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An Improved Method of Undetermined Coefficients

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Abstract: An exact solution method in terms of an infinite power series is developed for linear ordinary differential equations with polynomial coefficients. The method is general and applicable to a wide range of equations of any *N-th* order presented in normal form. The final solution is defined by a linear combination of *S* functions $f_j(x)$ $j=1,...,S$ expressed in the form of a power series, and by additional $(S\text{-}N)$ number of accompanied relationships for unknown constants. Each term of the series $f_j(x)$ is defined by a finite number of operations involving matrix calculations. The term calculation is independent on unknown constants.

The method is also applicable to any system of *R* differential equations in normal form with the restriction that all equations in the system are of the same order. The advantage over the "classical" method of undetermined coefficients is that here the recursions are of first order even for higher-order differential equations. The general solution is expressed explicitly in the form allowing for application of any form of initial, boundary, or combined conditions. The paper presents the development of the method and, as examples, its application to solving selected second- and third-order differential equations.

Keywords: Hermite equation, method of undetermined coefficients, polynomial coefficients, power series solutions.

1. INTRODUCTION

We consider a homogenous system of *R* – linear ordinary differential equations (ODE), each of *N-th* order, written in normal matrix form [2, 5]

$$\boldsymbol{Y}^{(N)} + \boldsymbol{w}_{N-1}(x)\boldsymbol{Y}^{(N-1)} + ... + \boldsymbol{w}_1(x)\boldsymbol{Y}^{I} + \boldsymbol{w}_0(x)\boldsymbol{Y} = 0. \tag{1.1}$$

Bold letters indicate square matrices (lower case) and column vectors (capital letters).

$\boldsymbol{Y} = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_R(x) \end{bmatrix}$ - is the column vector of dependent variables,

and $\boldsymbol{Y}^{(K)} = \begin{bmatrix} y_1^{(K)}(x) \\ y_2^{(K)}(x) \\ \vdots \\ y_R^{(K)}(x) \end{bmatrix}$ - is the vector of their *K-th* order ($K=0,1,2,...,N$) derivatives.

We consider the cases where the nonconstant coefficients are polynomials in the form

$$\boldsymbol{w}_K(x) = \boldsymbol{w}_{K,m_K} \cdot x^{m_K} + \boldsymbol{w}_{K,m_K-1} \cdot x^{m_K-1} + ... + \boldsymbol{w}_{K,1} \cdot x + \boldsymbol{w}_{K,0} \tag{1.2}$$

where $\boldsymbol{w}_{K,m_K-i}$, $i = 0,1,...,m_K$ – are the $R\times R$ matrices of constant coefficients and m_K is the degree of the polynomial representing the matrix $\boldsymbol{w}_K$. It is assumed that $\boldsymbol{w}_0(x) \neq \boldsymbol{0}$.

We are looking for analytic solutions of the system (1.1). The general solution of (1.1) is expressed in the form of a power series, and additionally, the matrix calculation is applied for the method of undetermined coefficients [1].

2. IMPROVED METHOD OF UNDETERMINED COEFFICIENTS

Based on the existence theorem for equations with analytic coefficients [2], the solution of the system (1.1) can be sought in the form of an infinite power series, convergent for all real *x*

$$\boldsymbol{Y}(x) = \sum_{j=0}^{\infty} \boldsymbol{A}_j x^j. \tag{2.1}$$

The symbol $\boldsymbol{A}_j$ indicates $R\times 1$ column vectors of the constant series coefficients. The vector made up of *K-th* order derivatives can be expressed in the form

$$\boldsymbol{Y}^{(K)}(x) = \sum_{j=0}^{\infty} \frac{(j+K)!}{j!} \boldsymbol{A}_{j+K} x^j. \tag{2.2}$$

The determination of the vector $\boldsymbol{Y}(x)$ is based on an algorithm consisting of a finite number of matrix operations used for computing any vector $\boldsymbol{A}_j$ of constants in the series (2.1) and (2.2). The algorithm developed below is derived without any approximate assumptions, and in this way, the presented method is formally exact. Any vector of unknown constants (with a sufficiently large index) is dependent on the fixed finite number of initial constants (dependent on boundary or initial conditions).

Applying the method of undetermined coefficients, we substitute series (2.1) and (2.2) into the system (1.1), differentiate term by term, and equate to zero the coefficients

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(vectors) of each power x^j of x. In this way, we obtain successive equations of the form

$$\text{for } j \geq S \qquad \boldsymbol{A}_j = \sum_{k=1}^{S} \boldsymbol{a}_k(j)\boldsymbol{A}_{j-k}. \tag{2.3}$$

where $R\times R$ matrices $\boldsymbol{a}_k(j)$ are dependent on the matrices of the polynomial coefficients w_{K,m_K-i} defined by (1.2) and the index j. S indicates the number of matrices $\boldsymbol{a}_k(j)$ presented in the equation (2.3). This number depends on the order N of the differential equations (1.1) and on the degrees m_k of the polynomials $\boldsymbol{w}_K(x)$

$$S = N\text{-}\min\{K\text{-}m_K\}, \text{ for } K=0,1,\ldots,N\text{-}1. \tag{2.4}$$

