On the order of the recursion relation of Motzkin numbers of higher rank

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Abstract. For Motzkin paths with up- and down-steps of heights 1 and 2, the minimal recursion is of order 6, not of order 4, as conjectured by Schork.

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The classical Motzkin numbers count the numbers of Motzkin paths: We consider in the Cartesian plane $\mathbb{Z}\times\mathbb{Z}$ those lattice paths starting at $(0,0)$ that use an up-step $(1,1)$, a down-step $(1,-1)$, and a level-step $(1,0)$. Motzkin paths of length n are built of these, lead to $(n,0)$ and never go below the x-axis.

Now we consider *higher rank Motzkin numbers*, as suggested by Schork [2]: There are up-steps $(1,1)$, $(1,2)$, ..., $(1,r)$ with respective weights $a_1,\dots,a_r$, down-steps $(1,-1)$, $(1,-2)$, ..., $(1,-r)$ with respective weights $c_1,\dots,c_r$, and a level-step $(1,0)$ with weight b.

Let us first consider the classical case $r=1$. The generating function $M(z)$ of these paths satisfies the equation

$$M = 1 + bzM + azMczM,$$

whence

$$\frac{1-bz-\sqrt{1-2bz+b^2z^2-4az^2c}}{2az^2c}.$$

This equation is obtained by a *decomposition* of the Motzkin paths with respect to the first return to the x-axis.

Schork's first problem is to find a recursion for the numbers $m_n=[z^n]M(z)$. (The coefficient of z^n in the power series expansion of $M(z)$, i.e., the number of (weighted) Motzkin paths of length n.)

This can be automatically solved with Maple's program gfun (written by Salvy et al.): The procedure algeqtodiffeq translates the (algebraic) equation for $M(z)$ into an equivalent differential equation:

$$2+(3bz-b^2z^2+4az^2c-2)M \\ +(-z+2bz^2-z^3b^2+4z^3ac)M'=0.$$

The procedure diffeqtorec translates the differential equation into a recursion:

$$(-b^2+4ac)(n+1)m_n \\ +(5b+2bn)m_{n+1}-(n+4)m_{n+2}=0,$$

which solves already this first problem.[1]

Now let us move to the instance $r=2$. Let us assume that the weights are all 1, so that we are just interested to count the number of (generalized) Motzkin paths. In the paper [1] we find the equation for the generating function:

$$z^4M^4-z^2(1+z)M^3 \\ +z(2+z)M^2-(1+z)M+1=0.$$

Thus (again with gfun)

$$-4-100z^2+56z \\ +(3750z^6-5000z^5+250z^4 \\ +700z^3+160z^2-92z+4)M \\ +(-328z^2+32z-15250z^6-20z^3 \\ +4750z^4+11250z^7-650z^5)M'$$

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[1]After sending a draft of this note to M. Schork, he informed me that he could now also establish this recurrence together with Mansour and Sun.

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$$+ (5625z^8 - 7750z^7 - 1200z^6 + 3880z^5 - 395z^4 - 186z^3 + 26z^2)M'' + (625z^9 - 875z^8 - 250z^7 + 610z^6 - 91z^5 - 23z^4 + 4z^3)M''' = 0$$

and

$$625(n+3)(n+2)(n+1)m_n - 125(n+3)(n+2)(7n+27)m_{n+1} - 50(n+3)(5n^2+24n+23)m_{n+2} + (41890 + 30860n + 7540n^2 + 610n^3)m_{n+3} + (-6844 - 5151n - 1214n^2 - 91n^3)m_{n+4} - (n+7)(23n^2+301n+976)m_{n+5} + 2(2n+13)(n+8)(n+7)m_{n+6} = 0.$$

(This recursion also appears in [1].)

Bruno Salvy has kindly informed me that this recursion of order 6 is *minimal.*

Schork [2] conjectured that there should be a $(2r+1)$-term recursion (=order $2r$). Thus, the conjecture does not hold.

References

[1] C. Banderier and P. Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoret. Comput. Sci.*, 281(1-2):37–80, 2002. Selected papers in honour of Maurice Nivat.

[2] M. Schork. On the recursion relation of Motzkin numbers of higher rank. *Online Journal of Analytic Combinatorics*, 2:#3, 2007.

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