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Vibrations of Elastically Restrained Circular Plates Resting on Partial Winkler Foundation

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Abstract: This work describes a study of vibration characteristics of thin circular plates with elastic edge support and resting on partial Winkler-type elastic foundation. The foundation is described by the Winkler model, which is called as single parameter foundation. The exact analytical method is used to derive the frequency equation of the circular plate with elastic edge support-conditions and resting on partial elastic foundation system. Parametric investigations on the behavior of circular plates with elastic edge support and resting on partial elastic foundation have been carried out with respect to various values of transverse stiffness parameter, foundation parameter for a variety of boundary conditions. Extensive data is tabulated so that pertinent conclusions can be arrived at on the influence of translational edge restraint, and the foundation modulus parameters of the Winkler foundation on the natural frequencies of uniform isotropic circular plates. A comparison of the results obtained here in this paper with those available in the literature shows an excellent agreement.

Keywords: Circular plate, restrained edge, translational spring stiffness, elastic foundation.

1. INTRODUCTION

The structural behavior of circular plates on elastic foundation is of great interest for the design on many engineering problems. Such plate systems can be found in many engineering applications, ranging from more conventional civil engineering, mechanical engineering and marine engineering to an aerospace engineering. Research work in this area has been discussed in a series of papers by Lessia [1, 2] and Bert [3, 4]. The vibration of a circular plate supported laterally by an elastic foundation was studied by Leissa in Ref. [5] from which he deduced that the effect of a Winkler foundation merely increases the square of the natural frequency of the plate by a constant. Laura *et al.* [6], while studying the case of a circular plate partially embedded in a Winkler foundation, found that a simple frequency relation like the above no longer holds good, and thus reached a similar conclusion. The most general soil model used in practical applications is the Winkler model [7] in which the soil layer is represented by unconnected closely spaced elastic springs.

The present study of circular plates on elastic foundations with elastic edge support finds useful applications in foundation designs of large storage tanks, deep-sea pressure vessels and heavy machines [8].

However, studies of the vibration of plates considering the combined effects of elastic foundations and elastic constraints are relatively scarce in the literature [9]. The vibration characteristics of plates resting on an elastic medium are different from those of the plates supported only on the boundary. There are many difficulties which very often arise due to complexity and uncertainty of boundary conditions. This uncertainty could be due to practical engineering applications where the edge of the plate does not fall into the classical boundary conditions. The accepted fact is that the condition on a periphery often tends to be part away between the classical boundary conditions (simply supported, free, pinned) and non-classical boundary conditions (elastic edge restraints) [10]. Therefore, when the boundary conditions of the plate deviate from classical cases, elastic edge restraints need to be considered. The present study considers the problem of vibrations of circular plates elastically restrained against translation and resting on partial elastic foundation i.e. on partial Winkler foundation. In this paper, exact solutions for first Eigen-frequencies of thin circular plates for various values of non-dimensional parameters are presented in graphical and tabular form, which may be useful for engineers in practice as well as to researchers as benchmark results for checking the relative accuracy of the approximate results obtained through alternate methods of solution.

2. DEFINITION OF THE PROBLEM

Consider a thin isotropic, circular plate of radius R, uniform thickness h, Young's modulus E, flexural rigidity D and Poisson's ratio ν as shown in Fig. (**1**). The plate edge is considered to be elastically restrained in translation and partially supported on Winkler foundation. The plate is also assumed to be made of linearly elastic, homogeneous and isotropic material. However, the effects of shear

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deformation and rotary inertia are neglected in the present paper as the plate considered is quite thin. The problem at hand is to determine the frequencies of a circular plate with elastically restrained edge and resting on partial elastic foundation.

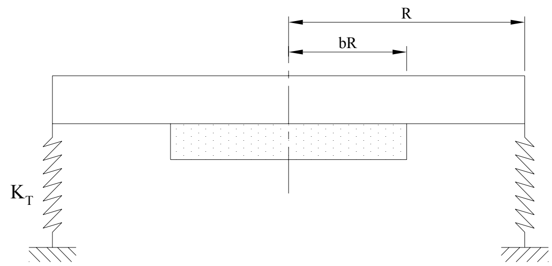

Fig. (1). A thin circular plate with rotational K_R and translational K_T elastic edge constraints and supported on partial elastic foundation.