If (as has been assumed earlier) $\boldsymbol{w}_0\neq 0$, then $S \geq N$. For the cases where $S > N$ (see the examples) in addition to the equation (2.3), we have $S\text{-}N$ equations obtained from equating to zero the coefficients of x^j for $j=0,\ldots,S\text{-}N$. Usually, these equations can be obtained from (2.3) by substituting a fixed value for j. If $S=N$, then all the recurrence relationships are expressed by (2.3). To find a relationship between any vector $\boldsymbol{A}_j$ and the finite number of initial constants, we group all the vectors $\boldsymbol{A}_j$ with R elements into larger vectors $\boldsymbol{B}_j$ with $R\cdot S$ elements according to the following formula:

$$\boldsymbol{B}_{nS} = \begin{bmatrix} \boldsymbol{A}_{nS} \\ \boldsymbol{A}_{nS+1} \\ \vdots \\ \boldsymbol{A}_{nS+S-1} \end{bmatrix}, \text{ where } n=0,1,2,\ldots \tag{2.5}$$

and nS means the product of n and S.

The vector for $n=0$

$$\mathrm{B}_0 = \begin{bmatrix} \mathrm{A}_0 \\ \mathrm{A}_1 \\ \vdots \\ \mathrm{A}_{S-1} \end{bmatrix} \tag{2.6}$$

is the vector of initial constants. Substituting (2.3) for the vectors $\boldsymbol{A}_{nS+i}$, $i=1,\ldots,S\text{-}1$ grouped in $\boldsymbol{B}_{nS}$, and using each time the relationships defined earlier, we are able to depend all these vectors on vectors $\boldsymbol{A}_{(n-1)S+i}$ grouped in $\boldsymbol{B}_{(n-1)S}$. Due to the form of the equation (2.3), the relationship between the succeeding vectors $\boldsymbol{B}_{nS}$ and $\boldsymbol{B}_{(n-1)S}$ is linear:

$$\boldsymbol{B}_{nS} = \boldsymbol{b}(nS)\,\boldsymbol{B}_{(n-1)S},\ n\geq 1. \tag{2.7}$$

The $(R\cdot S)\times(R\cdot S)$ matrix $\boldsymbol{b}(nS)$ can be presented as the matrix consisting of S^2 $R\times R$ submatrices $\boldsymbol{b}_{ij}(nS)$ placed in S rows and S columns

$$\boldsymbol{b}(nS) = \begin{bmatrix} \boldsymbol{b}_{11}(nS) & \boldsymbol{b}_{12}(nS) & \cdots & \boldsymbol{b}_{1S}(nS) \\ \boldsymbol{b}_{21}(nS) & \boldsymbol{b}_{22}(nS) & \cdots & \boldsymbol{b}_{2S}(nS) \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{b}_{S1}(nS) & \boldsymbol{b}_{S2}(nS) & \cdots & \boldsymbol{b}_{SS}(nS) \end{bmatrix}. \tag{2.8}$$

Submatrices $\boldsymbol{b}_{ij}(nS)$ are defined using the following relationships

$$\boldsymbol{b}_{1j}(nS) = \boldsymbol{a}_{S+1-j}(nS),\ j=1,\ldots,S$$

and for $i = 2,\ldots,S$ (2.9)

$$\boldsymbol{b}_{ij}(nS) = \sum_{k=1}^{i-1} \boldsymbol{a}_k(nS+i\text{-}1)\boldsymbol{b}_{(i-k),j}(nS) + \boldsymbol{a}_{(S+i-j)}(nS+i\text{-}1).$$

Matrices $\boldsymbol{a}_k$ for $k = 1,\ldots,S$ are expressed by (2.3), but for $k = S+1, \ldots, 2S\text{-}1$, zero matrices $\boldsymbol{a}_k = \boldsymbol{0}$ are used. It should be noted that, on the right-hand side of the equations (2.9), each time there are terms that were defined earlier. The equation (2.7) expresses the one-term recurrence relationship, which will be used to determine any vector $\boldsymbol{B}_{nS}$ as dependent on the initial vector $\boldsymbol{B}_0$. Making substitutions one after the other according to (2.7), we obtain the equation

$$n\geq 1\ \boldsymbol{B}_{nS} = \boldsymbol{b}(nS)\boldsymbol{b}((n\text{-}1)S)\ldots\boldsymbol{b}(S)\boldsymbol{B}_0. \tag{2.10}$$

Introducing a new matrix given as the product of the following matrices

$$\boldsymbol{c}(nS) = \boldsymbol{b}(nS)\boldsymbol{b}((n\text{-}1)S)\ldots\boldsymbol{b}(S), \tag{2.11}$$

(2.10) can now be written shortly

$$n\geq 1\ \boldsymbol{B}_{nS} = \boldsymbol{c}(nS)\boldsymbol{B}_0. \tag{2.12}$$

Applying the grouping of vectors $\boldsymbol{A}_j$, defined by (2.5) and (2.6), we can express the vector of the dependent variables as follows:

$$Y(x) = \left[x^0, x^1, \ldots, x^{S-1}\right]\cdot \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{S-1} \end{bmatrix} + \ldots + \left[x^{nS}, x^{nS+1}, \ldots, x^{nS+S-1}\right]\cdot \begin{bmatrix} A_{nS} \\ A_{nS+1} \\ \vdots \\ A_{nS+S-1} \end{bmatrix} + \ldots \tag{2.13}$$

