] by studying the potential generated by a massive straight segment with a parabolic profile of mass distribution. About rings, Harry *et al.* [24] calculated the potential due to uniform disk and deduced that of a homogeneous ring. Broucke *et al.* [25] established the potential of a homogeneous circular ring and studied the properties and perturbations of orbits around a central planet surrounded by that ring. Fred *et al.* [26] established the expressions for both the potential and the field of a disk. They suggested the formulas of a ring. In our present work we propose a new idea by using an anisotropic mass density for a circular ring. In section 2 we will establish the integral expression of the potential generated by an anisotropic ring in space. In section 3 we give the analytical expression of the potential in the plane perpendicular to the ring in terms of the complete elliptical integral of the first kind K(k) and second kind E(k). We plot the level curves of this potential. In section 4 we give the set of differential equations by using the Hamiltonian formulation. Finally, in section 5 we solve the differential equations of motion and discuss different dynamical states of the test particle around the ring, we draw some orbits and discuss their gravitational features.

2. INTEGRAL EXPRESSION OF THE POTENTIAL

We consider a circular ring of radius a and total mass M, located in the (xoy) plane (Fig. **1**).

The density of the ring is given by:

$$\lambda(\theta)=\lambda_0\left(1+b\cos^2\frac{\theta}{2}\right) \tag{1}$$

In which λ_0 and b are positive constants.

The total mass of the ring is given by: $M=a\lambda_0\pi(b+2)$ and $b>-2$

If b>0 the line density is (Fig. **2**):

- Maximum for $\theta=0$: $\lambda=\lambda_0(1+b)$

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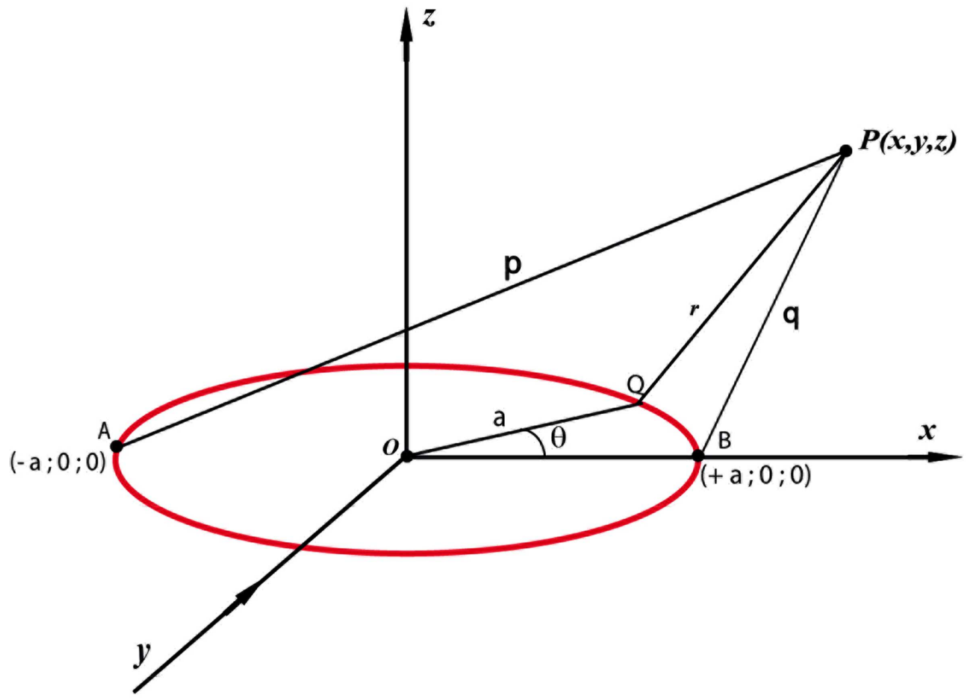

Fig. (1). Ring in the plane (xoy).