3. MATHEMATICAL FORMULATION

The geometrical and loading configuration of the plate is axi-symmetric and consequently, deflection shape of the plate will be axially symmetric as well. Consider a circular plate of radius R being supported in the interior by a foundation of radius bR as shown in Fig. (**1**). Let the subscript **I** denote the outer region $b \le \bar{r} \le 1$ and the subscript **II** denote the inner region $0 \le \bar{r} \le b$. Here, all lengths are normalized with respect to R. The non-dimensional radius at the outer edge is 1 and at the inner edge is b. As per the classical Kirchhoff's plate theory [11, 12], the fourth order differential equation describing the free flexural vibrations of a thin circular uniform plate for region **I**, in polar coordinates (r,θ) is given by:

$$\nabla^4 w_I - k^4 w_I = 0 \tag{1}$$

where $k^4 = R^4\omega^2\rho / D$ which is the non-dimensional frequency parameter.

The plate Eq. (5) for region **II** is:

$$\nabla^4 w_{II} - k^4 w_{II} + \lambda^4 w_{II} = 0 \tag{2}$$

where $\lambda^4 = R^4 K_w / D$ which represents the no-dimensional foundation stiffness parameter.

Let the solution to the Eq. (1) be represented as:

$$w = u(r)\cos(n\theta) \tag{3}$$

where r is the radius normalized w.r.t R, n is the number of nodal diameters. The function u a linear combination of the Bessel functions $J_n(kr), Y_n(kr), I_n(kr) \,\&\, K_n(kr)$ is:

$$u_I(r) = C_1 J_n(kr) + C_2 Y_n(kr) + C_3 I_n(kr) + C_4 K_n(kr) \tag{4}$$

where $C_1, C_2, C_3 \,\&\, C_4$ are constants.

$J_n(.)\, and Y_n(.)$ are the Bessel functions of the first and second kinds of order n respectively.

$I_n(.)\, and K_n(.)$ are the modified Bessel functions of the first and second kinds of order n respectively.

Substituting Eq. (4) in Eq. (3) gives the following:

$$w_I(r,\theta) = \begin{bmatrix} C_1 J_n(kr) + C_2 Y_n(kr) + \\ C_3 I_n(kr) + C_4 K_n(kr) \end{bmatrix} \cos(n\theta) \tag{5}$$

Unlike Eq. (1), the general solution to Eq. (2), is more complicated and the following three special cases are considered in this paper.

Case (i) If $k \succ \lambda$, the solution to Eq. (2) is:

$$u_{II}(r) = C_5 J_n(k_1 r) + C_6 I_n(k_1 r) \tag{6}$$

where $k_1 = (k^4 - \lambda^4)^{\frac{1}{4}}$.

Substituting Eq. (6) in Eq. (3), we get:

$$w_{II}(r,\theta) = \left[C_5 J_n(k_1 r) + C_6 I_n(k_1 r)\right]\cos(n\theta) \tag{7}$$

Case (ii) If $k = \lambda$, the solution to Eq. (2) is given by:

$$u_{II}(r) = C_5 r^n + C_6 r^{n+2} \tag{8}$$

Substitution of Eq. (8) in Eq. (3) gives the following:

$$w_{II}(r,\theta) = \left[C_5 r^n + C_6 r^{n+2}\right]\cos(n\theta) \tag{9}$$

Case (iii) If $k < \lambda$, the solution to Eq. (2) is given by:

$$u_{II}(r) = C_5\,\mathrm{Re}[J_n(\sqrt{ik_2 r})] + C_6\,\mathrm{Im}[\sqrt{ik_2 r}] \tag{10}$$

where $k_2 = (\lambda^4 - k^4)^{\frac{1}{4}}$.

Substitution of Eq. (10) in Eq. (3) gives the following:

$$w_{II}(r,\theta) = \left[C_5\,\mathrm{Re}[J_n(\sqrt{ik_2 r})] + C_6\,\mathrm{Im}[\sqrt{ik_2 r}]\right] \cos(n\theta) \tag{11}$$

For an elastically restrained circular plate, the boundary conditions at the edge of the plate in terms of rotational and translational stiffness are given by the following expressions:

$$v_r(r,\theta) = -K_T w_I(r,\theta) \tag{12}$$

$$M_r(r,\theta) = K_R \frac{\partial w}{\partial r}(r,\theta) \tag{13}$$

where the shearing force and bending moment as per Kelvin-Kirchhofff theory are defined as follows:

$$V_r = -D\left[\frac{\partial}{\partial r}\nabla^2 w + (1-\nu)\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta}\right)\right] \tag{14}$$

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right] \tag{15}$$

B.C. A : For a circulate plate with outer edge elastically restrained against rotation only, the Eqs. (12) and (13) become:

$$v_r(r,\theta) = -K_T w_I(r,\theta) \tag{12a}$$

$$M_r(r,\theta) = 0 \tag{13a}$$

From Eqs.(12a) & (14)

$$\begin{bmatrix} \frac{\partial}{\partial r}\nabla^2 w_I(r,\theta) + \\ (1-\nu)\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial^2 w_I(r,\theta)}{\partial r\partial\theta^2} - \frac{1}{r^2}\frac{\partial w_I(r,\theta)}{\partial\theta}\right) \end{bmatrix} = \frac{K_T}{D} w_I(r,\theta) \tag{14a}$$

