Riccati Equation Solution Method for the Computation of the Solutions of $X + A^TX^{-1}A = Q$ and $X - A^TX^{-1}A = Q$ †

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Abstract: A test is developed for checking the existence of a finite number of solutions of the matrix equations $X + A^TX^{-1}A = Q$ and $X - A^TX^{-1}A = Q$, when A is a nonsingular matrix. An algebraic method for computing all the solutions of these matrix equations is proposed. The method is based on the algebraic solution of the corresponding discrete time Riccati equations. The number of solutions is also derived. The extreme solutions are computed as well.

Keywords: Matrix theory, matrix equations, eigenvalue-eigenvector, Riccati equation.

1. INTRODUCTION

The central issue of this paper is the investigation and the computation of all the solutions of the matrix equations

$$X + A^T X^{-1} A = Q \tag{1}$$

and

$$X - A^T X^{-1} A = Q \tag{2}$$

where A is an $n \times n$ matrix, A^T denotes the transpose of A and Q is an $n \times n$ Hermitian positive definite matrix. These equations arise in many applications in various research areas including control theory, ladder networks, dynamic programming, stochastic filtering and statistics: see [1, 2] for references concerning equation (1) and [3] for references concerning equation (2).

Concerning equation (1), we denote a solution of $X + A^T X^{-1} A = Q$ as X^+. It is well known [4, Theorem 5.1] that, the existence of the Hermitian positive definite solution of the equation (1) depends on the numerical radius of matrix $Q^{-1/2}AQ^{-1/2}$. In particular, it is required that

$$r(Q^{-1/2}AQ^{-1/2}) \le \frac{1}{2},$$

where

$$r(A) = \max\{|x^* A x|: \text{ for every } x \in \mathbb{C}^n \text{ with } x^* x = 1\}$$

denotes the *numerical radius* of A. It is also known [4-6] that, if (1) has a Hermitian positive definite solution X^+, then there exist minimal and maximal solutions X^+_{min} and X^+_{max}, respectively, such that $0 < X^+_{min} \le X^+ \le X^+_{max}$ for any Hermitian positive definite solution X^+. Here, if X and Y are Hermitian matrices, then $X \le Y$ means that $Y - X$ is a nonnegative definite matrix and $X < Y$ means that $Y - X$ is a positive definite matrix.

Concerning equation (2), we denote a solution of $X - A^T X^{-1} A = Q$ as X^-. It is well known [3, 6] that there always exists a unique Hermitian positive definite solution, which is the maximal solution X^-_{max} of (2), and, if A is nonsingular, then there exists a unique Hermitian negative definite solution, which is the minimal solution X^-_{min} of (2). The minimal and maximal solutions are referred as the extreme solutions.

These equations have been studied recently by many authors [1-12]: the theoretical properties such as necessary and sufficient conditions for the existence of a positive definite solution have been investigated and numerical methods for computing the extreme solutions of these equations have been proposed; the available methods are recursive algorithms based mainly on the fixed point iteration and on applications of the Newton's algorithm or the cyclic reduction method. Furthermore, these equations may have Hermitian positive definite solutions or non-Hermitian solutions. They also may have definite or indefinite number of solutions. For example as referred in [2], when $A = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the equation (1) has indefinite number of solutions formed as $X^+ = \begin{bmatrix} 1 & -a \\ a^{-1} & 1 \end{bmatrix}$, where a is any real number $a \neq 0$. Finally, the existence of a finite number of Hermitian positive definite solutions of equation (1) has been studied in [4], when A is nonsingular and $Q = I$, where I is the identity matrix. In fact, the existence of a finite number of solutions of $X + A^T X^{-1} A = I$ depends on matrix A and is calculated, when A is a nonsingular complex matrix

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or A is a normal matrix ($AA^T = A^T A$). Also, the finite number of real symmetric positive definite solutions of $X + A^T X^{-1} A = I$ is calculated, when A is real and invertible.

In this paper, we focus in the computation of all the solutions of the matrix equations (1) and (2), when A is nonsingular. In Section 2, an algebraic method for computing all the solutions of these equations is proposed. The method is based on the algebraic solution of the corresponding discrete time Riccati equations. The number of solutions is also derived. In Section 3, a test for checking the existence of a finite number of solutions is developed. The number of Hermitian, symmetric, real and the extreme solutions are computed as well. In Section 4, simulation results are given to illustrate the efficiency of the proposed method and finally, concluding remarks are given in Section 5.

2. RICCATI EQUATION SOLUTION METHOD FOR THE COMPUTATION OF ALL THE SOLUTIONS

2.1. All the Solutions of Equation $X + A^T X^{-1} A = Q$

Here, A is an $n \times n$ *nonsingular* matrix and the *numerical radius* $r(Q^{-1/2} A Q^{-1/2}) \leq \frac{1}{2}$.

Working as in [7] and [4], we are able to derive a Riccati equation, which is equivalent to the matrix equation (1). In fact, the matrix equation (1) can be written as

$$X = Q - A^T X^{-1} A,$$

whereby arises

$$\begin{aligned} X &= Q - A^T [Q - A^T X^{-1} A]^{-1} A \\ &= Q - A^T A^{-1} [A^{-T} Q A^{-1} - X^{-1}]^{-1} A^{-T} A \\ &= Q + A^T A^{-1} [X^{-1} + (-A^{-T} Q A^{-1})]^{-1} (A^T A^{-1})^T \end{aligned}$$

and thus the following equivalent related Riccati equation is derived

$$P = C^+ + F^+ (P^{-1} + G^+)^{-1} (F^+)^T, \tag{3}$$

with

$$F^+ = A^T A^{-1}, \quad C^+ = Q \quad \text{and} \quad G^+ = -A^{-T} Q A^{-1}. \tag{4}$$

It becomes obvious that the matrix equation (1) is equivalent to the related Riccati equation (3) and that the two equations have equivalent solutions. Thus the unique maximal solution of (1) coincides with the unique Hermitian positive definite solution of the related Riccati equation:

$$X^+_{max} = P \tag{5}$$

It is clear that we are able to solve (1), if we know the solution of the related Riccati equation. The solution of the related Riccati equation can be derived using the algebraic solution proposed in [13] and [14]. More specifically, from the Riccati equation's parameters the following symplectic matrix is formed:

$$\Phi^+ = \begin{bmatrix} A^{-1} A^T & -A^{-1} Q A^{-1} \\ Q A^{-1} A^T & A^T A^{-1} - Q A^{-1} Q A^{-1} \end{bmatrix} \tag{6}$$

Note that a symplectic matrix M is defined as $M^T J M = J$, where $J = \begin{bmatrix} O & -I \\ I & O \end{bmatrix}$.

Since Φ^+ is a symplectic matrix, it can be written as

$$\Phi^+ = W^+ L^+ (W^+)^{-1}, \tag{7}$$

where all the eigenvalues of Φ^+ are non-zero and can be placed in the diagonal matrix

$$L^+ = \begin{bmatrix} \Lambda_1^+ & O \\ O & \Lambda_2^+ \end{bmatrix} \tag{8}$$

and W^+ is the matrix, which contains the corresponding eigenvectors of matrix Φ^+,

$$W^+ = \begin{bmatrix} W_{11}^+ & W_{12}^+ \\ W_{21}^+ & W_{22}^+ \end{bmatrix}. \tag{9}$$

Rewriting the equality in (7) as $W^+ L^+ = \Phi^+ W^+$, we conclude the following equalities:

$$W_{11}^+ \Lambda_1^+ = A^{-1} A^T (W_{11}^+ - A^{-T} Q A^{-1} W_{21}^+) \tag{10}$$

$$W_{21}^+ \Lambda_1^+ = Q A^{-1} A^T (W_{11}^+ - A^{-T} Q A^{-1} W_{21}^+) + A^T A^{-1} W_{21}^+ \tag{11}$$

$$W_{12}^+ \Lambda_2^+ = A^{-1} A^T (W_{12}^+ - A^{-T} Q A^{-1} W_{22}^+) \tag{12}$$

$$W_{22}^+ \Lambda_2^+ = Q A^{-1} A^T (W_{12}^+ - A^{-T} Q A^{-1} W_{22}^+) + A^T A^{-1} W_{22}^+ \tag{13}$$