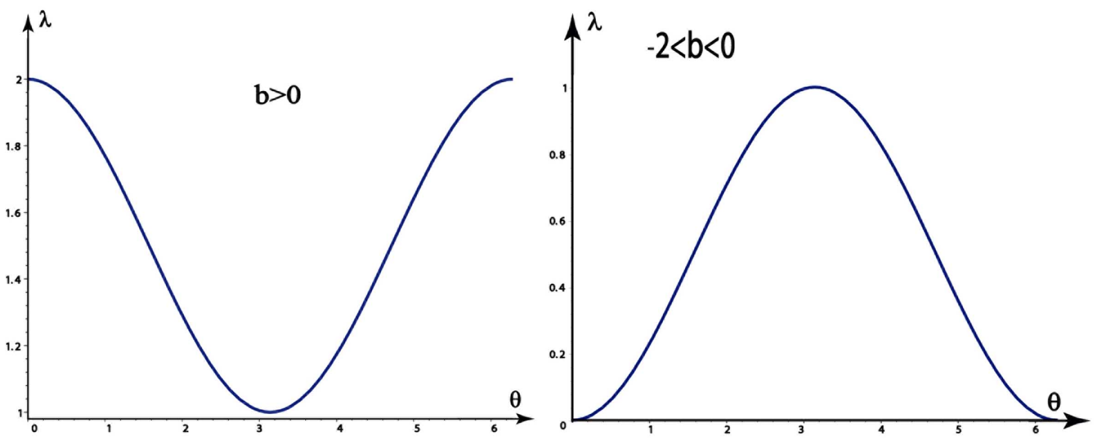

Fig. (2). Variation of the density.

- Minimum for $\theta = \pi$: $\lambda = \lambda_0$

If -2<b<0 the line density is (Fig. **2**):

- Maximum for $\theta = \pi$: $\lambda = \lambda_0$

- Minimum for $\theta = 0$: $\lambda = \lambda_0 (1 + b)$

Since the distribution is inhomogeneous its center of mass is not in O. By symmetry, the center of mass is at the position:

$$\left(x_G = \frac{ab}{2(b+2)} ; y_G = 0 \right) \tag{2}$$

The gravitational potential generated by the ring at a point P (x, y, z) is expressed by (Fig. **1**):

$$U(P) = -G \int_{ring} \frac{dm}{r} \tag{3}$$

- Expression of r :

r is the distance between the element dm centered at Q, (Fig. **1**) and P.

$$r = QP = \sqrt{(x - a\cos\theta)^2 + (y - a\sin\theta)^2 + z^2} \tag{4}$$

We introduce two new auxiliary functions p and q defined by:

- ✓ p is the largest distance between P and the ring given by:

$$p = \sqrt{(x+a)^2 + y^2 + z^2}$$

- ✓ q is the smallest distance between P and the ring given by:

$$q = \sqrt{(x-a)^2 + y^2 + z^2}$$

The substitution of p and q in (4) gives:

$$r = QP = \sqrt{\frac{p^2 + q^2}{2} - \frac{p^2 - q^2}{2}\cos\theta - 2ay\sin\theta} \tag{5}$$