From Eqs.(13a) & (15):

$$\begin{bmatrix} \dfrac{\partial^2 w_I(r,\theta)}{\partial r^2} + \\ \nu\left(\dfrac{1}{r}\dfrac{\partial w_I(r,\theta)}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 w_I(r,\theta)}{\partial \theta^2}\right) \end{bmatrix} = 0 \tag{15a}$$

The plate is continuous in terms of displacement, slope and moment at $r = b$. Therefore, the boundary conditions are:

$$w_I(b) = w_{II}(b) \tag{16}$$

$$w_I'(b) = w_{II}'(b) \tag{17}$$

$$w_I''(b) = w_{II}''(b) \tag{18}$$

$$w_I'''(b) = w_{II}'''(b) \tag{19}$$

Then for region **I**, from Eqs. (5) and (17), we get the following expression:

$$\begin{aligned} &\left[\frac{k^2}{4}J_{m2} + \frac{k\nu}{2}J_{m1} - \left(\frac{k^2}{2} + \nu n^2\right)J_n(k)\right]C_1 \\ &+\left[\frac{k^2}{4}Y_{n2} + \frac{k\nu}{2}Y_{n1} - \left(\frac{k^2}{2} + \nu n^2\right)Y_n(k)\right]C_2 \\ &+\left[\frac{k^2}{4}I_{p2} + \frac{k\nu}{2}I_{p1} + \left(\frac{k^2}{2} - \nu n^2\right)I_n(k)\right]C_3 \\ &+\left[\frac{k^2}{4}k_{q2} - \frac{k\nu}{2}k_{q1} + \left(\frac{k^2}{2} - \nu n^2\right)k_n(k)\right]C_4 = 0 \end{aligned} \tag{20}$$

<u>B.C. B</u>: For a circulate plate with outer edge elastically restrained against translation only, Eqs. (12) and (13) become:

$$v_r(r,\theta) = 0 \tag{12b}$$

$$M_r(r,\theta) = K_R \frac{\partial w}{\partial r}(r,\theta) \tag{13b}$$

Then for region (**I**, *n*) from Eqs. (5) and (16), we get the following expression:

$$\begin{aligned} &\begin{bmatrix} \frac{k^3}{8}J_{m3} + \frac{k^2}{4}J_{m2} - \frac{k}{2}\left(\frac{3}{4}k^2 + n^2(2-\nu) + 1\right)J_{m1} \\ +\left(n^2(2-\nu) - \frac{k^2}{2} - \frac{K_T}{D}\right)J_n(k) \end{bmatrix} C_1 + \\ &\begin{bmatrix} \frac{k^3}{8}Y_{n3} + \frac{k^2}{4}Y_{n2} - \frac{k}{2}\left(\frac{3}{4}k^2 + n^2(2-\nu) + 1\right)Y_{n1} \\ +\left(n^2(2-\nu) - \frac{k^2}{2} - \frac{K_T}{D}\right)Y_n(k) \end{bmatrix} C_2 + \\ &\begin{bmatrix} \frac{k^3}{8}I_{p3} + \frac{k^2}{4}I_{p2} + \frac{k}{2}\left(\frac{3}{4}k^2 + n^2(-2+\nu) - 1\right)I_{p1} \\ +\left(n^2(2-\nu) + \frac{k^2}{2} - \frac{K_T}{D}\right)I_n(k) \end{bmatrix} C_3 + \\ &\begin{bmatrix} \frac{k^3}{8}k_{Q3} + \frac{k^2}{4}K_{Q2} + \frac{k}{2}\left(-\frac{3}{4}k^2 + n^2(2-\nu) + 1\right)K_{Q1} \\ +\left(n^2(2-\nu) + \frac{k^2}{2} - \frac{K_T}{D}\right)K_n(k) \end{bmatrix} C_4 = 0 \end{aligned} \tag{21}$$

where

$$\begin{aligned} &J_{m1} = J_{n-1}(k) - J_{n+1}(k); J_{m2} = J_{n-2}(k) + J_{n+2}(k); \\ &J_{m3} = J_{n-3}(k) - J_{n+3}(k); Y_{n1} = Y_{n-1}(k) - Y_{n+1}(k); \\ &Y_{n2} = Y_{n-2}(k) + Y_{n+2}(k); Y_{n3} = Y_{n-3}(k) - Y_{n+3}(k); \\ &I_{p1} = I_{n-1}(k) + I_{n+1}(k); I_{p2} = I_{n-2}(k) + I_{n+2}(k); \\ &I_{p3} = I_{n-3}(k) + I_{n+3}(k); K_{q1} = K_{n-1}(k) + K_{n+1}(k); \\ &K_{q2} = K_{n-2}(k) + K_{n+2}(k); K_{q3} = K_{n-3}(k) + K_{n+3}(k); \end{aligned}$$