If we substitute the relationship (2.12) into (2.13), we obtain

$$Y(x) = \left[x^0, x^1, \ldots, x^{S-1}\right]B_0 + \ldots + \left[x^{nS}, x^{nS+1}, \ldots, x^{nS+S-1}\right]c(nS)B_0 + \ldots \tag{2.14}$$

or

$$\boldsymbol{Y}(x) = \left[f_0, f_1, \ldots, f_{S-1}\right]\boldsymbol{B}_0. \tag{2.15}$$

$\boldsymbol{f}_j(x)$ denotes the $R\times R$ matrices of the functions defined as

$$\boldsymbol{f}_j(x) = \boldsymbol{i}x^j + \sum_{n=1}^{\infty}\left[\sum_{k=1}^{S} \boldsymbol{c}_{k,(j+1)}(nS)\,x^{nS+k-1}\right], \text{ for } j=0,\ldots,S\text{-}1 \tag{2.16}$$

in which $\boldsymbol{i}$ – is the $R\times R$ identity matrix.

The $K\text{-}th$ order derivatives of the functions $\boldsymbol{f}_j$ can be easily calculated by differentiating (2.16) term by term

$$f^{(K)}{}_j(x) = j\cdot(j-1)\cdot\ldots\cdot(j-K+1)\cdot \mathrm{i}x^{j-K} +$$

$$+\sum_{n=1}^{\infty}\left[\sum_{k=1}^{S}(nS+k-1)\cdot(nS+k-2)\cdot\ldots\cdot(nS+k-K)\mathrm{c}_{k,(j+1)}(nS)x^{nS+k-1-K}\right]$$

$$\text{for } j=0,\ldots,S\text{-}1 \tag{2.17}$$

In the two previous formulae, $\boldsymbol{c}_{ij}(nS)$ denotes submatrices defined by the distribution analogous to (2.8). According to (2.16), each function matrix $\boldsymbol{f}_j$ is expressed as the power series beginning with a different power of x. This means [2] that these functions are linearly independent, and that the vector of sought-after functions $\boldsymbol{Y}(x)$ defined by (2.15) is the general solution of the differential equations (1.1), together with S-N (if positive) number of accompanying algebraic equations defined by (2.3) for $j=0,...,S-N$.

3. EXAMPLE - SECOND ORDER DIFFERENTIAL EQUATION

In the literature, [5] the series solutions are mostly applicable to homogenous linear ordinary differential equations. To illustrate the application of the developed algorithm, we consider the following second-order differential equation:

$$y^{II}+(w_{12}\cdot x^2+w_{11}\cdot x+w_{10})y^I+(w_{02}\cdot x^2+w_{01}\cdot x+w_{00})y=0. \qquad (3.1)$$

First we have to define the recursion formula analogues to (2.3) involving S terms on the right side. According to (2.4), we expect to get

$$S = 2-\min\{(1-2),(0-2)\} = 4, \text{ (for } N=2, K=0, \text{ and } m_K=2). \qquad (3.2)$$

Substituting (2.1) and (2.2) into (3.1) and rearranging the term numbering in every series, we obtain

$$\begin{aligned}&\sum_{j=0}^{\infty}(j+2)(j+1)\mathbf{A}_{j+2}x^j + w_{12}\sum_{j=2}^{\infty}(j-1)\mathbf{A}_{j-1}x^j + \\ &+ w_{11}\sum_{j=1}^{\infty} j\mathbf{A}_j x^j + w_{10}\sum_{j=0}^{\infty}(j+1)\mathbf{A}_{j+1}x^j + \\ &+ w_{02}\sum_{j=2}^{\infty}\mathbf{A}_{j-2}x^j + w_{01}\sum_{j=1}^{\infty}\mathbf{A}_{j-1}x^j + w_{00}\sum_{j=0}^{\infty}\mathbf{A}_j x^j = 0.\end{aligned} \qquad (3.3)$$

Grouping the coefficients corresponding to the same powers of x and equating to zero produces

$$\begin{aligned}&\text{for } j=0 \quad 2A_2 + w_{10}A_1 + w_{00}A_0 = 0\\ &\text{for } j=1 \quad 6A_3 + 2w_{10}A_2 + (w_{11}+w_{00})A_1 + w_{01}A_0 = 0\\ &\text{for } j\geq 2 \ (j+2)(j+1)A_{j+2} + w_{10}(j+1)A_{j+1} + (w_{11}\cdot j + w_0)A_j + (w_{12}\cdot(j-1) + w_{01})A_{j-1} + w_{02}\cdot A_{j-2} = 0.\end{aligned} \qquad (3.4)$$

Substituting $m=j+2$ and rewriting the recursion formula (3.4) in the form (2.3), we find

$$\text{for } m\geq 4 \qquad A_m = \sum_{k=1}^{S=4} a_k(m)A_{m-k}, \qquad (3.5)$$

where

$$a_1(m) = \frac{-(m-1)w_{10}}{m(m-1)}, \quad a_2(m) = \frac{-((m-2)w_{11}+w_{00})}{m(m-1)},$$

$$a_3(m) = \frac{-((m-3)w_{12}+w_{01})}{m(m-1)}, \quad a_4(m) = \frac{-w_{02}}{m(m-1)}. \qquad (3.6)$$