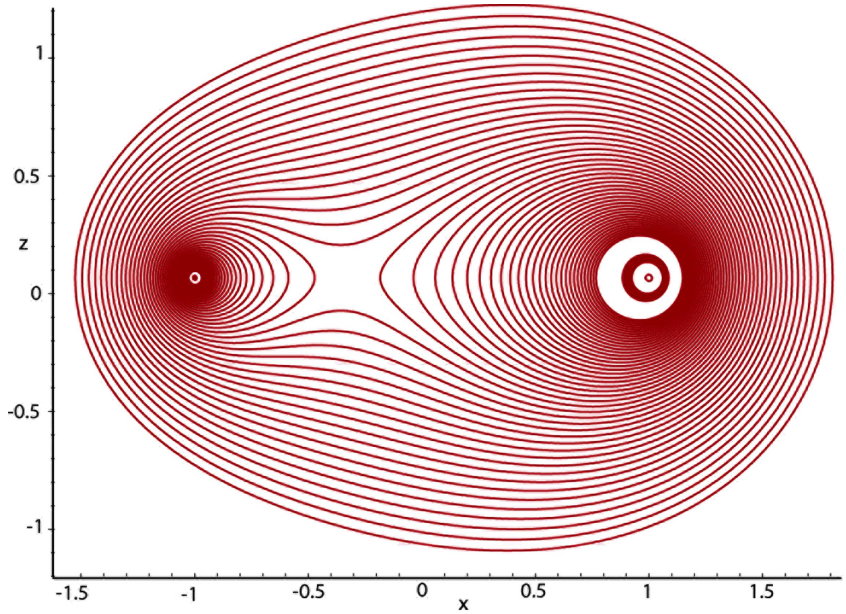

Fig. (3). Level cures of the potential function U=U(x,z).

And:

$$r=\sqrt{p^2\left(1-k^2\cos^2\frac{\theta}{2}\right)-2ay\sin\theta} \qquad (6)$$

With: $k^2=1-\dfrac{q^2}{p^2}<1$

<u>Expression of dm:</u>

$$U=-Ga\lambda_0\int_0^{2\pi}\frac{1+b\cos^2\frac{\theta}{2}}{\sqrt{\frac{p^2+q^2}{2}-\frac{p^2-q^2}{2}\cos\theta-2ay\sin\theta}}d\theta \qquad (7)$$

Substituting the expressions (6) and (7) in (3), the potential generated by the ring is:

$$U=-Ga\lambda_0\int_0^{2\pi}\frac{1+b\cos^2\frac{\theta}{2}}{\sqrt{\frac{p^2+q^2}{2}-\frac{p^2-q^2}{2}\cos\theta-2ay\sin\theta}}d\theta \qquad (8)$$

If we put $2\varphi=\pi-\theta$

The expression (6) becomes:

$$r=\sqrt{p^2(1-k^2\sin^2\varphi)-2ay\sin(2\varphi)}$$

Finally, the gravitational potential generated by the inhomogeneous ring at any point P in space is given by:

$$U=-2Ga\lambda_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(1+b\sin^2\varphi).d\varphi}{\sqrt{p^2\left(1-k^2\sin^2\varphi\right)-2ay\sin2\varphi}} \qquad (9)$$

3. ANALYTICAL EXPRESSION OF THE POTENTIAL IN (XOZ) PLANE

In this section we will derive the analytical expression of the potential given by (9), in the (xoz) plane:

$$U=-\frac{2Ga\lambda_0}{p}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(1+b\sin^2\varphi).d\varphi}{\sqrt{\left(1-k^2\sin^2\varphi\right)}} \qquad (10)$$

$$U=-\frac{4G.a\lambda_0}{p}\int_0^{\frac{\pi}{2}}\frac{1+b\sin^2\varphi}{\sqrt{\left(1-k^2\sin^2(\varphi)\right)}}d\varphi$$

$$U=-\frac{4Ga\lambda_0}{p}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\left(1-k^2\sin^2\varphi\right)}}+\frac{4Gab\lambda_0}{pk^2}\int_0^{\frac{\pi}{2}}\frac{-k^2\sin^2\varphi}{\sqrt{\left(1-k^2\sin^2\varphi\right)}}d\varphi$$

$$U=-\frac{4Ga\lambda_0}{p}K(k)+\frac{4Ga\lambda_0 b}{p.k^2}\left[-\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\left(1-k^2\sin^2(\varphi)\right)}}d\varphi+\int_0^{\frac{\pi}{2}}\frac{1-k^2\times\sin^2(\varphi)}{\sqrt{\left(1-k^2\sin^2(\varphi)\right)}}d\varphi\right]$$

Where K (k) is the complete elliptic integral of first kind [27].

$$U=-\frac{4Ga\lambda_0}{p}K(k)-\frac{4Ga\lambda_0 b}{p.k^2}\left[K(k)-\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2(\varphi)}\,d\varphi\right]$$

$$U=-\frac{4Ga\lambda_0}{p}K(k)-\frac{4Ga\lambda_0 b}{p.k^2}\left[K(k)-E(k)\right]$$

Where, E(k) is the complete elliptic integral of second kind.