Since all the eigenvalues of the symplectic matrix Φ^+ are non-zero, it is clear that $0 \notin \sigma(\Lambda_1^+)$ and $0 \notin \sigma(\Lambda_2^+)$, where $\sigma(A)$ denotes the spectrum of A. Moreover, if the block matrices W_{11}^+, W_{12}^+, W_{21}^+, W_{22}^+ are nonsingular, then, after some algebra, from (10) and (11) we take:

$$\begin{aligned} &W_{21}^+ (W_{11}^+)^{-1} \\ &\quad = Q + A^T A^{-1} \left[W_{11}^+ (W_{21}^+)^{-1} - A^{-T} Q A^{-1} \right]^{-1} A^{-T} A \end{aligned} \tag{14}$$

From (12) and (13) we take:

$$\begin{aligned} &W_{22}^+ (W_{12}^+)^{-1} \\ &\quad = Q + A^T A^{-1} \left[W_{12}^+ (W_{22}^+)^{-1} - A^{-T} Q A^{-1} \right]^{-1} A^{-T} A \end{aligned} \tag{15}$$

Using (4) in (14) and (15), we are able to rewrite the last two equations as

$$W_{21}^+ (W_{11}^+)^{-1} = C^+ + F^+ \left[(W_{21}^+ (W_{11}^+)^{-1})^{-1} + G^+ \right]^{-1} (F^+)^T,$$

$$W_{22}^+ (W_{12}^+)^{-1} = C^+ + F^+ \left[(W_{22}^+ (W_{12}^+)^{-1})^{-1} + G^+ \right]^{-1} (F^+)^T,$$

whereby it is obvious that both the quantities $W_{21}^{+}(W_{11}^{+})^{-1}$ and $W_{22}^{+}(W_{12}^{+})^{-1}$ satisfy the Riccati equation (3) and consequently satisfy the matrix equation (1). Thus, the following proposition is proved.

Proposition 1. *Let the block matrices* W_{11}^{+}, W_{12}^{+}, W_{21}^{+}, W_{22}^{+} *in* (9) *be nonsingular and arise from every permutation of columns of* W^{+} in (9), *which consist of the eigenvectors of* Φ^{+} *in* (6). *The solutions of* (1) *are formed by*

$$X_1^{+} = W_{21}^{+}(W_{11}^{+})^{-1} \tag{16}$$

$$X_2^{+} = W_{22}^{+}(W_{12}^{+})^{-1}. \tag{17}$$

At this point, we observe that the solutions calculated by (16) can be derived from any arrangement of the first n eigenvectors $\begin{bmatrix} W_{11}^{+} \\ W_{21}^{+} \end{bmatrix}$ and that the other solutions calculated by (17) can be derived from any arrangement of the next n eigenvectors $\begin{bmatrix} W_{12}^{+} \\ W_{22}^{+} \end{bmatrix}$.

Proposition 2. *The solutions in* (16), (17) *of equation* (1) *do not depend on the permutation of the first n columns of* W^{+} *in* (9), *which are eigenvectors of* Φ^{+} *in* (6).

Proof. Let

$$E_{rt} = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & \text{O} & \\ & & \ddots & & & & & & \\ & & & 0 & & 1 & & & \\ & & & & \ddots & & & & \\ & & & 1 & & 0 & & & \\ & & & & & & \ddots & & \\ & \text{O} & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix} \tag{18}$$

be the $n \times n$ identity matrix, in which the $t-$column is permutated by the $r-$column. Let Φ^{+} be the symplectic matrix in (6) and $\upsilon_1, \upsilon_2, \dots, \upsilon_n, \upsilon_{n+1}, \dots, \upsilon_{2n}$ be the eigenvectors of Φ^{+} in (7), where

$$W^{+} = \begin{bmatrix} \upsilon_1 & \dots & \upsilon_i & \dots & \upsilon_j & \dots & \upsilon_n & \upsilon_{n+1} & \dots & \upsilon_k & \dots \upsilon_\ell & \dots & \upsilon_{2n} \end{bmatrix} = \begin{bmatrix} W_{11}^{+} & W_{12}^{+} \\ W_{21}^{+} & W_{22}^{+} \end{bmatrix}$$

and

$$\breve{W}^{+} = \begin{bmatrix} \upsilon_1 & \dots & \upsilon_j & \dots & \upsilon_i & \dots & \upsilon_n & \upsilon_{n+1} & \dots & \upsilon_\ell & \dots \upsilon_k & \dots & \upsilon_{2n} \end{bmatrix} = \begin{bmatrix} \breve{W}_{11}^{+} & \breve{W}_{12}^{+} \\ \breve{W}_{21}^{+} & \breve{W}_{22}^{+} \end{bmatrix}.$$

Suppose that X_1^{+}, X_2^{+} constitute two solutions of (1) and due to (16), (17), they are written $X_1^{+} = W_{21}^{+}(W_{11}^{+})^{-1}$ and $X_2^{+} = W_{22}^{+}(W_{12}^{+})^{-1}$ respectively. Since $E_{ij}^{-1} = E_{ij}$ and $E_{k\ell}^{-1} = E_{k\ell}$ we take

$$\begin{aligned} X_1^{+} &= W_{21}^{+}(W_{11}^{+})^{-1} = W_{21}^{+}E_{ij}E_{ij}^{-1}(W_{11}^{+})^{-1} \\ &= (W_{21}^{+}E_{ij})(W_{11}^{+}E_{ij})^{-1} = \breve{W}_{21}^{+}(\breve{W}_{11}^{+})^{-1} \end{aligned}$$

and

$$\begin{aligned} X_2^{+} &= W_{22}^{+}(W_{12}^{+})^{-1} = W_{22}^{+}E_{k\ell}E_{k\ell}^{-1}(W_{12}^{+})^{-1} \\ &= (W_{22}^{+}E_{k\ell})(W_{12}^{+}E_{k\ell})^{-1} = \breve{W}_{22}^{+}(\breve{W}_{12}^{+})^{-1}. \end{aligned}$$

Consequently, every permutation of the first n eigenvectors $\begin{bmatrix} W_{11}^{+} \\ W_{21}^{+} \end{bmatrix}$ constitutes the same solution for (1) and every permutation of the following n eigenvectors $\begin{bmatrix} W_{12}^{+} \\ W_{22}^{+} \end{bmatrix}$ constitutes the same solution for (1). □

Furthermore, we are going to investigate every permutation of the eigenvalues of Φ^{+} in a diagonal matrix. In fact, as referred in [14], the symplectic matrix Φ^{+} can be written as

$$\Phi^{+} = \hat{W}^{+}\hat{L}^{+}(\hat{W}^{+})^{-1}, \tag{19}$$

where $\hat{L}^{+}$ is the matrix, which contains all the eigenvalues of matrix Φ^{+} as L^{+} in (8) and formulated as

$$\hat{L}^{+} = \begin{bmatrix} \hat{\Lambda}^{+} & \text{O} \\ \text{O} & (\hat{\Lambda}^{+})^{-1} \end{bmatrix}, \tag{20}$$

with $\hat{\Lambda}^{+}$ the diagonal matrix, which contains the eigenvalues of matrix Φ^{+}, that lie outside the unit circle. Note that the eigenvalues of matrix Φ^{+} occur in reciprocal pairs, due to the fact that is a symplectic matrix. In (19), $\hat{W}^{+}$ is the matrix, that contains the eigenvectors of matrix Φ^{+}, which correspond to the eigenvalues arranged in the diagonal matrix $\hat{L}^{+}$ in (20):

$$\hat{W}^{+} = \begin{bmatrix} \hat{W}_{11}^{+} & \hat{W}_{12}^{+} \\ \hat{W}_{21}^{+} & \hat{W}_{22}^{+} \end{bmatrix} \tag{21}$$

Using the matrix of eigenvectors $\hat{W}^{+}$ in (21), Proposition 2 and (16) of Proposition 1, we have:

$$X_1^{+} = P = \hat{W}_{21}^{+}(\hat{W}_{11}^{+})^{-1} \tag{22}$$

Moreover, by (5) and (22), the unique maximal solution of (1) is computed as

$$X_{max}^{+} = \hat{W}_{21}^{+}(\hat{W}_{11}^{+})^{-1},$$

that has been proved in a different way in [7]. By (17) the other solution of (1) is computed $X_2^{+} = \hat{W}_{22}^{+}(\hat{W}_{12}^{+})^{-1}$, which is the minimal solution of (1), as proved in [7]:

$$X_{min}^{+} = \hat{W}_{22}^{+}(\hat{W}_{12}^{+})^{-1}$$

Consequently, *the extreme solutions* X_{max}^{+}, X_{min}^{+} of $X + A^T X^{-1} A = Q$ are derived, doing the *specific arrangement* in the matrices L^{+}, W^{+}, which are formed as in (20)-(21).