<u>Case (i)</u>: Considering the case (i) i.e. $k > \lambda$, for the region (**II**, *n*), the solution to Eq. (2) is given by Eq.(7) and for this case the boundary conditions, Eqs. (16), (17), (18) and (19) give us the following:

$$\begin{aligned} &J_n(kb)C_1 + Y_n(kb)C_2 + I_n(kb)C_3 + K_n(kb)C_4 \\ &-J_n(k1b)C_5 - I_n(k1b)C_6 = 0 \end{aligned} \tag{22}$$

$$\begin{aligned} &\frac{k}{2}J_{m1}'C_1 + \frac{k}{2}Y_{n1}'C_2 + \frac{k}{2}I_{p1}'C_3 - \frac{k}{2}K_{q1}'C_4 - \\ &\frac{k_1}{2}J_{m11}'C_5 - \frac{k}{2}I_{p11}'C_6 = 0 \end{aligned} \tag{23}$$

$$\begin{aligned} &\frac{k^2}{4}\left(J_{m2}' - 2J_n(kb)\right)C_1 + \frac{k^2}{4}\left(Y_{n2}' - 2Y_n(kb)\right)C_2 + \\ &\frac{k^2}{4}\left(I_{p2}' + 2I_n(kb)\right)C_3 + \frac{k^2}{4}\left(K_{q2}' + 2K_n(kb)\right)C_4 \\ &-\frac{k_1^2}{4}\left(J_{m22}' - 2J_n(k_1b)\right)C_5 - \\ &\frac{k_1^2}{4}\left(I_{p22}' + 2I_n(k_1b)\right)C_6 = 0 \end{aligned} \tag{24}$$

$$\begin{aligned} &\frac{k^3}{8}\left[J_{m3}' - 3J_{m1}'\right]C_1 + \frac{k^3}{8}\left[Y_{n3}' - 3Y_{n1}'\right]C_2 + \\ &\frac{k^3}{8}\left[I_{p3}' + 3I_{p1}'\right]C_3 - \frac{k^3}{8}\left[K_{q3}' + 3K_{q1}'\right]C_4 - \\ &\frac{k_1^3}{8}\left[J_{m33}' - 3J_{m11}'\right]C_5 - \\ &\frac{k_1^3}{8}\left[I_{p33}' + 3I_{p11}'\right]C_6 = 0 \end{aligned} \tag{25}$$

<u>Case (ii)</u>: Considering the case (ii) i.e. $k = \lambda$, for the region (**II**, *n*), the solution to Eq. (2) is given by Eq. (9) and for this case the boundary conditions, Eqs. (16), (17), (18) and (19) give us the following:

$$\begin{aligned} &J_n(kb)C_1 + Y_n(kb)C_2 + I_n(kb)C_3 \\ &+K_n(kb)C_4 - b^nC_5 - b^{n+2}C_6 = 0 \end{aligned} \tag{26}$$

$$\begin{aligned} &\frac{k}{2}J_{m1}'C_1 + \frac{k}{2}Y_{n1}'C_2 + \frac{k}{2}I_{p1}'C_3 - \\ &\frac{k}{2}K_{q1}'C_4 - nb^{n-1}C_5 - (n+2)b^{n+1}C_6 = 0 \end{aligned} \tag{27}$$

$$\begin{aligned} &\frac{k^2}{4}\left(J_{m2}' - 2J_n(kb)\right)C_1 + \frac{k^2}{4}\left(Y_{n2}' - 2Y_n(kb)\right)C_2 \\ &+\frac{k^2}{4}\left(I_{p2}' + 2I_n(kb)\right)C_3 + \frac{k^2}{4}\left(K_{q2}' + 2K_n(kb)\right)C_4 \\ &-\left(n(n-1)b^{n-2}\right)C_5 - (n+1)(n+2)b^nC_6 = 0 \end{aligned} \tag{28}$$