Finally we reach the analytical expression of the gravitational potential generated by the anisotropic circular ring in the (xoz) plane.

$$U=-\frac{4G\lambda_0 a}{pk^2}\left[(k^2+b)K(k)-bE(k)\right] \qquad (11)$$

The expression (11) represents the analytical form of the potential.

The effect of non-uniform distribution is owing to b.

For b = 0 we find the particular case of a homogeneous circular ring studied by Broucke *et al.* in [11].

-Fig. **(3)** shows the plot of equipotential contours, the analysis of this figure shows the existence of a hyperbolically unstable point corresponding to maximum potential. This maximum corresponds to a maximum of the density.

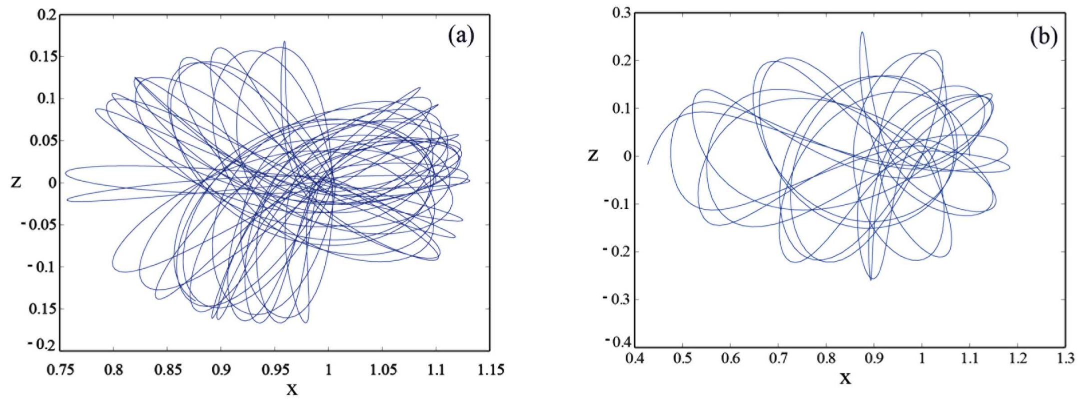

Fig. (4). Curve in (xoz) plane for b=0 :v_{0z}=2 and v_{0z}=3.

4. DYNAMICAL STUDY

We study the dynamical behavior of a test particle, with unit mass, in the field of the inhomogeneous ring. The Hamiltonian of the test particle is given by:

$$H=\frac{1}{2}(p_1^2+p_2^2)-\frac{4G\lambda_0 aK(k)}{p}+\frac{4G\lambda_0 ab(E(k)-K(k))}{pk^2} \tag{12}$$

The equations of motion are given by:

$$\begin{cases}\ddot{x}=-\dfrac{\partial U}{\partial x}=-4G\lambda_0 a\left\{\left[E(k)\dfrac{\partial}{\partial x}\left(\dfrac{b}{pk^2}\right)+\left(\dfrac{b}{pk^2}\right)\left(\dfrac{E(k)-K(k)}{k}\right)\dfrac{\partial k}{\partial x}\right]\right. \\ \left.-K(k)\dfrac{\partial}{\partial x}\left(\dfrac{1}{p}+\dfrac{b}{pk^2}\right)-\left(\dfrac{1}{p}+\dfrac{b}{pk^2}\right)\left(\dfrac{E(k)-(1-k^2)K(k)}{k(1-k^2)}\right)\dfrac{\partial k}{\partial x}\right\}\end{cases} \tag{13}$$