In the following, by Proposition 1 and 2, we lead to the computation of the number of solutions concerning equation (1).

Theorem 3. *Let A be an n × n nonsingular matrix, with*

$$r(Q^{-1/2}AQ^{-1/2}) \leq \frac{1}{2},$$

and the block matrices $W_{11}^{+}, W_{12}^{+}, W_{21}^{+}, W_{22}^{+}$ *be nonsingular, which arise from every permutation of columns of* W^{+} *in* (9). *The finite number of solutions* ($n.s$) of (1) *is equal to*

$$n.s = \frac{(2n)!}{(n!)^2}. \tag{23}$$

Proof. Since $r(Q^{-1/2}AQ^{-1/2}) \leq \frac{1}{2}$, the equation (1) has a Hermitian positive definite solution, (see [4, Theorem 5.1]) and by [14] the matrix Φ^{+} in (6) for a specific arrangement of eigenvalues (recall the eigenvalues belong outside or inside the unit circle) can be written as in (19), it is obvious that, the equation (1) has at least a Hermitian positive definite solution, namely the maximal one. Moreover, the symplectic matrix Φ^{+} in (7) has $2n$ eigenvectors, which can be used to compute all the solutions by (16)-(17). The finite number of the possible permutations of these eigenvectors are $(2n)!$. By Proposition 2 each solution has multiplicity equal to n. Hence, the number of solutions of (1) computed using the Riccati equation solution method is:

$$n.s = \binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2} \qquad \square$$

Remark 1. Note that in Theorem 3, if we omit the nonsingularity of all the block matrices W_{ij}^{+}, $i, j = 1, 2$ which arise from every permutation of the n first with the n following columns of W^{+} in (9), then $n.s$ in (23) is an upper bound for the number of solutions of (1). Recall that the existence of the solutions of (1) depends on $r(Q^{-1/2}AQ^{-1/2}) \leq \frac{1}{2}$. Consequently, the existence of the extreme solutions and the nonsingularity of the corresponding matrices $W_{11}^{+} \equiv \hat{W}_{11}^{+}$, $W_{12}^{+} \equiv \hat{W}_{12}^{+}$ are ensured by the assumptions of Theorem 3, therefore the solutions are computed by (16)-(17). The nonsingularity of all the block matrices W_{ij}^{+}, $i, j = 1, 2$ is needed for the existence of all the other (not extreme) solutions of (1), which are given by formulas in (16)-(17).

2.2. All the Solutions of Equation $X - A^T X^{-1} A = Q$

Let A be a *nonsingular* matrix. Working as in [7] and [3], we are able to derive a Riccati equation, which is equivalent to the matrix equation (2). In fact, the matrix equation (2) can be written as

$$X = Q + A^T X^{-1} A,$$

whereby arises

$$\begin{aligned} X &= Q + A^T[Q + A^T X^{-1} A]^{-1} A \\ &= Q + A^T A^{-1}[A^{-T} Q A^{-1} + X^{-1}]^{-1} A^{-T} A \\ &= Q + A^T A^{-1}[X^{-1} + A^{-T} Q A^{-1}]^{-1} (A^T A^{-1})^T \end{aligned}$$

and thus the following equivalent related Riccati equation is derived

$$P = C^{-} + F^{-}(P^{-1} + G^{-})^{-1}(F^{-})^T \tag{24}$$

with

$$F^{-} = A^T A^{-1}, \quad C^{-} = Q \quad \text{and} \quad G^{-} = A^{-T} Q A^{-1}. \tag{25}$$

It becomes obvious that the matrix equation (2) is equivalent to the related Riccati equation (24) and that the two equations have equivalent solutions. Thus the unique maximal solution of (2) coincides with the unique Hermitian positive definite solution of the related Riccati equation:

$$X_{max}^{-} = P \tag{26}$$

It is clear that we are able to solve (2), if we know the solution of the related Riccati equation. The solution of the related Riccati equation can be derived using the algebraic solution proposed in [13] and [14]. More specifically, from the Riccati equation's parameters the following symplectic matrix is formed:

$$\Phi^{-} = \begin{bmatrix} A^{-1}A^T & A^{-1}QA^{-1} \\ QA^{-1}A^T & A^T A^{-1} + QA^{-1}QA^{-1} \end{bmatrix} \tag{27}$$

Since Φ^{-} is a symplectic matrix, it can be written as

$$\Phi^{-} = W^{-} L^{-} (W^{-})^{-1}, \tag{28}$$

where all the eigenvalues of Φ^{-} are non-zero and can be placed in the diagonal matrix

$$L^{-} = \begin{bmatrix} \Lambda_1^{-} & O \\ O & \Lambda_2^{-} \end{bmatrix}, \tag{29}$$

and W^{-} is the matrix, which contains the corresponding eigenvectors of matrix Φ^{-},

$$W^{-} = \begin{bmatrix} W_{11}^{-} & W_{12}^{-} \\ W_{21}^{-} & W_{22}^{-} \end{bmatrix}. \tag{30}$$

Rewriting the equality in (28) as $W^- L^- = \Phi^- W^-$, and following the same way of proof as in paragraph 2.1, we conclude :

$$W_{21}^-(W_{11}^-)^{-1} = Q + A^T A^{-1}\left[W_{11}^-(W_{21}^-)^{-1} + A^{-T}QA^{-1}\right]^{-1} A^{-T}A \tag{31}$$

$$W_{22}^-(W_{12}^-)^{-1} = Q + A^T A^{-1}\left[W_{12}^-(W_{22}^-)^{-1} + A^{-T}QA^{-1}\right]^{-1} A^{-T}A \tag{32}$$

Using (25) in (31) and (32), it is obvious that both the quantities $W_{21}^-(W_{11}^-)^{-1}$ and $W_{22}^-(W_{12}^-)^{-1}$ satisfy the Riccati equation (24) and consequently satisfy the matrix equation (2). Thus, the following proposition is proved.

Proposition 4. *Let the block matrices* $W_{11}^-, W_{12}^-, W_{21}^-, W_{22}^-$ *in* (30) *be nonsingular and arise from every permutation of columns of* W^- *in* (30), *which consist of the eigenvectors of* Φ^- *in* (27). *The solutions of* (2) *are formed by*

$$X_1^- = W_{21}^-(W_{11}^-)^{-1} \tag{33}$$

$$X_2^- = W_{22}^-(W_{12}^-)^{-1}. \tag{34}$$

At this point, we observe that the solutions calculated by (33) can be derived from any arrangement of the first n eigenvectors $\begin{bmatrix} W_{11}^- \\ W_{21}^- \end{bmatrix}$ and that the other solutions calculated by (34) can be derived from any arrangement of the next n eigenvectors $\begin{bmatrix} W_{12}^- \\ W_{22}^- \end{bmatrix}$ Thus, following the same method as in paragraph 2.1, the following proposition is proved.

Proposition 5. *The solutions in* (33), (34) *of equation* (2) *do not depend on the permutation of the first n columns of* W^- *in* (30), *which are eigenvectors of* Φ^- in (27).