$$\begin{aligned} &\frac{k^3}{8}\left[J_{m3}' - 3J_{m1}'\right]C_1 + \frac{k^3}{8}\left[Y_{n3}' - 3Y_{n1}'\right]C_2 \\ &+\frac{k^3}{8}\left[I_{p3}' + 3I_{p1}'\right]C_3 - \frac{k^3}{8}\left[K_{q3}' + 3K_{q1}'\right]C_4 - \\ &\left[n(n-1)(n-2)b^{n-3}\right]C_5 - \left[n(n+1)(n+2)b^{n-1}\right]C_6 = 0 \end{aligned} \tag{29}$$

where

$$J'_{m1} = J_{n-1}(kb) - J_{n+1}(kb); J'_{m2} = J_{n-2}(kb) + J_{n+2}(kb);$$
$$J'_{m3} = J_{n-3}(kb) - J_{n+3}(kb) Y'_{n1} = Y_{n-1}(kb) - Y_{n+1}(kb);$$
$$Y'_{n2} = Y_{n-2}(kb) + Y_{n+2}(kb); Y'_{n3} = Y_{n-3}(kb) - Y_{n+3}(kb)$$
$$I'_{p1} = I_{n-1}(kb) + I_{n+1}(kb); I'_{p2} = I_{n-2}(kb) + I_{n+2}(kb);$$
$$I'_{p3} = I_{n-3}(kb) + I_{n+3}(kb) K'_{q1} = K_{n-1}(kb) + K_{n+1}(kb);$$
$$K'_{q2} = K_{n-2}(kb) + K_{n+2}(kb); K'_{q3} = K_{n-3}(kb) + K_{n+3}(kb)$$

Case (iii) : Considering the case (iii), for $k < \lambda$, for region (**II**, n), the solution to Eq. (2) is given by Eq. (11) and for this case the boundary conditions Eqs. (16), (17), (18) and (19) give us the following:

$$J_n(kb)C_1 + Y_n(kb)C_2 + I_n(kb)C_3 + K_n(kb)C_4 - \mathrm{Re}[J_n(\sqrt{ik_2b})]C_5 - \mathrm{Im}[J_n(\sqrt{ik_2b})]C_6 = 0 \quad (30)$$

$$J'_n(kb)C_1 + Y'_n(kb)C_2 + I'_n(kb)C_3 - K'_n(kb)C_4 - \mathrm{Re}[J'_{m1}(\sqrt{ik_2b})]C_5 - \mathrm{Im}[J'_{m1}(\sqrt{ik_2b})]C_6 = 0 \quad (31)$$

$$J''_n(kb)C_1 + Y''_n(kb)C_2 + I''_n(kb)C_3 - K''_n(kb)C_4 - \mathrm{Re}[J''_{m1}(\sqrt{ik_2b})]C_5 - \mathrm{Im}[J''_{m1}(\sqrt{ik_2b})]C_6 = 0 \quad (32)$$

$$J'''_n(kb)C_1 + Y'''_n(kb)C_2 + I'''_n(kb)C_3 - K'''_n(kb)C_4 - \mathrm{Re}[J'''_{m1}(\sqrt{ik_2b})]C_5 - \mathrm{Im}[J'''_{m1}(\sqrt{ik_2b})]C_6 = 0 \quad (33)$$

Therefore, the set of Eqs. (20), (21), (30)-(33), represent for the case $k < \lambda$.

4. SOLUTION

For the given values of n, v, b, T_{11}, R_{11} & λ the above set of equations gives an exact characteristic equation for non-trivial solutions of the coefficients C_1, C_2, C_3, C_4, C_5 & C_6. For non-trivial solution, the determinant of $[C]_{6x6}$ must vanish. This eigenvalue problem was solved using Mathematica computer software with symbolic capabilities.

5. RESULTS & DISCUSSIONS

There is a lot of flexibility in the code developed in Mathematica. It is used to determine the frequency parameter for any range of translational constraints. This code is also implanted for various plate materials by adjusting Poisson's ratio. Since Poisson's ratio occurs as a parameter in most of the equations, the effect of this ratio on the roots of the equations is also considered. The findings are presented in both tabular and graphical form. The frequencies are calculated for various radius radii b, translational spring stiffness parameter.

The frequency values for the plate with elastically restrained edge against translation and resting on partial foundation, at various values of the translational stiffness parameter, T_{11} and for constant λ, have been calculated and the results are shown in the Table **1** and graphically in Fig. (**2**). The frequency has increased considerably as increase in translational constraint. The results also listed in Table **2** and graphically in Fig. (**3**), for different values of foundation constraint (λ) by keeping translational constraint (T_{11}) constant. For large foundation stiffness the curves will be asymptotic in nature. The results are shown in Fig. (**4**) for different values of Translational and foundation constraints. It is observed that the frequency is increases as two constraints increase simultaneously. It was found that the $n = 0$ axisymmetric mode gives the fundamental frequency. When b = 0, the foundation is absent and the frequency is governed by the elastically restrained edge plate, i.e. k = 2.1834. When $b = 1$, the plate has full foundation support, and the frequency is k_0 = 2.1834. The effect of Translational constraint and Foundation constraint on frequency is shown in Figs. (**2**, **3**) respectively. Also the combined effect of translational and Foundation constraints are shown in Fig. (**4**).