$$\begin{cases}\ddot{z}=-\dfrac{\partial U}{\partial z}=-4G\lambda_0 a\left\{\left[E(k)\dfrac{\partial}{\partial z}\left(\dfrac{b}{pk^2}\right)+\left(\dfrac{b}{pk^2}\right)\left(\dfrac{E(k)-K(k)}{k}\right)\dfrac{\partial k}{\partial z}\right]\right. \\ \left.-K(k)\dfrac{\partial}{\partial z}\left(\dfrac{1}{p}+\dfrac{b}{pk^2}\right)-\left(\dfrac{1}{p}+\dfrac{b}{pk^2}\right)\left(\dfrac{E(k)-(1-k^2)K(k)}{k(1-k^2)}\right)\dfrac{\partial k}{\partial z}\right\}\end{cases} \tag{14}$$

After calculation and arrangement of expressions (13) and (14) we find:

$$\begin{cases}\ddot{x}=\alpha E(k)+\beta(E(k)-K(k))+\gamma K(k)+\delta(E(k)-(1-k^2)K(k)) \\ \ddot{z}=\eta E(k)+\mu(E(k)-K(k))+\psi K(k)+\xi(E(k)-(1-k^2)K(k))\end{cases}$$

So:

$$\begin{cases}\ddot{x}=(\gamma-\beta-\delta(1-k^2))K(k)+(\alpha+\beta+\delta)E(k) \\ \ddot{z}=(\psi-\mu-\xi(1-k^2))K(k)+(\eta+\mu+\xi)E(k)\end{cases}$$

With:

$$\begin{cases}\alpha=-4G.\lambda_0.a\times\dfrac{\partial}{\partial x}\left(\dfrac{b}{p.k^2}\right)=4G\lambda_0 a\left[\dfrac{4.ba(z^2+(x+a)a)}{k^4p^5}\right] \\ \beta=-4G.\lambda_0.a\times\left(\dfrac{b}{p.k^3}\right)\times\dfrac{\partial k}{\partial x}=-4G\lambda_0 a\left(\dfrac{2ba(a^2-x^2+z^2)}{p^5k^4}\right) \\ \gamma=+4G.\lambda_0.a\times\dfrac{\partial}{\partial x}\left(\dfrac{1}{p}+\dfrac{b}{p.k^2}\right)=-4G.\lambda_0.a\times\dfrac{1}{k^4p^3}\left[4ab+k^2(x+a)(k^2-b)\right] \\ \delta=+4G.\lambda_0.a\times\left(\dfrac{1}{p}+\dfrac{b}{p.k^2}\right)\times\left(\dfrac{1}{k(1-k^2)}\right)\times\dfrac{\partial k}{\partial x}=4G\lambda_0 a\left(\dfrac{2a(pk^2+bp)(a^2-x^2+z^2)}{p^6k^4(1-k^2)}\right)\end{cases}$$

and

$$\begin{cases}\eta=-4G.\lambda_0.a\times\dfrac{\partial}{\partial z}\left(\dfrac{b}{p.k^2}\right)=-4G.\lambda_0.a\times\left(\dfrac{b.z}{k^2.p^3}\right) \\ \mu=-4G.\lambda_0.a\times\left(\dfrac{b}{p.k^3}\right)\times\dfrac{\partial k}{\partial z}=4G\lambda_0 a\left(\dfrac{bz}{p^3k^2}\right)=-\eta \\ \psi=4G.\lambda_0.a\times\dfrac{\partial}{\partial z}\left(\dfrac{1}{p}+\dfrac{b}{p.k^2}\right)=4G\lambda_0 a\left(\dfrac{z}{p^3}\left(\dfrac{b}{k^2}-1\right)\right) \\ \xi=4G.\lambda_0.a\times\left(\dfrac{1}{p}+\dfrac{b}{p.k^2}\right)\times\left(\dfrac{1}{k(1-k^2)}\right)\times\dfrac{\partial k}{\partial z}=-4G\lambda_0 a\left(\dfrac{(pk^2+bp)z}{p^4k^2(1-k^2)}\right)\end{cases}$$

Finally:

$$\begin{cases}\ddot{x}=AK(k)+BE(k) \\ \ddot{z}=CK(k)+DE(k)\end{cases} \tag{15}$$

With:

$$A=(\gamma-\beta-\delta(1-k^2))$$
$$B=(\alpha+\beta+\delta)$$
$$C=(\psi-\mu-\xi(1-k^2))$$
$$D=(\eta+\mu+\xi)$$

The system (15) represents the dynamical equations of motion of the test particle in the gravitational field generated by the inhomogeneous ring.