Furthermore, in the proof of the formulas of the extreme solutions in [7], we did not observe that these are derived by the specific permutation of the eigenvalues of the diagonal matrix Φ^-. In fact, as referred in [14], the symplectic matrix Φ^- can be written as

$$\Phi^- = \hat{W}^- \hat{L}^- (\hat{W}^-)^{-1}, \tag{35}$$

where $\hat{L}^-$ is the matrix, which contains all the eigenvalues of matrix Φ^- as L^- in (29) and formulated

$$\hat{L}^- = \begin{bmatrix} \hat{\Lambda}^- & O \\ O & (\hat{\Lambda}^-)^{-1} \end{bmatrix} \tag{36}$$

with $\hat{\Lambda}^-$ the diagonal matrix, which contains the eigenvalues of matrix Φ^-, that lie outside the unit circle. In (35), $\hat{W}^-$ is the matrix, that contains the eigenvectors of matrix Φ^-, which correspond to the eigenvalues arranged in the diagonal matrix $\hat{L}^-$ in (36):

$$\hat{W}^- = \begin{bmatrix} \hat{W}_{11}^- & \hat{W}_{12}^- \\ \hat{W}_{21}^- & \hat{W}_{22}^- \end{bmatrix} \tag{37}$$

It is obvious that, using the matrix of eigenvectors $\hat{W}^-$ in (37), Proposition 5, the formulas in (33)-(34) of Proposition 4, the unique maximal and minimal solutions of (2) are computed as:

$$X_{max}^- = \hat{W}_{21}^-(\hat{W}_{11}^-)^{-1} \tag{38}$$

$$X_{min}^- = \hat{W}_{22}^-(\hat{W}_{12}^-)^{-1} \tag{39}$$

Consequently, *the extreme solutions* X_{max}^-, X_{min}^- of $X - A^T X^{-1} A = Q$ are derived, doing the *specific arrangement* in the matrices L^-, W^-, which are formed as in (36)-(37).

By Proposition 4 and 5 and following the same way of the proof as in paragraph 2.1, we lead to the computation of the number of solutions concerning equation (2).

Theorem 6. *Let* A *be an* $n \times n$ *nonsingular matrix and the block matrices* $W_{11}^-, W_{12}^-, W_{21}^-, W_{22}^-$, *which arise from every permutation of columns of* W^- *in* (30), *be nonsingular. The finite number of solutions* ($n.s$) *of* (2) *is equal to*

$$n.s = \frac{(2n)!}{(n!)^2}.$$

Remark 2. Note that in Theorem 6, if we omit the nonsingularity of all the block matrices W_{ij}^-, $i, j = 1, 2$, which arise from every permutation of the n first with the n following columns of W^- in (30), then $n.s$ in (23) is an upper bound for the number of solutions of (2). Recall that for an A nonsingular matrix, the equation (2) has always extreme Hermitian solutions, whereby the nonsingularity of the corresponding matrices $W_{11}^- \equiv \hat{W}_{11}^-$, $W_{12}^- \equiv \hat{W}_{12}^-$ is implied, therefore the solutions are computed by (33) and (34) or by (38)-(39). The nonsingularity of all the block matrices W_{ij}^-, $i, j = 1, 2$, is needed for the existence of all the other (not extreme) solutions of (2), which are given by formulas in (33)-(34).

3. EXISTENCE OF A FINITE NUMBER OF SOLUTIONS

3.1. Existence of a Finite Number of Solutions for $X + A^T X^{-1} A = Q$

The existence of a finite number of solutions of equation (1), when A is nonsingular and $Q = I$, depends on matrix A, as stated in [4]. In fact, it depends on the eigenvalues of the matrix:

$$H = \begin{bmatrix} O & -A^{-1} \\ A^T & -A^{-1} \end{bmatrix} \quad (40)$$

More precisely: if $\dim(V(\lambda_i(H))) = 1$ for all eigenvalues $\lambda_i(H)$, $i = 1, 2, \dots, 2n$ of H, with $|\lambda_i(H)| \neq 1$, then, there exists a finite number of Hermitian solutions of $X + A^T X^{-1} A = I$. Note that $V(\lambda_i(H))$ denotes the eigenspace corresponding to eigenvalue $\lambda_i(H)$ of the matrix H and $\dim(V(\lambda_i(H)))$ denotes the dimension of $V(\lambda_i(H))$. Moreover, the number of Hermitian positive definite solutions (h.p.d.n.s) of $X + A^T X^{-1} A = I$ is given by [4]

$$h.p.d.n.s = \prod_{j=1}^{m} (n_j + 1), \quad (41)$$

where m is the number of eigenvalues $\lambda_i(H)$ of H lying outside the unit circle, with algebraic multiplicity n_j, $j = 1, 2, \dots, m$. Also, as referred in [4, Proposition 8.2], the number of real symmetric positive definite solutions (r.p.d.n.s) is equal to

$$r.p.d.n.s = \prod_{j=1}^{p+q} (n_j + 1), \quad (42)$$

where p is the number of real eigenvalues lying outside the unit circle, q is the number of complex conjugate pairs of eigenvalues lying outside the unit circle, with algebraic multiplicity n_j, $j = 1, 2, \dots, p+q$.

This result, that is referred in the special case, where $Q = I$, can be extended for the general case, when $Q \neq I$. It is known [4] that the matrix equation (1) can be reduced to a corresponding matrix equation, with $Q = I$. Multiplying on the right and on the left both sides of (1) by $Q^{-1/2}$, (recall that Q is a Hermitian positive definite matrix, hence it is nonsingular), we have :

$$Q^{-1/2} X Q^{-1/2} + Q^{-1/2} A^T X^{-1} A Q^{-1/2} = I$$

Setting in the last equation

$$Y = Q^{-1/2} X Q^{-1/2} \quad \text{and} \quad R = Q^{-1/2} A Q^{-1/2} \quad (43)$$

we obtain

$$Y + R^T Y^{-1} R = I, \quad (44)$$

where the matrix R is nonsingular, since A is nonsingular. By (43), it is obvious that the solution X^+ of (1) can be computed as

$$X^+ = Q^{1/2} Y^+ Q^{1/2}, \quad (45)$$

where Y^+ is the corresponding Hermitian solution of (44). Since the existence of a finite number of Hermitian positive definite solutions of (44) depends on the matrix, which is formed as in (40), we have

$$H^+ = \begin{bmatrix} O & -R^{-1} \\ R^T & -R^{-1} \end{bmatrix},$$

and substituting the matrix R by (43) we obtain:

$$H^+ = \begin{bmatrix} O & -Q^{1/2} A^{-1} Q^{1/2} \\ Q^{-1/2} A^T Q^{-1/2} & -Q^{1/2} A^{-1} Q^{1/2} \end{bmatrix} \quad (46)$$

It is easy to verify that the matrix H^+ is symplectic. Using the above statements we conclude the following results, which constitute a generalization of Corollary 6.6 and of Proposition 8.2 in [4] as well as of the relationships in (41)-(42).

Corollary 7. *Suppose that the matrix equation* $X + A^T X^{-1} A = Q$ *has a Hermitian positive definite solution. If* $\dim(V(\lambda_i(H^+))) = 1$ *for every eigenvalue* $\lambda_i(H^+)$, $i = 1, 2, \dots, 2n$ *of* H^+ *in* (46), *with* $|\lambda_i(H^+)| \neq 1$, *then there exists a finite number of solutions of* (1).

Let $\lambda_1(H^+), \lambda_2(H^+), \dots, \lambda_m(H^+)$ *be the distinct eigenvalues of* H^+ *lying outside the unit circle, with algebraic multiplicity* n_j, $j = 1, 2, \dots, m$. *The number of Hermitian positive definite solutions* ($h.p.d.n.s$) *of* (1) *is equal to*

$$h.p.d.n.s = \prod_{j=1}^{m} (n_j + 1). \quad (47)$$

Corollary 8. *Let A be an n × n nonsingular real matrix, Q be real and* H^+ *be the matrix in* (46). *Let* $\lambda_1(H^+), \lambda_2(H^+), \dots, \lambda_p(H^+)$ *be the distinct real eigenvalues of* H^+ *lying outside the unit circle,* q *be the number of complex conjugate pairs of eigenvalues lying outside the unit circle, with algebraic multiplicity* n_j, $j = 1, 2, \dots, p+q$. *The number of real symmetric positive definite solutions* ($r.p.d.n.s$) *of* (1) *is equal to*

$$r.p.d.n.s = \prod_{j=1}^{p+q} (n_j + 1). \quad (48)$$

The existence of a finite number of solutions of the matrix equation (1) can be expressed with respect to the eigenvalues of Φ^+ instead of the eigenvalues of H^+, as it is presented in the following.