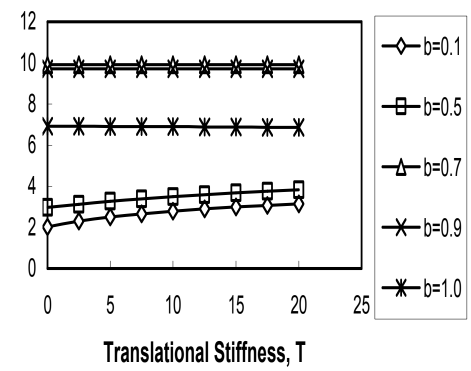

Fig. (2). Effect of translational stiffness parameter, T_{11} on first natural frequency parameter, k for $\lambda = 10$.

Table 1. **First Frequency Parameters for Different Translational Stiffness Ratio for λ = 10 & ν = 0.33**

T_{11}	b = 0.1	b = 0.2	b = 0.3	b = 0.4	b = 0.5	b = 0.6	b = 0.7	b = 0.8	b = 0.9	b = 1
0	2.02235	2.00269	1.65954	3.17252	2.97074	2.59295	9.91088	9.82665	9.71751	6.9251
2.5	2.30653	2.28733	2.06005	3.32198	3.1293	2.81866	9.91072	9.82583	9.71713	6.91746
5	2.50643	2.48571	2.29898	3.45341	3.26574	2.99905	9.91056	9.82492	9.71666	6.90972
7.5	2.66201	2.63915	2.47235	3.5711	3.38604	3.15054	9.91039	9.824	9.71627	6.90198
10	2.78959	2.7643	2.60887	3.67793	3.49379	3.28191	9.91023	9.82319	9.71579	6.89404
12.5	2.89784	2.86973	2.72118	3.77599	3.59168	3.39823	9.91007	9.82227	9.71541	6.8862
15	2.99165	2.96052	2.81616	3.86678	3.68139	3.5029	9.9099	9.82135	9.71492	6.87817
17.5	3.07421	3.03995	2.89819	3.95139	3.76433	3.5981	9.90974	9.82044	9.71444	6.87013
20	3.1477	3.11033	2.96997	4.03073	3.84139	3.68573	9.90957	9.81952	9.71404	6.86199

Table 2. First Frequency for Different Foundation Stiffness Parameter for T_{11} & ν = 0.33

λ	*b* = 0.1	*b* = 0.2	*b* = 0.3	*b* = 0.4	*b* = 0.5	*b* = 0.6	*b* = 0.7	*b* = 0.8	*b* = 0.9	*b* = 1
0	1.85759	1.85759	1.85759	1.85759	1.85759	1.85759	1.85759	1.85759	1.85759	1.85759
7.5	2.95671	2.82178	2.81512	2.66589	6.00336	4.8109	4.54729	4.3199	6.40789	6.36732
10	2.78959	2.7643	2.60887	3.67793	3.49379	3.28191	9.91023	9.82319	9.71579	6.89404
12.5	2.74279	2.72015	3.46242	3.2138	7.75637	4.58806	4.45195	11.6979	11.5023	11.3633
15	2.72551	2.58	3.0792	2.94435	3.83438	3.65997	6.43791	5.90339	12.7186	12.6973
17.5	2.71662	3.36935	2.9918	3.62631	3.4595	4.69047	4.25264	8.95145	12.9626	13.52
20	2.70984	2.96343	2.83938	3.31891	4.39574	3.9389	5.6791	18.1682	17.9198	13.6294

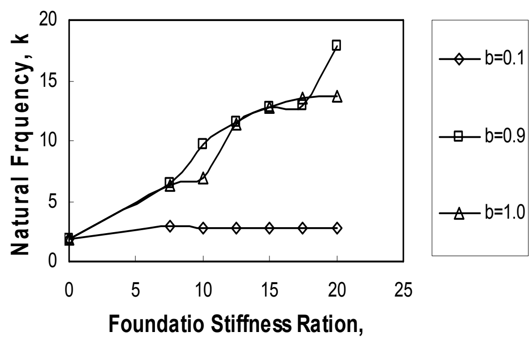

Fig. (3). Effect of foundation stiffness parameter, λ on first natural frequency parameter, k for $T_{11} = 10$.