These equations are coupled and highly nonlinear, they require then a numerical resolution.

5. NUMERICAL INTEGRATION

The system (15) could be worked out by a perturbative or numerical method. We adopted, in a first time, the last one to investigate this way.

To gain deep insight about the dynamical behavior of the test particle in the field of the inhomogeneous ring, we have to solve the system (15). In this system of differential equations, the unknown variables are x and z. We derive some curves in (xoz) plane.

For different values of b, we test many initial conditions about v_{0z} and v_{0x} separately.

This allows us to deduce the gravitational influence on the dynamical behavior of the test particle.

5.1. Effect of *b* on The Critical Value of v_{0z}

Fig. (**4.a**): $x_0=1{,}1$; $z_0=0$; $v_{0x}=0$; $v_{0z}=2$; $b=0$

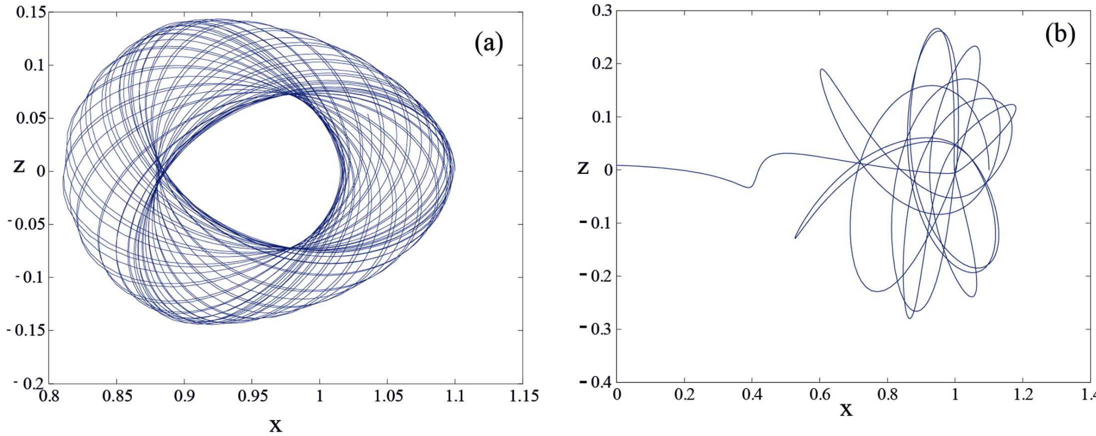

Fig. (5). Curve in (xoz) plane for b=1 :v_{0z}=3 and v_{0z}=4,62.