Theorem 9. *Let A be an n × n nonsingular matrix, with* $r(Q^{-1/2} A Q^{-1/2}) \leq \frac{1}{2}$, Φ^+ *be the matrix in* (6) *and the block matrices* W_{11}^+, W_{12}^+, W_{21}^+, W_{22}^+ *in* (9) *be nonsingular. If* $\dim(V(\lambda_i(\Phi^+))) = 1$ *for every eigenvalue* $\lambda_i(\Phi^+)$, $i = 1, 2, \dots, 2n$ *of* Φ^+, *with* $|\lambda_i(\Phi^+)| \neq 1$, *then, there exists a* *finite number of Hermitian positive definite solutions* (*h.p.d.n.s*) *of* (1). *The number of Hermitian positive definite solutions of* (1) *is equal to*

$$h.p.d.n.s = \prod_{j=1}^{m}(n_j + 1)\,. \tag{49}$$

The number of non-Hermitian solutions (*n.h.n.s*) *of* (1) *is*

$$n.h.n.s = n.s - \prod_{j=1}^{m}(n_j + 1)\,. \tag{50}$$

In particular, if A and Q are real matrices, then among the h.p.d.n.s there exist real symmetric solutions (*r.p.d.n.s*) *and the number of r.p.d.n.s of* (1) *is*

$$r.p.d.n.s = \prod_{k=1}^{p+q}(n_k + 1)\,. \tag{51}$$

In above, m is the number of the distinct eigenvalues of Φ^{+}, *that lie outside the unit circle, with algebraic multiplicity* n_j, $j = 1, 2, \dots, m$, *p is the number of real distinct eigenvalues of* Φ^{+} *lying outside the unit circle, q is the number of complex conjugate pairs of eigenvalues lying outside the unit circle, with algebraic multiplicity* n_k, $k = 1, 2, \dots, p+q$, *and n.s is given by* (23).

Proof. Due to the following relation

$$\Phi^{+} = \begin{bmatrix} Q^{-1/2} & \mathbf{O} \\ \mathbf{O} & Q^{1/2} \end{bmatrix} (-(H^{+})^2) \begin{bmatrix} Q^{-1/2} & \mathbf{O} \\ \mathbf{O} & Q^{1/2} \end{bmatrix}^{-1}, \tag{52}$$

it is clear that the matrices Φ^{+} and $-(H^{+})^2$ are similar, thus the relation between the eigenvalues of these two matrices is:

$$\lambda_i(\Phi^{+}) = -\lambda_i^2(H^{+})\,,\ i = 1, 2, \dots, 2n\,. \tag{53}$$

Moreover, by (53), the relation of the absolute value is: $\left|\lambda_i(H^{+})\right| = \sqrt{|\lambda_i(\Phi^{+})|}$. By the last equality it is obvious that,

$$\left|\lambda_i(H^{+})\right| \neq 1 \Leftrightarrow \left|\lambda_i(\Phi^{+})\right| \neq 1 \tag{54}$$

and

$$\left|\lambda_i(H^{+})\right| > 1 \Leftrightarrow \left|\lambda_i(\Phi^{+})\right| > 1\,. \tag{55}$$

Note that the eigenspaces, corresponding to eigenvalues $\lambda_i(H^{+})$ and $-\lambda_i^2(H^{+})$ of the matrices H^{+} and $-(H^{+})^2$, coincide, hence, $\dim(V(\lambda_i(H^{+}))) = \dim(V(-\lambda_i^2(H^{+})))$. Also, if we combine the above equality with the similarity of matrices Φ^{+} and $-(H^{+})^2$, we conclude that

$$\dim(V(\lambda_i(\Phi^{+}))) = \dim(V(\lambda_i(H^{+})))\,. \tag{56}$$

The assumption and (56) yields

$$\dim(V(\lambda_i(\Phi^{+}))) = \dim(V(\lambda_i(H^{+}))) = 1\,.$$

The last statement and (54) lead to the conclusion that, the first part of Corollary 7 can be applied, i.e., there exists a finite of Hermitian solutions of (1). Furthermore, combining the assumption with (55), it is obvious that the second part of Corollary 7 is applied, hence, $h.p.d.n.s = \prod_{j=1}^{m}(n_j + 1)$, where n_j is the algebraic multiplicity of the eigenvalue $\lambda_j(\Phi^{+})$ of Φ^{+}, for every $j = 1, 2, \dots, m$, with $\left|\lambda_j(\Phi^{+})\right| > 1$.

Also, by (53) we have that any real (complex) eigenvalue $\lambda_i(H^{+})$ of the matrix H^{+} is real (complex) eigenvalue of the matrix Φ^{+}, respectively, thus the hypotheses and (55) lead to application of Corollary 8, whereby arises the number of real symmetric solutions in (51). Furthermore, due to the assumptions for the block matrices W_{ij}^{+}, $i, j = 1, 2$, the number of non-Hermitian solutions of (1) in (50) yields combining Theorem 3 with the equation (49). □

Remark 3. Note that in Theorem 9, m, p, q are connected by the relationship $m = p + 2q$.

3.2. Existence of a Finite Number of Solutions for $X - A^T X^{-1} A = Q$

The equation (2) can be formed as (1), if we substitute the nonsingular matrix A with $\widehat{A}\mathbf{i}$, and then we have

$$X + \widehat{A}^T X^{-1} \widehat{A} = Q\,. \tag{57}$$

According to Theorem 9, the existence of a finite number of solutions of equation (57) depends on the matrix, which is formed as (6) and is

$$\widehat{\Phi}^{+} = \begin{bmatrix} \widehat{A}^{-1}\widehat{A}^T & -\widehat{A}^{-1}Q\widehat{A}^{-1} \\ Q\widehat{A}^{-1}\widehat{A}^T & \widehat{A}^T\widehat{A}^{-1} - Q\widehat{A}^{-1}Q\widehat{A}^{-1} \end{bmatrix}. \tag{58}$$

Theorem 10. *Let A be an n × n nonsingular matrix* Φ^{-} *be the matrix in* (27) *and the block matrices* W_{11}^{-}, W_{12}^{-}, W_{21}^{-}, W_{22}^{-} *in* (30) *be nonsingular. If* $\dim(V(\lambda_i(\Phi^{-}))) = 1$ *for every eigenvalue* $\lambda_i(\Phi^{-})$, $i = 1, 2, \dots, 2n$ *of* Φ^{-}, *with* $\left|\lambda_i(\Phi^{-})\right| \neq 1$, *then, there exists a finite number of Hermitian solutions* (*h.n.s*) *of* (2). *The number of Hermitian solutions of* (2) *is equal to*

$$h.n.s = \prod_{j=1}^{m}(n_j + 1)\,, \tag{59}$$

and the number of non-Hermitian solutions (*n.h.n.s*) *of* (2) *is*

$$n.h.n.s = n.s - \prod_{j=1}^{m}(n_j + 1)\,. \tag{60}$$

In particular, if A and Q are real matrices, then among the h.n.s there exist real symmetric solutions (*r.n.s*) *and the number of r.n.s of* (2) *is*

$$r.n.s = \prod_{k=1}^{p+q} (n_k + 1) . \qquad (61)$$

In above, m is the number of the distinct eigenvalues of Φ^-, *that lie outside the unit circle, with algebraic multiplicity* n_j, $j = 1, 2, \ldots, m$, p *is the number of real distinct eigenvalues of* Φ^- *lying outside the unit circle, q is the number of complex conjugate pairs of eigenvalues lying outside the unit circle, with algebraic multiplicity* n_k, $k = 1, 2, \ldots, p+q$, *and n.s is given by* (23).

Proof. It is obvious that, when we substitute in (58) $\hat{A}$ with $-A\mathbf{i}$, arises

$$\hat{\Phi}^+ = \begin{bmatrix} A^{-1}A^T & A^{-1}QA^{-1} \\ QA^{-1}A^T & A^T A^{-1} + QA^{-1}QA^{-1} \end{bmatrix} = \Phi^- , \qquad (62)$$

i.e., the matrix $\hat{\Phi}^+$ is identified with the matrix Φ^- in (27). Hence, it becomes obvious that the solution of (2) can be derived through the solution of equation of type (1) in (57) and combining the conclusions of Theorem 9. In fact, if $\dim(V(\lambda_i(\Phi^-))) = \dim(V(\lambda_i(\hat{\Phi}^+))) = 1$, for all eigenvalues $\lambda_i(\Phi^-)$ of the matrix Φ^- with $|\lambda_i(\Phi^-)| = |\lambda_i(\hat{\Phi}^+)| \neq 1$, $i = 1, 2, \ldots, 2n$, then there exists a finite number of Hermitian solutions of $X - A^T X^{-1} A = Q$, due to the existence of finite number of Hermitian positive definite solutions of $X + \hat{A}^T X^{-1} \hat{A} = Q$.