The next step is to create matrices $\boldsymbol{b(4n)}$ and $\boldsymbol{c(4n)}$ following the formulae (2.9) and (2.11). The entire process can be easily programmed, and approximations of functions $f_j(x)$ (2.16) can be calculated for finite n terms after numerical values are substituted for coefficients w_{ij} in (3.6). Using the first two equations (3.4) for relating A_3 and A_2 to A_1 and A_0, we can express the general solution of (3.1) as follows:

$$y(x) = A_0\cdot F_0(x) + A_1\cdot F_1(x), \qquad (3.7)$$

where

$$\begin{aligned}F_0(x) &= f_0(x) - \frac{w_{00}}{2}f_2(x) + \frac{w_{10}w_{00}-w_{01}}{6}f_3(x),\\ F_1(x) &= f_1(x) - \frac{w_{10}}{2}f_2(x) + \frac{w_{10}{}^2-w_{11}-w_{00}}{6}f_3(x).\end{aligned} \qquad (3.8)$$

Functions $f_j(x)$, $j=0,1,2,3$ are defined by (2.16) with $S=4$. Figs. (**1** and **2**) show diagrams of functions $f_j(x)$ and $F_i(x)$ for two particular equations (3.1)

$$y^{II} + (x^2\pm 2x+1)y^I + (x^2\pm 2x+1)y = 0. \qquad (3.9)$$

Each time, a different sign for the coefficients w_{11} and w_{01} is taken. The presented diagrams were obtained using the MathCAD program [3]. The first 100 terms ($n_{max}=100$) for series (2.16) were calculated.

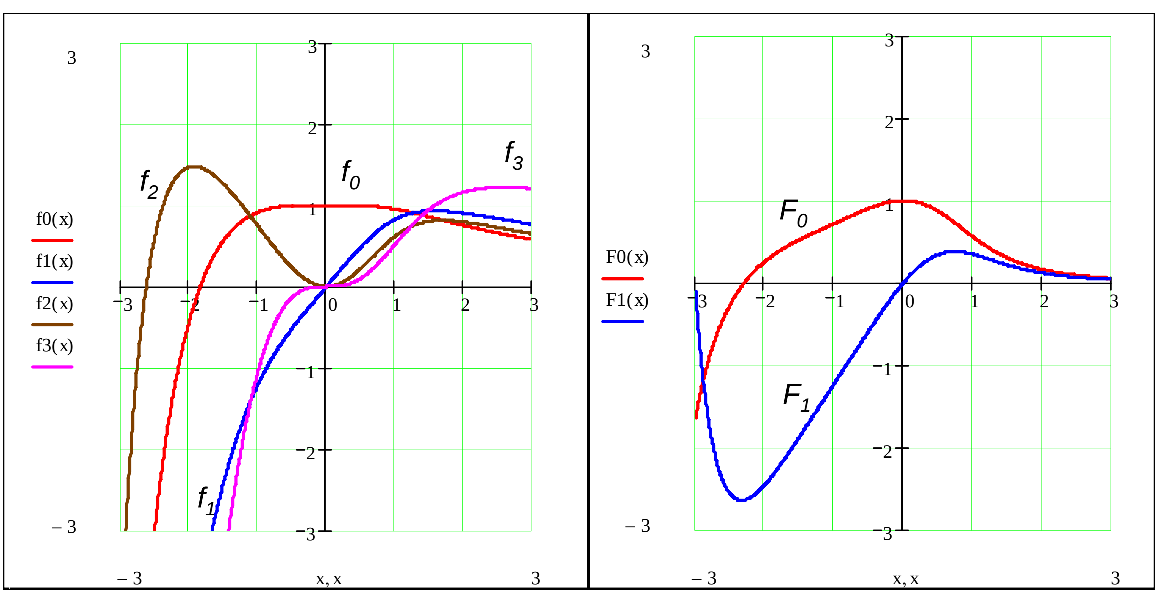


Fig. (1). Functions $f_i(x)$ and $F_i(x)$ for equation $y^{II} + (x^2+2x+1)y^I + (x^2+2x+1)y = 0$.