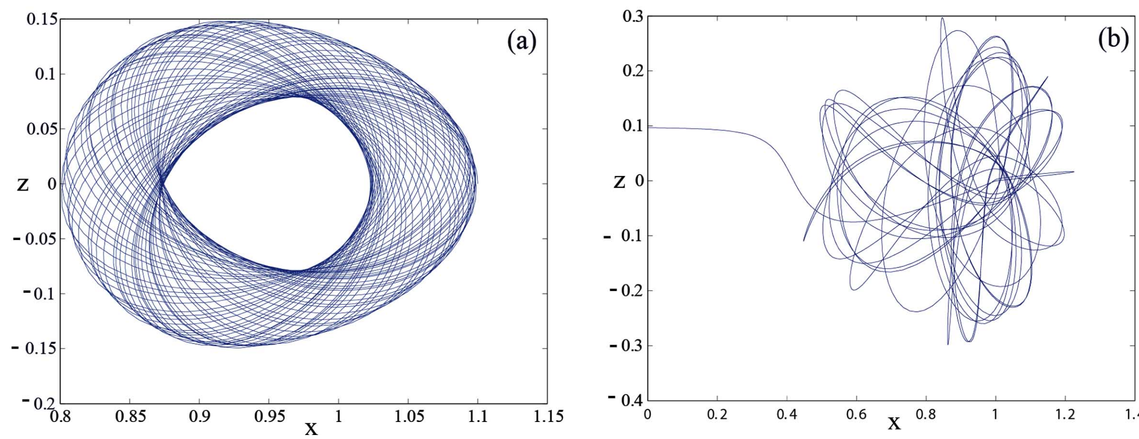

Fig. (6). Curve in (xoz) plane for b=2 :v_{0z}=4 and v_{0z}=5,82.

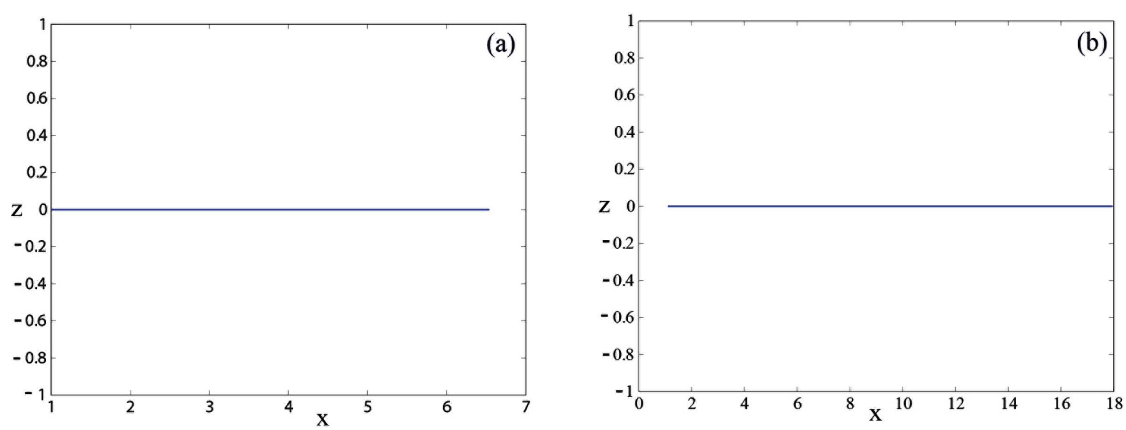

Fig. (7). Curve in (xoz) plane for b=0 :v_{0x}=7 and v_{0x}=7,8.

Fig. (**4.b**): $x_0 = 1,1\ ;\ z_0 = 0\ ;\ v_{0x} = 0\ ;\ v_{0z} = 3\ ;\ b = 0$

Fig. (**5.a**): $x_0 = 1,1\ ;\ z_0 = 0\ ;\ v_{0x} = 0\ ;\ v_{0z} = 3\ ;\ b = 1$

Fig. (**5.b**): $x_0 = 1,1\ ;\ z_0 = 0\ ;\ v_{0x} = 0\ ;\ v_{0z} = 4,62\ ;\ b = 1$

Fig. (**6.a**): $x_0 = 1,1\ ;\ z_0 = 0\ ;\ v_{0x} = 0\ ;\ v_{0z} = 4\ ;\ b = 2$

Fig. (**7.b**): $x_0 = 1,1\ ;\ z_0 = 0\ ;\ v_{0x} = 0\ ;\ v_{0z} = 5,82\ ;\ b = 2$

5.1.1. Analysis of Curves

For different values of b, there exist a critical value v_{0zc} of v_{0z} beyond which the test particle is shifted from a bounded state to a free state. The Table **1** gives an overview of this situation. We notice that the value of v_{0zc} grow with that of *b*, this is due to the fact that when the density is important, the escape becomes more difficult.