Clearly, since $\lambda_i(\Phi^-) = \lambda_i(\hat{\Phi}^+)$, $i = 1, 2, \ldots, 2n$, the eigenvalues $\lambda_1(\Phi^-), \lambda_2(\Phi^-), \ldots, \lambda_m(\Phi^-)$ lie outside the unit circle, when $\lambda_1(\hat{\Phi}^+), \lambda_2(\hat{\Phi}^+), \ldots, \lambda_m(\hat{\Phi}^+)$ lie outside the unit circle. Also, any real (complex) eigenvalue $\lambda_i(\hat{\Phi}^+)$ of $\hat{\Phi}^+$ is real (complex) eigenvalue of Φ^-; therefore, applying the conclusions of Theorem 9 for the matrix $\hat{\Phi}^+$, the equalities in (59) and (61) are derived, immediately. Furthermore, due to the assumptions for the block matrices W_{ij}^-, $i, j = 1, 2$, the number of non-Hermitian solutions of (2) in (60) yields combining Theorem 6 with the equation (59). □

Remark 4. Note that in Theorem 10, m, p, q are connected by the relationship $m = p + 2q$.

4. SIMULATION RESULTS

Simulation results are given to illustrate the efficiency of the proposed method. The proposed methods compute accurate solutions as verified through the following simulation examples, using Matlab 6.5.

Example 1. This example concerns the scalar equation $X + A^T X^{-1} A = Q$, with

$$A = 0.25 , \; Q = 1 .$$

Since A is a nonsingular matrix, Riccati equation solution method can be applied. The matrix Φ^+ in (6) has $\sigma(\Phi^+) = \{-0.0718, \; -13.9282\}$, and the corresponding eigenvectors are the columns of the matrix

$$W^+ \equiv W_1^+ = \begin{bmatrix} 0.9978 & 0.7312 \\ 0.0668 & 0.6822 \end{bmatrix}.$$

Consequently, Φ^+ has $m = 1$ eigenvalue outside the unit circle, which is a real eigenvalue $p = 1$, with multiplicity $n_1 = 1$ and no pair of complex eigenvalues ($q = 0$). Since all the hypotheses of Theorem 9 are verified, by (49) and (51) the number of Hermitian positive definite solutions is equal to the number of real symmetric positive definite solutions:

$$h.p.d.n.s = \prod_{j=1}^{m} (n_j + 1) = r.p.d.n.s = \prod_{k=1}^{p+q} (n_k + 1) = 2$$

Furthermore, $n = 1$ and by (23) the method computes $n.s = \dfrac{(2n)!}{(n!)^2} = 2$ solutions of the matrix equation $X + A^T X^{-1} A = Q$ and according to Theorem 9 the number of non-Hermitian solutions is given by (50), i.e., the non-Hermitian solutions are

$$n.h.n.s = n.s - \prod_{j=1}^{m} (n_j + 1) = 2 - 2 = 0 .$$

Thus, there exist only Hermitian-real symmetric solutions, which are the extreme solutions:

$X_1^+ = X_{max}^+ = 0.9330$ (maximal solution)

$X_2^+ = X_{min}^+ = 0.0669$ (minimal solution)

It is also verified that: $0 < X_{min}^+ < X_{max}^+$.

Example 2. This example concerns the equation $X + A^T X^{-1} A = Q$ and is taken from [11]. Consider the equation (1), with

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad Q = \begin{bmatrix} 6 & 5 \\ 5 & 8.6 \end{bmatrix}.$$

Since A is a real and nonsingular matrix with $r(Q^{-1/2} A Q^{-1/2}) = 0.4957$, Riccati equation solution method can be applied. The matrix Φ^+ in (6) has

$$\sigma(\Phi^+) = \{ \lambda_{1,2}(\Phi^+) = -2.0531 \pm 0.8506\mathbf{i}, \; \lambda_{3,4}(\Phi^+) = -0.4157 \pm 0.1722\mathbf{i} \},$$

with $|\lambda_{1,2}(\Phi^+)| = 2.2223$, $|\lambda_{3,4}(\Phi^+)| = 0.45$, and the corresponding eigenvectors of Φ^+ are the columns of the matrix $W^+ \equiv W_2^+$.

$$W^+ \equiv W_2^+ = \begin{bmatrix} -0.2639+0.1316\mathbf{i} & -0.2639-0.1316\mathbf{i} & -0.0008+0.1369\mathbf{i} & -0.0008-0.1369\mathbf{i} \\ 0.3286-0.0727\mathbf{i} & 0.3286+0.0727\mathbf{i} & 0.3275-0.0377\mathbf{i} & 0.3275+0.0377\mathbf{i} \\ -0.2358+0.3364\mathbf{i} & -0.2358-0.3364\mathbf{i} & 0.2453+0.1127\mathbf{i} & 0.2453-0.1127\mathbf{i} \\ 0.7944 & 0.7944 & 0.8943 & 0.8943 \end{bmatrix}.$$

The matrix Φ^+ has $m=2$ eigenvalues outside the unit circle, $p=0$ real eigenvalue, $q=1$ pair of complex eigenvalues outside the unit circle, the algebraic multiplicity of all the eigenvalues is $n_1 = n_2 = 1$. Moreover, the 2 × 2 block matrix W_{ij}^+, $i, j = 1,2$, that arises from every permutation of columns of $W^+ \equiv W_2^+$, is nonsingular. Thus, the hypotheses of Theorem 9 are verified and by (49) we compute $h.p.d.n.s = \prod_{j=1}^{m}(n_j+1) = 4$, and by (51) $r.p.d.n.s = \prod_{k=1}^{p+q}(n_k+1) = 2$.

Furthermore, by (23) of Theorem 3, for $n=2$, the method computes $n.s = \frac{(2n)!}{(n!)^2} = 6$ solutions of the matrix equation $X + A^T X^{-1} A = Q$. According to Theorem 9 the number of non-Hermitian solutions is given by (50) and is equal to $n.h.n.s = n.s - \prod_{j=1}^{m}(n_j+1) = 6-4 = 2$.

All the solutions are:

$$X_1^+ = X_{max}^+ = \begin{bmatrix} 3.8832 & 2.4009 \\ 2.4009 & 4.346 \end{bmatrix} \text{(maximal solution)}$$

$$X_2^+ = \begin{bmatrix} 1.7648-0.9409\mathbf{i} & 0.4002-0.35\mathbf{i} \\ 0.4002-0.35\mathbf{i} & 2.5708+0.1275\mathbf{i} \end{bmatrix}$$

$$X_3^+ = \begin{bmatrix} 2 & 0.8+0.4\mathbf{i} \\ 0.8-0.4\mathbf{i} & 2.9 \end{bmatrix}$$

$$X_4^+ = \begin{bmatrix} 2 & 0.8-0.4\mathbf{i} \\ 0.8+0.4\mathbf{i} & 2.9 \end{bmatrix}$$

$$X_5^+ = \begin{bmatrix} 1.7648+0.9409\mathbf{i} & 0.4002+0.35\mathbf{i} \\ 0.4002+0.35\mathbf{i} & 2.5708-0.1275\mathbf{i} \end{bmatrix}$$

$$X_6^+ = X_{min}^+ = \begin{bmatrix} 1.0301 & 0.7516 \\ 0.7516 & 2.7326 \end{bmatrix} \text{(minimal solution)}$$

It is also verified that: $0 < X_{min}^+ < X_{max}^+$.