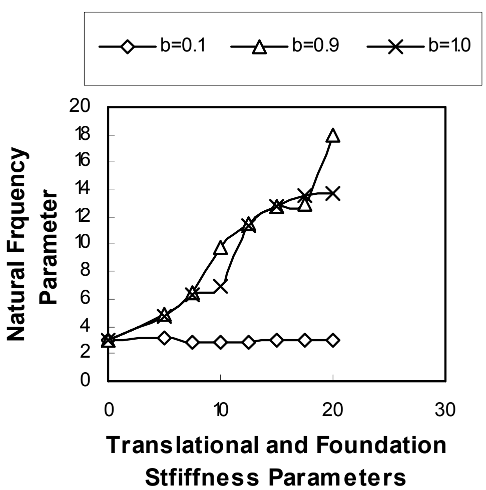

Fig. (4). Effect of translational stiffness parameter, T_{11} and foundation stiffness parameter λ on natural frequency parameter, k.

The frequencies for different plate materials, for various values of transverse, rotational and foundation parameters are computed and the results are given in Table **3**. It is observed that for any value of foundation parameter (λ), frequencies are independent on Poisson ratio, as shown in Fig. (**5**). And also it was observed that for any value of T_{11}, frequencies are independent on Poisson ratio.

Table 3. Frequencies for Different Poisson Ratios

ν	T_{11} = 10, λ = 10	T_{11} = 1000, λ = 10	T_{11} = 10, λ = 1000
0	2.77951	4.37181	2.78038
0.1	2.7831	4.39014	2.78338
0.2	2.7865	4.40768	2.78608
0.3	2.78959	4.42472	2.78857
0.4	2.79248	4.44116	2.79087
0.5	2.79517	4.4571	2.79306

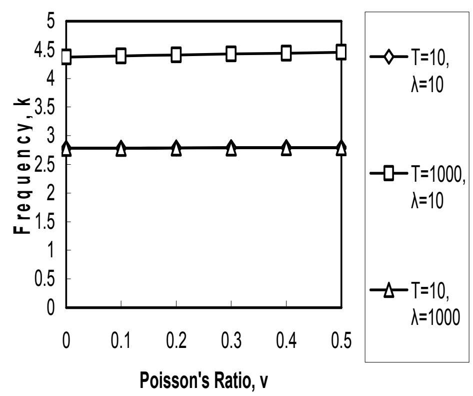

Fig. (5). Effect of poisson ratio, ν on frequency parameters, k.

The results of this kind are scarce in the literature. However, the results are compared with the following. If $R_{11} \to 0$ & $T_{11} \to \infty$, then the problem at hand becomes a simply supported boundary condition as shown in Fig. (**6**). The results are listed in Table **4**. It was found that the $n = 0$ axisymmetric mode gives the fundamental frequency. When $b = 0$, the foundation is absent and the circular simply supported plate governs the frequency, i.e., k = 2.22152. When $b = 1$, the plate has full foundation support and the frequency is k_0 = 2.22152. Table **5**, presents the comparison of frequency parameters k, for the plate with simply supported edges as shown in Fig. (**6**) (by setting the translational restraints with $R_{11} \to 0$ & $T_{11} \to \infty$), against those obtained by Wang [13] and Laura *et al* [6] by Ritz and finite element methods respectively. Results presented in this paper can be seen to be in excellent agreement with the those results available in the literature.

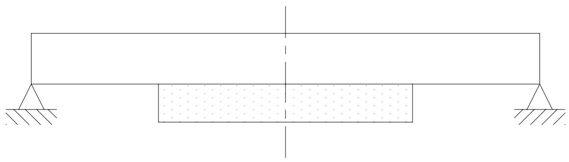

Fig. (6). A thin circular plate with simply supported edge and supported on partial elastic foundation.

Table 4. Frequency for Different Foundation Stiffness Ratio for ν = 0.33

λ	*b* = 0.1	*b* = 0.3	*b* = 0.4	*b* = 0.9
0	2.2215	2.2215	2.2215	2.22152
20	3.9522	4.5808	6.0081	14.6284
50	4.1065	5.1786	6.336	29.1056
100	4.2517	5.3877	6.228	36.7841
500	4.1747	5.4437	6.3845	39.3756
1000	4.2049	5.4673	6.4141	38.8204
2000	4.214	5.3866	6.41	37.7465
5000	4.2186	5.4738	6.4151	39.1173
7500	4.2165	5.4756	6.4103	39.1006
10000	4.2176	5.4708	6.4183	39.0791

Table 5. Comparison of Exact Values with Approximate Values from Ref. [6, 13, 14] for Simply Supported Edge Plate

b	0.3	0.3	0.6	0.6
λ	2.1147	3.1623	2.1147	3.1623
***k* [Present]**	2.33844	2.67264	2.51384	3.17202
***k* [12]**	2.33844	2.67274	2.51304	3.17204
Ritz [6]*	2.339	2.677	2.514	3.1724
F.E [6]*	2.349	2.702	2.536	3.2249
***k* [13]**	2.33844	2.67264	2.51384	3.17202

* The results are approximate.