5.2. Effect of *b* on the Critical Value of v_{0x}

Fig. (**8.a**): $x_0 = 1,1\ ;\ z_0 = 0\ ;\ v_{0x} = 7\ ;\ v_{0z} = 0\ ;\ b = 0$

Fig. (**8.b**): $x_0 = 1,1\ ;\ z_0 = 0\ ;\ v_{0x} = 7,8\ ;\ v_{0z} = 0\ ;\ b = 0$

Fig. (**9.a**): $x_0 = 1,1\ ;\ z_0 = 0\ ;\ v_{0x} = 9,5\ ;\ v_{0z} = 0\ ;\ b = 1$

Fig. (**9.b**): $x_0 = 1,1\ ;\ z_0 = 0\ ;\ v_{0x} = 9,6\ ;\ v_{0z} = 0\ ;\ b = 1$

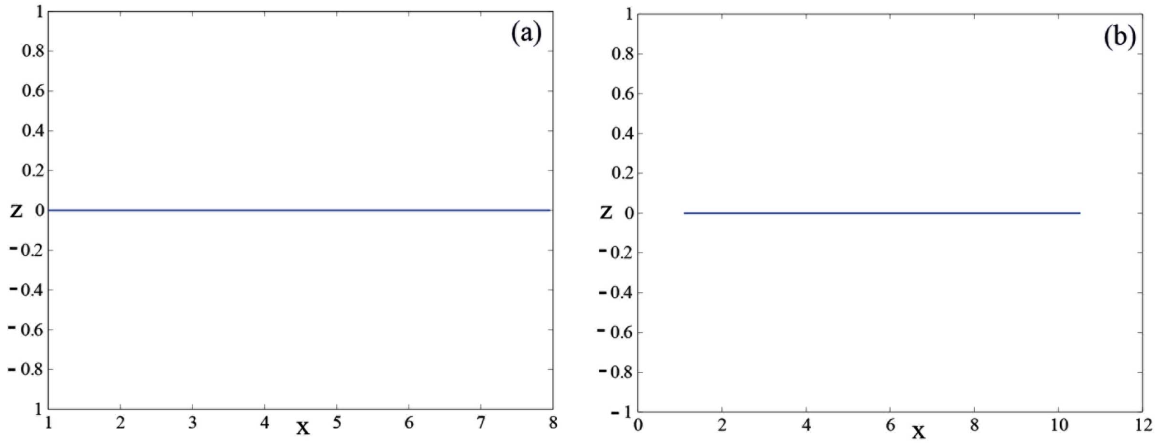

Fig. (8). Curve in (xoz) plane for b=1 :v_{0x}=9,5 and v_{0x}=9,6.

Table 1. Effect of b on the Critical Value of V_{0z}

Value of b	0	1	2
Critical value of V_{0z}	3	4,6	5,8

Table 2. Effect of b on The Critical Value oF V_{0x}

Value of b	0	1	2
Critical value of v_{0x}	7,42	9,6	infinite

5.2.1. Analysis of Curves

For different values of b, there exist a critical value v_{0xc} of v_{0x} beyond which the test particle is shifted from a collision state to a free state. The Table **2** give an overview of this situation.

We notice that the value of v_{0xc} grow with that of b, this is due to the fact that when the density is important the escape becomes more difficult.

6. CONCLUSION

In this work we gave a new idea about the anisotropic mass distribution of a body bent into a circular shape, as in the ring of Saturn. The non-homogenous distribution of density is consistent with the photographical exploration made by many missions to Saturn like Cassini-Huygens. We established then the analytical expression in term of elliptical integrals of first and second kind, of the potential generated by that distribution. For this profile of density, we found many kind of orbits depending on the initial conditions. After reaching these results, we explored the gravitational behaviour of a test particle in the field of a ring fixed in space. This was studied by Hamiltonian formulation. In a next future, we plan to study the case of a rotating ring.

CONFLICT OF INTEREST

The author confirms that this article content has no conflicts of interest.

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Declared none.

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