Example 3. This example concerns the equation $X + A^T X^{-1} A = Q$ and is taken from [11]. Consider the equation (1), with

$$A = \begin{bmatrix} 0.37 & 0.13 & 0.12 \\ -0.30 & 0.34 & 0.12 \\ 0.11 & -0.17 & 0.29 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.20 & -0.30 & 0.10 \\ -0.30 & 2.10 & 0.20 \\ 0.10 & 0.20 & 0.65 \end{bmatrix}.$$

Since A is a real and nonsingular matrix with $r(Q^{-1/2}AQ^{-1/2}) = 0.4998$, Riccati equation solution method can be applied. The matrix Φ^+ in (6) has

$$\sigma(\Phi^+) = \{\lambda_{1,2}(\Phi^+) = -1.785 \pm 10.0583\mathbf{i},\ \lambda_3(\Phi^+) = -1.0673,$$
$$\lambda_4(\Phi^+) = -0.937,\quad \lambda_{5,6}(\Phi^+) = -0.0171 \pm 0.0964\mathbf{i}\},$$

with $|\lambda_{1,2}(\Phi^+)| = 10.2155$, $|\lambda_3(\Phi^+)| = 1.0673$, $|\lambda_4(\Phi^+)| =$ 0.937, and $|\lambda_{5,6}(\Phi^+)| = 0.0979$. The corresponding eigenvectors of Φ^+ are the columns of the matrix $W^+ \equiv W_3^+$.

The matrix Φ^+ has $m=3$ eigenvalues outside the unit circle, from that, $p=1$ real eigenvalue, $q=1$ pair of complex eigenvalues outside the unit circle, the algebraic multiplicity of all the eigenvalues is equal to 1. Moreover, the 3 × 3 block matrix W_{ij}^+, $i, j = 1,2$, that arises from every permutation of columns of $W^+ \equiv W_3^+$, is nonsingular. Thus, the hypotheses of Theorem 9 are verified and we compute

$$h.p.d.n.s = \prod_{j=1}^{m}(n_j+1) = 8 \quad \text{and} \quad r.p.d.n.s = \prod_{k=1}^{p+q}(n_k+1) = 4$$

by (49) and (51), respectively. The real symmetric positive definite solutions are X_1^+, (maximal solution), X_2^+, X_{19}^+, X_{20}^+, (minimal solution), in particular:

$$X_1^+ = X_{max}^+ = \begin{bmatrix} 0.9463 & -0.1987 & -0.0596 \\ -0.1987 & 1.8674 & 0.3252 \\ -0.0596 & 0.3252 & 0.4158 \end{bmatrix}$$

$$W^+ \equiv W_3^+ = \begin{bmatrix} -0.0722-0.3856\mathbf{i} & -0.0722+0.3856\mathbf{i} & -0.2126 & 0.2188 & -0.6830 & -0.6830 \\ 0.2719-0.0995\mathbf{i} & 0.2719+0.0995\mathbf{i} & 0.1348 & -0.1427 & -0.2545-0.4975\mathbf{i} & -0.2545+0.4975\mathbf{i} \\ 0.2638+0.3360\mathbf{i} & 0.2638-0.3360\mathbf{i} & -0.8977 & 0.9024 & 0.4519+0.0638\mathbf{i} & 0.4519-0.0638\mathbf{i} \\ -0.1380-0.3652\mathbf{i} & -0.1380+0.3652\mathbf{i} & -0.1744 & 0.1652 & -0.0670+0.0329\mathbf{i} & -0.0670-0.0329\mathbf{i} \\ 0.6078 & 0.6078 & 0.0019 & -0.0057 & 0.0089-0.0734\mathbf{i} & 0.0089+0.0734\mathbf{i} \\ 0.2024+0.1303\mathbf{i} & 0.2024-0.1303\mathbf{i} & -0.3168 & 0.3001 & 0.0445+0.0048\mathbf{i} & 0.0445-0.0048\mathbf{i} \end{bmatrix}$$

$$X_2^{+} = \begin{bmatrix} 0.9323 & -0.1894 & -0.073 \\ -0.1894 & 1.8613 & 0.334 \\ -0.073 & 0.334 & 0.4031 \end{bmatrix}$$

$$X_{19}^{+} = \begin{bmatrix} 0.2074 & -0.0484 & 0.138 \\ -0.0484 & 0.1517 & 0.0321 \\ 0.138 & 0.0321 & 0.325 \end{bmatrix}$$

$$X_{20}^{+} = X_{min}^{+} = \begin{bmatrix} 0.2004 & -0.0498 & 0.1266 \\ -0.0498 & 0.1514 & 0.0297 \\ 0.1266 & 0.0297 & 0.3066 \end{bmatrix}$$

It is verified that: $\mathbf{0} < X_{min}^{+} \leq X^{+} \leq X_{max}^{+}$ for every Hermitian positive definite solution X^{+}, since

$$\mathbf{0} < X_{min}^{+} \leq X_2^{+},\ X_{19}^{+} \leq X_{max}^{+}.$$

Furthermore, by (23) of Theorem 3, for $n = 3$, the method computes $n.s = \dfrac{(2n)!}{(n!)^2} = 20$ solutions of the matrix equation $X + A^T X^{-1} A = Q$. According to Theorem 9 the number of non-Hermitian solutions is given by (50) and is equal to $n.h.n.s = n.s - \prod_{j=1}^{m}(n_j + 1) = 20 - 8 = 12$.

Example 4. This example concerns the scalar equation $X - A^T X^{-1} A = Q$, with

$$A = 2, \quad Q = 1.$$

Since A is a nonsingular matrix, Riccati equation solution method can be applied. The matrix Φ^- in (27) has $\sigma(\Phi^-) = \{0.6096,\ 1.6404\}$, and the corresponding eigenvectors are the columns of the matrix

$$W^- \equiv W_4^- = \begin{bmatrix} -0.5393 & -0.3637 \\ 0.8421 & -0.9315 \end{bmatrix}.$$

Consequently, Φ^- has $m = 1$ eigenvalue outside the unit circle, which is a real eigenvalue $p = 1$, with multiplicity $n_1 = 1$ and $q = 0$. Since all the hypotheses of Theorem 10 are verified, by (59) and (61) the number of Hermitian solutions and especially of real solutions is equal to

$$h.n.s = \prod_{j=1}^{m}(n_j + 1) = 2, \quad r.n.s = \prod_{k=1}^{p+q}(n_k + 1) = 2,$$

respectively. Furthermore, $n = 1$ and by Theorem 6, the method computes $n.s = \dfrac{(2n)!}{(n!)^2} = 2$ solutions of the matrix equation $X - A^T X^{-1} A = Q$. According to Theorem 10 the number of non-Hermitian solutions is given by (60) and is equal to

$$n.h.n.s = n.s - \prod_{j=1}^{m}(n_j + 1) = 2 - 2 = 0.$$

Thus, there exist only Hermitian-real symmetric solutions, which are the extreme solutions:

$X_1^- = X_{max}^- = 2.5616$ (maximal solution)

$X_2^- = X_{min}^- = -1.5616$ (minimal solution)

Example 5. This example concerns the equation $X - A^T X^{-1} A = Q$ and is taken from [6] and [11]. Consider the equation (2), with

$$A = \begin{bmatrix} 50 & 20 \\ 10 & 60 \end{bmatrix}, \quad Q = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}.$$

Since A is a real and nonsingular matrix, Riccati equation solution method can be applied. The matrix Φ^- in (27) has

$$\sigma(\Phi^-) = \{\lambda_{1,2}(\Phi^-) = 1.0404 \pm 0.198\mathbf{i},\ \lambda_{3,4}(\Phi^-) = 0.9276 \pm 0.1766\mathbf{i}\},$$

with $\left|\lambda_{1,2}(\Phi^-)\right| = 1.0591$, $\left|\lambda_{3,4}(\Phi^-)\right| = 0.9442$, and the corresponding eigenvectors of Φ^- are the columns of the matrix $W^- \equiv W_5^-$.