If $R_{11} \to \infty$ & $T_{11} \to \infty$, then the problem at hand becomes a clamped boundary condition as shown in Fig. (**7**). The results are listed in Table **6**. It was found that the $n = 0$ axi-symmetric mode gives the fundamental frequency. When $b = 0$, the foundation is absent and the circular clamped plate governs frequency, i.e., $k = 3.19622$.

When $b = 1$, the plate has full foundation support and the frequency is $k_0 = 3.19622$. Table **7** shows a comparison of our exact values with the values obtained by Wang [13] and Laura *et al*. [6] by Ritz and finite element methods respectively. Excellent agreement has been found between these results.

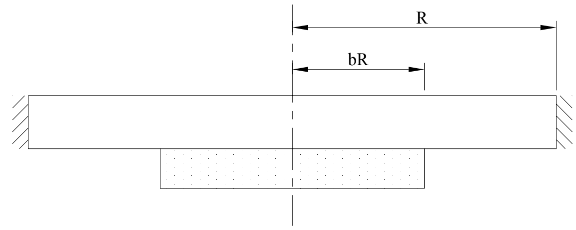

Fig. (7). A thin circular plate with clamped edge and supported on partial elastic foundation.

Table 6. Frequency for Different Foundation Stiffness Ratio for ν = 0.33

λ	*b* = 0.3	*b* = 0.4	*b* = 0.8	*b* = 1.0
0	3.19622	3.19622	3.1962	3.19622
20	5.73303	7.42844	18.151	13.536
50	6.38558	7.77718	21.658	33.8797
100	6.63007	7.64012	22.987	56.2845
500	6.69625	7.82554	23.572	124.841
1000	6.72454	7.86099	23.474	249.707
2000	6.62889	7.85603	23.633	499.426
5000	6.73224	7.86222	23.649	1002.67
10000	6.72868	7.86598	23.636	1200.98
20000	6.73245	7.86527	23.597	1600.78

Table 7. Comparison of Exact Values with Approximate Values from Ref. [6, 12] for Clamped Edge Plate

b	0.3	0.3	0.6	0.6
λ	2.1147	3.1623	2.1147	3.1623
***k* [Present]**	3.25572	3.46125	3.32705	3.73673
***k* [12]**	3.25573	3.46124	3.32706	3.73672
Ritz [6]*	3.2558	3.4615	3.3275	3.7367
F.E [6]*	3.2558	3.4771	3.1237	3.6576

* The results are approximate.

6. CONCLUSIONS

The flexural vibration behaviour of a circular plate supported along its edge by elastically restrained springs against rotation and translational and supported partially on a Winkler-type foundation has been studied in this paper. Mathematica computer software was used in obtaining the results for first frequency values of this circular plate.

The values of first frequencies are presented in both tabular and graphical form for various values of translational spring stiffness parameters [R_{11} & T_{11}] at the edges that simulate a clamped edge when $R_{11} \to \infty \,\&\, T_{11} \to \infty$, or a simply supported edge when $R_{11} \to 0$ & $T_{11} \to \infty$.

Graphical plots of first frequencies are presented for a wide range of rotational, translational and foundation constraints. The wide range of results provided in this paper could be potentially utilized for vibration control and in structural design. It is observed that the influence of foundation parameter on frequency is more predominant than that of translational parameter or rotational parameter.

Comparison of results obtained here with those available in literature for some special cases, demonstrates excellent accuracy and numerical stability of the present method. In this paper the characteristic equations solved are exact ones and therefore the frequency results can be calculated to any desired accuracy. These exact solutions can be used as benchmark solutions to check numerical or approximate results obtained through other methods of solution.

NOTATIONS

h	=	Thickness of a plate
R	=	Radius of a plate
b	=	Non-dimensional radius of support
ν	=	Poisson's ratio
E	=	Young's modulus of a material
ρ	=	Density of a material
ω	=	Angular frequency
D	=	Flexural rigidity of a material
K_{R1}	=	Rotational spring stiffness at outer edge
K_{T2}	=	Translational spring stiffness at outer edge
R_{11}	=	Non-dimensional rotational flexibility Parameter at outer edge
T_{22}	=	Non-dimensional translational flexibility Parameter at outer edge
λ	=	Non-dimensional foundation parameter

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