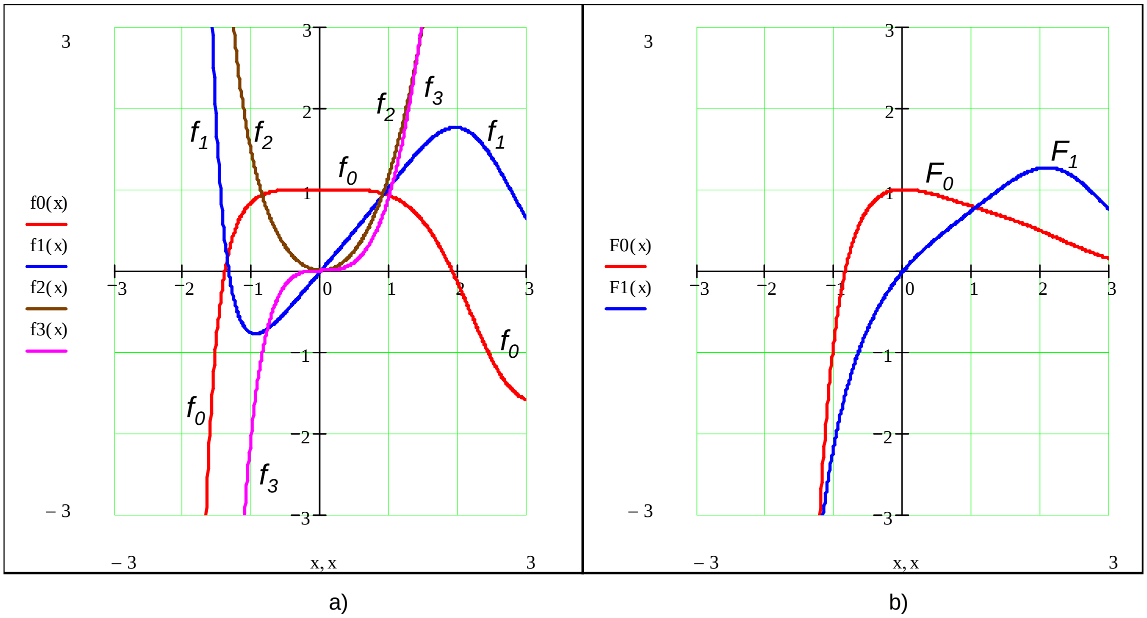


Fig. (2). Functions $f_j(x)$ and $F_i(x)$ for equation $y^{II} + (x^2\text{-}2x\text{+}1)y^I + (x^2\text{-}2x\text{+}1)y= 0$.

Using (2.17), we can calculate the derivatives of the solution functions $F_i(x)$. As an example, Fig. (**3a**) presents first-order and Fig. (**3b**) second-order derivatives of the solution functions shown in Fig. (**2b**).

4. EXAMPLE - HERMITE EQUATION

To show in more detail the development of the functions $f_j(x)$, we consider a special case of (3.1). After substituting for coefficients w_{ij}

$$w_{12}=0,\quad w_{11}=-2,\ w_{10}=0,\ w_{02}=0,\ w_{01}=0,\ w_{00}=2p. \tag{4.1}$$

where p is a constant, (3.1) takes the form of the Hermite equation [4]

$$y^{II} - 2xy^I + 2py= 0. \tag{4.2}$$

For $p=0,1,2,...$ the equation (4.2) has polynomial solutions called Hermite polynomials H_p [4]. We will show how these particular solutions are obtained using the algorithm described above.

Substituting the values (4.1) into (3.6), we readily find

$$a_1(n4)=0,\ \ a_2(n4)=\frac{2(4n\text{-}2)-2p}{4n(4n\text{-}1)},\ \ a_3(n4)=0,\ \ a_4(n4)=0. \tag{4.3}$$

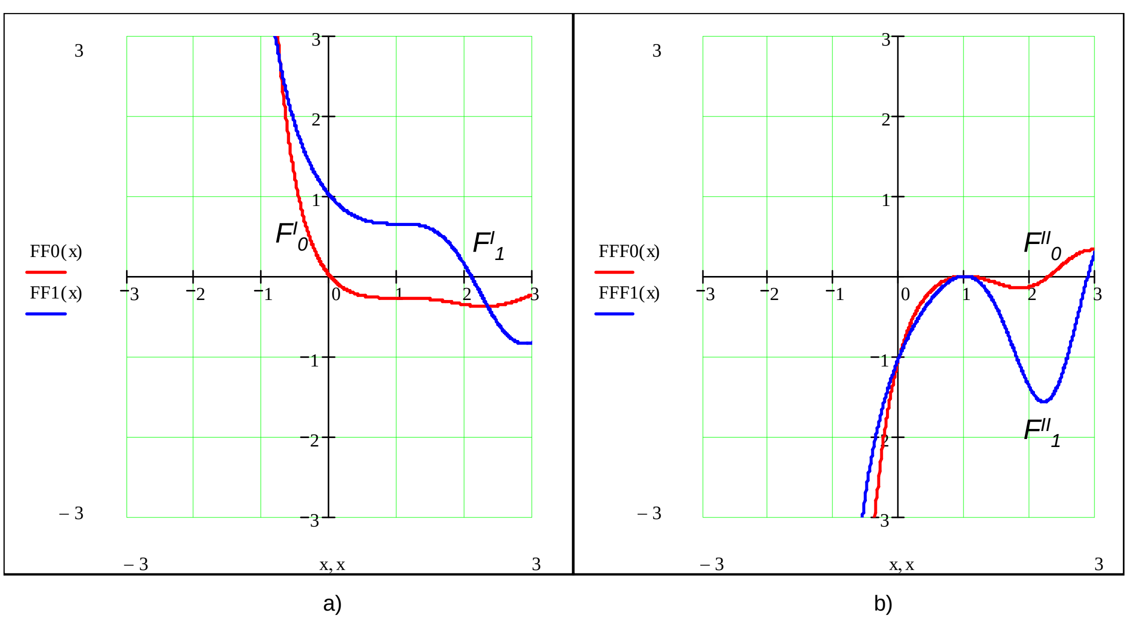


Fig. (3). Derivatives of functions $F_i(x)$ for equation $y^{II} + (x^2\text{-}2x\text{+}1)y^I + (x^2\text{-}2x\text{+}1)y= 0$.