The matrix Φ^- has $m = 2$ eigenvalues outside the unit circle, with $n_1 = n_2 = 1$, where $p = 0$ and $q = 1$, and the 2×2 block matrices W_{ij}^-, $i, j = 1, 2$, that arise from every permutation of columns of $W^- \equiv W_5^-$, are nonsingular. Thus, the hypotheses of Theorem 10 are verified and we compute

$$h.n.s = \prod_{j=1}^{m}(n_j + 1) = 4, \quad r.n.s = \prod_{k=1}^{p+q}(n_k + 1) = 2,$$

by (59) and (61), respectively. Furthermore, $n = 2$ and by Theorem 6, the method computes $n.s = \dfrac{(2n)!}{(n!)^2} = 6$ solutions of the matrix equation $X - A^T X^{-1} A = Q$. According to

$$W^- \equiv W_5^- = \begin{bmatrix} -0.0005 + 0.0128\mathbf{i} & -0.0005 - 0.0128\mathbf{i} & -0.0147 - 0.0036\mathbf{i} & -0.0147 + 0.0036\mathbf{i} \\ 0.0125 - 0.0033\mathbf{i} & 0.0125 + 0.0033\mathbf{i} & 0.0003 + 0.0124\mathbf{i} & 0.0003 - 0.0124\mathbf{i} \\ 0.1760 + 0.6091\mathbf{i} & 0.1760 - 0.6091\mathbf{i} & 0.7127 & 0.7127 \\ 0.7731 & 0.7731 & 0.1912 - 0.6746\mathbf{i} & 0.1912 + 0.6746\mathbf{i} \end{bmatrix}.$$

Theorem 10 the number of non-Hermitian solutions is given by (60) and is equal to $n.h.n.s = n.s - \prod_{j=1}^{m}(n_j + 1) = 6 - 4 = 2$.

All the solutions are:

$$X_1^- = X_{max}^- = \begin{bmatrix} 51.7994 & 16.0999 \\ 16.0999 & 62.2516 \end{bmatrix} \text{ (maximal solution)}$$

$$X_2^- = \begin{bmatrix} -447.7701 + 50.2254\mathbf{i} & -80.6855 + 485.5228\mathbf{i} \\ -80.6855 + 485.5228\mathbf{i} & 492.1128 + 231.345\mathbf{i} \end{bmatrix}$$

$$X_3^- = \begin{bmatrix} -4 & 52.6118\mathbf{i} \\ -52.6118\mathbf{i} & 8 \end{bmatrix}$$

$$X_4^- = \begin{bmatrix} -4 & -52.6118\mathbf{i} \\ 52.6118\mathbf{i} & 8 \end{bmatrix}$$

$$X_5^- = \begin{bmatrix} -447.7701 - 50.2254\mathbf{i} & -80.6855 - 485.5228\mathbf{i} \\ -80.6855 - 485.5228\mathbf{i} & 492.1128 - 231.345\mathbf{i} \end{bmatrix}$$

$$X_6^- = X_{min}^- = \begin{bmatrix} -48.7004 & -14.0819 \\ -14.0819 & -58.3596 \end{bmatrix} \text{ (minimal}$$

solution)

Example 6. This example concerns the equation $X - A^T X^{-1} A = Q$, with

$$A = \begin{bmatrix} 0.37 & 0.13 & 0.12 \\ -0.30 & 0.34 & 0.12 \\ 0.11 & -0.17 & 0.29 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.20 & -0.30 & 0.10 \\ -0.30 & 2.10 & 0.20 \\ 0.10 & 0.20 & 0.65 \end{bmatrix}.$$

This example has the same matrices A, Q as Example 3, thus A is nonsingular, hence the Riccati equation solution method can be applied. The matrix Φ^- in (27) has

$$\sigma(\Phi^-) = \{\lambda_{1,2}(\Phi^-) = 1.7421 \pm 14.3591\mathbf{i},\ \lambda_3(\Phi^-) = 6.1644,$$
$$\lambda_4(\Phi^-) = 0.1622,\quad \lambda_{5,6}(\Phi^-) = 0.0083 \pm 0.0686\mathbf{i}\},$$

with

$$\left|\lambda_{1,2}(\Phi^-)\right| = 14.4644,\ \left|\lambda_3(\Phi^-)\right| = 6.1644,\ \left|\lambda_4(\Phi^-)\right| = 0.1622$$

and $\left|\lambda_{5,6}(\Phi^-)\right| = 0.0691$.

The corresponding eigenvectors of Φ^- are the columns of the matrix $W^- \equiv W_6^-$.

The matrix Φ^- has $m = 3$ eigenvalues outside the unit circle, with $n_1 = n_2 = n_3 = 1$, from that, there exists $p = 1$ real eigenvalue and $q = 1$ pair of complex eigenvalues outside the unit circle. Moreover, the 3×3 block matrix W_{ij}^-, $i, j = 1, 2$, that arises from every permutation of columns of $W^- \equiv W_6^-$, is nonsingular. Thus, the hypotheses of Theorem 10 are verified and we compute

$$h.n.s = \prod_{j=1}^{m}(n_j + 1) = 8 \quad \text{and} \quad r.n.s = \prod_{k=1}^{p+q}(n_k + 1) = 4$$

by (59) and (61), respectively. The real symmetric solutions are X_1^-, (maximal solution), X_2^-, X_{19}^-, X_{20}^-, (minimal solution), in particular:

$$X_1^- = X_{max}^- = \begin{bmatrix} 1.3334 & -0.326 & 0.1502 \\ -0.326 & 2.2418 & 0.1589 \\ 0.1502 & 0.1589 & 0.7657 \end{bmatrix}$$

$$X_2^- = \begin{bmatrix} 0.6739 & -0.0012 & -0.4686 \\ -0.0012 & 2.0818 & 0.4637 \\ -0.4686 & 0.4637 & 0.185 \end{bmatrix}$$

$$X_{19}^- = \begin{bmatrix} 0.2742 & 0.0599 & 0.5212 \\ 0.0599 & -0.1192 & 0.0128 \\ 0.5212 & 0.0128 & 0.6357 \end{bmatrix}$$

$$X_{20}^- = X_{min}^- = \begin{bmatrix} -0.1311 & 0.0464 & -0.0378 \\ 0.0464 & -0.1196 & -0.0059 \\ -0.0378 & -0.0059 & -0.1353 \end{bmatrix}$$

Furthermore, $n = 3$ and by Theorem 6, the method computes $n.s = \frac{(2n)!}{(n!)^2} = 20$ solutions of the matrix equation $X - A^T X^{-1} A = Q$. According to Theorem 10 the number of non-Hermitian solutions is given by (60) and is equal to

$$n.h.n.s = n.s - \prod_{j=1}^{m}(n_j + 1) = 20 - 8 = 12.$$

CONCLUSIONS

A method is developed for checking the existence of a finite number of solutions of the matrix equations

$$W^- \equiv W_6^- = \begin{bmatrix} -0.0454 + 0.3372\mathbf{i} & -0.0454 - 0.3372\mathbf{i} & -0.2365 & -0.4498 & 0.6467 & 0.6467 \\ 0.2748 + 0.0718\mathbf{i} & 0.2748 - 0.0718\mathbf{i} & 0.0033 & 0.1736 & 0.2691 - 0.5236\mathbf{i} & 0.2691 + 0.5236\mathbf{i} \\ 0.1926 - 0.3217\mathbf{i} & 0.1926 + 0.3217\mathbf{i} & -0.6787 & -0.8596 & -0.4754 + 0.0127\mathbf{i} & -0.4754 - 0.0127\mathbf{i} \\ -0.1212 + 0.3779\mathbf{i} & -0.1212 - 0.3779\mathbf{i} & -0.4184 & 0.0995 & -0.0544 - 0.0248\mathbf{i} & -0.0544 + 0.0248\mathbf{i} \\ 0.6615 & 0.6615 & -0.0232 & -0.0366 & 0.0006 + 0.0626\mathbf{i} & 0.0006 - 0.0626\mathbf{i} \\ 0.1843 - 0.1842\mathbf{i} & 0.1843 + 0.1842\mathbf{i} & -0.5547 & 0.1322 & 0.0383 + 0.0014\mathbf{i} & -0.0383 - 0.0014\mathbf{i} \end{bmatrix}.$$

$X + A^T X^{-1} A = Q$ and $X - A^T X^{-1} A = Q$, when A is nonsingular. An algebraic method for computing of all the solutions of these matrix equations is proposed, namely the Riccati equation solution method, which is based on the algebraic solution of the corresponding discrete time Riccati equations. The method provides simple formulas for computing the accurate solutions of these matrix equations, as verified through simulation experiments. The method computes the extreme solutions as well. The number of solutions is also derived.

The existence and the computation *only* of the extreme solutions (Hermitian) of the matrix equations (1) and (2) have been studied in the bibliography (see [3, 4, 7, 10]). In the present paper, the necessary conditions for the existence of all the solutions (Hermitian ones as well as non-Hermitian ones) of these equations are presented, the number of solutions is computed and formulas for computing all these solutions are given.

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