Hence, the matrices (2.8) according to (2.9) are

$$\boldsymbol{b}(n4)=\begin{bmatrix}0 & 0 & a_2(n4) & 0\\ 0 & 0 & 0 & a_2(n4+1)\\ 0 & 0 & a_2(n4+2)a_2(n4) & 0\\ 0 & 0 & 0 & a_2(n4+3)a_2(n4+1)\end{bmatrix} \quad (4.4)$$

Subsequent multiplication of matrices (4.4) according to (2.10) produces matrices $\boldsymbol{c}(n4)$ with only four nonzero elements

$$\boldsymbol{c}(n4)=\begin{bmatrix}0 & 0 & c_{13}(n4) & 0\\ 0 & 0 & 0 & c_{24}(n4)\\ 0 & 0 & c_{33}(n4) & 0\\ 0 & 0 & 0 & c_{44}(n4)\end{bmatrix}. \quad (4.5)$$

According to (2.16), each function $f_j(x)$ is expressed as the series created by the summation of elements along the columns of subsequent matrices $\boldsymbol{c}(n4)$, each time multiplied by appropriate powers of x. Based on (4.5), the summation along the first two columns breaks off the series for the first two functions $f_j(x)$

$$f_0(x)=1,$$

$$f_1(x)=x,$$

$$f_2(x)=x^2+\sum_{n=1}^{\infty}\left[c_{13}(4n)x^{4n}+c_{33}(4n)x^{4n+2}\right],$$

$$f_3(x)=x^3+\sum_{n=1}^{\infty}\left[c_{24}(4n)x^{4n+1}+c_{44}(4n)x^{4n+3}\right]. \quad (4.6)$$

After applying (4.1) in (3.4) to relate constants A_2 and A_3 to A_0 and A_1, the general solution of the equation (4.2) has the form (3.7) with the following functions $F_i(x)$

$$F_0(x)=1-p\cdot f_2(x), \quad (4.7)$$

$$F_1(x)=x+\frac{1-p}{3}f_3(x).$$

For $p=0$, the first solution function (4.7) is identical to the Hermite polynomial of degree 0, $F_0(x)=H_0(x)=1$ [4]. For $p=1$, the second solution function (4.7) takes the form of the Hermite polynomial of degree 1, $F_1(x)=½H_1(x)=x$. Figs. (**4** and **5**) show the solutions for $p=2$ and $p=3$. Diagrams 4.1a and 4.2a present the functions $f_j(x)$ for equation (4.2), for $p=2$ and $p=3$, respectively. Diagrams 4.1b and 4.2b show the corresponding functions $F_i(x)$ and particular solutions $y_p(x)=H_p(x)$, marked with dotted lines. Using the formula (3.7), the particular solutions $y_p(x)$ are determined as curves passing through selected points lying on corresponding $H_p(x)$. The constants A_0 and A_1 in (3.7) were calculated by applying the corresponding boundary conditions. Assuming for $p=2$, $y(0)=-2$ and $y(1)=2$, we obtain as the particular solution the curve $y_2(x)$ coinciding with $H_2(x)=4x^2-2$ (Fig. **4b**). Analogously, for $p=3$, assuming $y(0)=0$ and $y(1)=-4$, we obtain in Fig. (**5b**) the particular solution $y_3(x)$ coinciding with $H_3(x)=8x^3-12x$. In both Figs. (**4b** and **5b**), zeros for the particular solutions $y_p(x)$ are also shown. In the same way, the particular solutions corresponding to the higher-degree Hermite polynomials can be determined as particular solutions of ODE (4.2) satisfying appropriate boundary conditions.

5. THIRD ORDER DIFFERENTIAL EQUATION AND COMBINED CONDITIONS

Now we consider a third-order differential equation of the form

$$y^{III}+(w_{22}\cdot x^2+w_{21}\cdot x+w_{20})y^{II}+(w_{11}\cdot x+w_{10})y^{I}+w_{00}y=0. \quad (5.1)$$

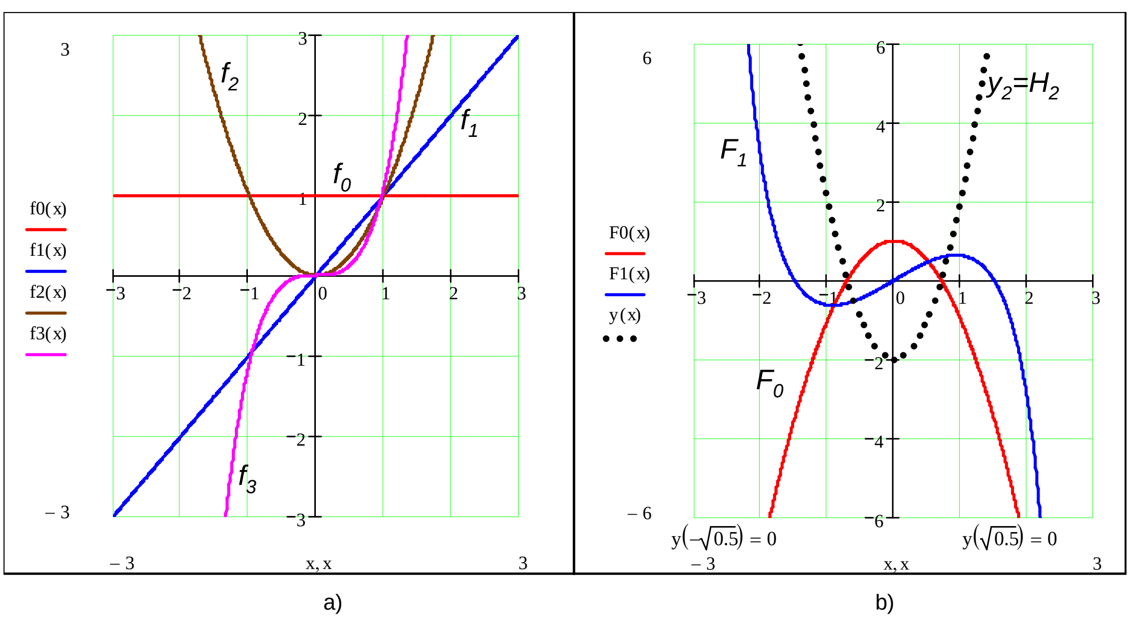
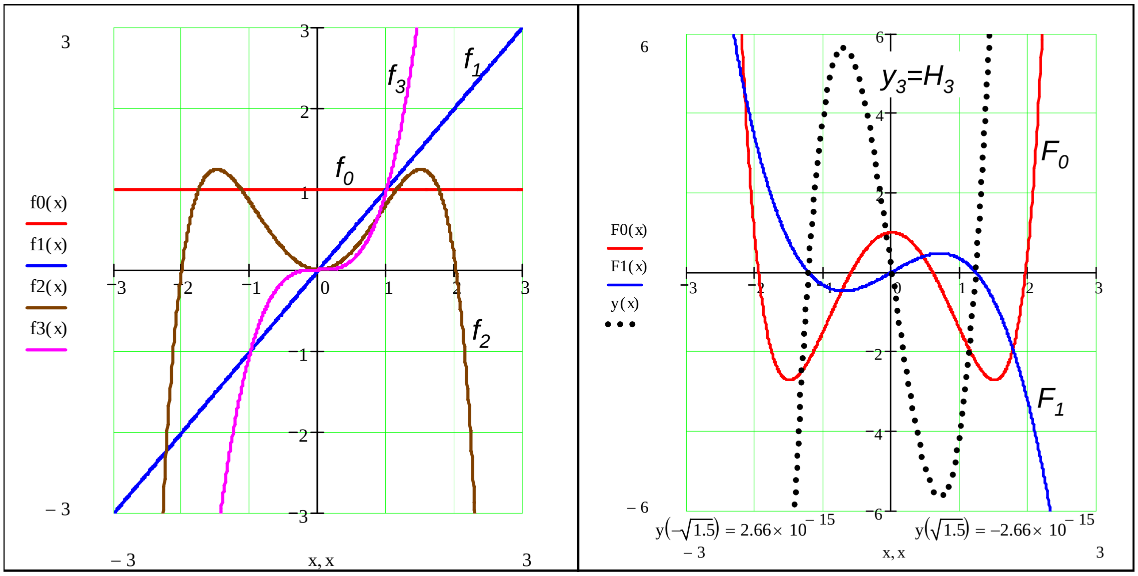


Fig. (4). Functions $f_j(x)$, $F_i(x)$ and particular solution $y_2(x)=H_2(x)$ for $p=2$.



a) b)

Fig. (5). Functions $f_j(x)$, $F_i(x)$ and particular solution $y_3(x)=H_3(x)$ for $p=3$.

This is one of the examples where number S of the "natural" functions $f_j(x)$ defined by (2.16) is equal to the order of the equation. According to (2.4), we have

$$S = N - min\{(0-0), (1-1), (2-2)\} = 3. \tag{5.2}$$

This means that a linear combination of functions $f_j(x)$ forms the general solution

$$y(x)= A_0 \cdot f_0(x)+A_1 \cdot f_1(x)+A_2 \cdot f_2(x). \tag{5.3}$$

To find functions $f_j(x)$, we will follow the procedure defined in Section 2 and illustrated in Section 3. First, substituting (2.1) and (2.2) into (5.1), rearranging term numbering in every series, and grouping coefficients corresponding to the same powers of x, we obtain

$$\text{for } m \geq 3 \qquad A_m = \sum_{k=1}^{S=3} a_k(m) A_{m-k}, \tag{5.4}$$

where

$$a_1(m) = \frac{-w_{20}}{m}, \quad a_2(m) = \frac{-((m-3)w_{21} + w_{10})}{m(m-1)},$$

$$a_3(m) = \frac{-((m-3)(m-4)w_{22} + (m-3)w_{11} + w_{00})}{m(m-1)(m-2)}. \tag{5.5}$$

The elements of the matrices $\boldsymbol{b}(n3)$ (2.8) are defined row by row using the general formula (2.9). In our case, we get for $n>1$ and $j=1,2,3$

$$\mathbf{b}_{1j}(n3) = \mathbf{a}_{(4-j)}(n3),$$

$$\mathbf{b}_{2j}(n3) = \mathbf{a}_1(n3+1)\mathbf{b}_{1j}(n3) + \mathbf{a}_{(5-j)}(n3+1), \tag{5.6}$$

$$\mathbf{b}_{3j}(n3) = \mathbf{a}_1(n3+2)\mathbf{b}_{2j}(n3) + \mathbf{a}_2(n3+2)\mathbf{b}_{1j}(n3) + \mathbf{a}_{(6-j)}(n3+2),$$

where $a_4(m)=a_5(m)=0$ for every m (e.g. $m=n3$, $n3+1,\ldots$). The functions $f_j(x)$ are

$$\text{for } j = 0,1,2 \qquad f_j(x)= x^j + \sum_{n=1}^{\infty}\left[\sum_{k=1}^{3} c_{k(j+1)}(n3)x^{3n+k-1}\right]. \tag{5.7}$$

The matrices $\boldsymbol{c}(n3)$ are determined as the successive multiplication of the matrices $\boldsymbol{b}(n3)$ defined by (5.6), conducted in agreement with (2.11). Fig. (**6**) shows two solutions for the example equations.

$$y^{III} + (x^2+2x+1)y^{II} + (x+1)y^{I} + y= 0 \tag{5.8}$$

$$y^{III} + (x^2-2x+1)y^{II} + (x+1)y^{I} + y= 0. \tag{5.9}$$

For each case (5.8) and (5.9) in Fig. (**6**), there are diagrams of the functions $f_j(x)$ and the particular solutions $y(x)$ obtained as curves crossing two selected points $(-3,-1)$, $(3,1)$ and satisfying the following combined condition.

$$y^{II}(2) + y^{I}(1) + y(-1) = 0. \tag{5.10}$$

Finding constants A_j satisfying the conditions of any form is straightforward (practically in the region where the power series (5.7) can be calculated) once we have the functions $f_j(x)$ (and their derivatives) defined by (5.7) and using (5.3).

6. SUMMARY

The method of power-series expansions is used mostly to study the functions defined by certain classes of linear differential equations. For most of these equations, application of the standard method of undetermined coefficients leads to two- up to three-term recursion formulae [4]. Due to the application of the matrix calculation, the method presented in this paper always leads to the one-term recursion formula expressed in the matrix form (2.12). A general solution is defined as a combination of functions $f_j(x)$ expressed by power series independent of unknown constants. The functions $f_j(x)$ are naturally related to the considered differential equation and are independent of initial or boundary conditions. The presented algorithm can be

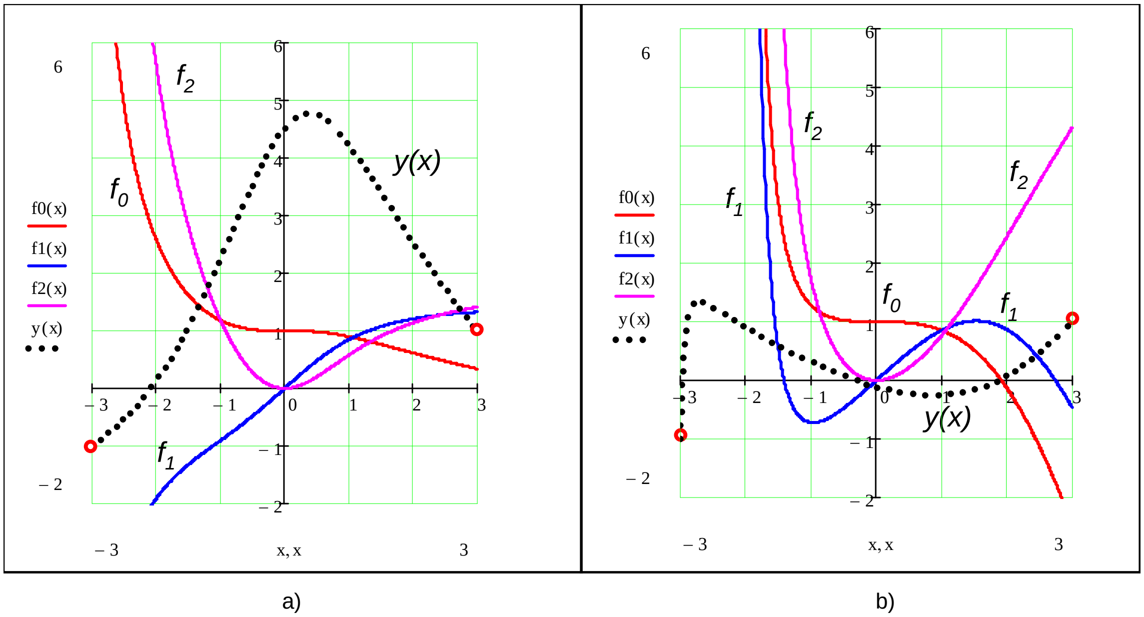


Fig. (6). Functions $f_j(x)$ and particular solutions $y(x)$ crossing two points *(-3,-1)*, *(3,1)* and satisfying combined condition (5.10). Solutions of equations **a**) (5.8) and **b**) (5.9).

applied to any system of equations of the form (1.1) without formal limitations regarding the order of equations or degree of polynomial coefficients.

The demonstrated method of applying a matrix calculation can be easily programmed using a computer. However, the numerical calculation of the power series for functions $f_j(x)$ is always limited to the interval around origin $a<x<b$, where $a<0$ and $b>0$, even though formally the series are convergent for all real x. For greater values of x, the calculation involves summation of large real components (positive and negative), many orders greater than their sum. It requires increasing the precision, which is strictly limited for computer calculations using general-purpose mathematical programs. Within the interval *(a,b)*, the values of functions $f_j(x)$ and their derivatives can be calculated before the initial or boundary conditions are stated. Therefore, any form of initial or boundary conditions (or their combination such as (5.10)) can be used to calculate particular solutions. The problem of numerical calculations outside the interval *(a,b)* with the application of multiple-precision modules needs further consideration.

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Received: April 17, 2009 Revised: May 07, 2008 Accepted: June 07, 2008