Application of VIM to the Nonlinear Vibrations of Multiwalled Carbon Nanotubes

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Abstract: The present work deals with applying the variational iteration method to the problem of the nonlinear vibrations of multiwalled carbon nanotubes embedded in an elastic medium. A multiple-beam model is utilized in which the governing equations of each layer are coupled with those of its adjacent ones *via* the van der Waals inter layer forces. The amplitude-frequency curves for large-amplitude vibrations of single-walled, double-walled and triple-walled carbon nanotubes are obtained. The influence of changes in material constants of the surrounding elastic medium and the effect of changes in nanotube geometrical parameters on the vibration characteristics are studied by comparing the results with those from the open literature. This comparison illustrates that the solutions obtained are of very high accuracy and clarifies the capability and the simplicity of the present method.

Keywords: Variational iteration method, nonlinear vibration, carbon nanotube, elastic medium.

1. INTRODUCTION

With the rapid development of nanotechnology, there appears an ever-increasing interest of scientists and researchers in this field of science. Nanomaterials, because of their exceptional mechanical, physical and chemical properties have been the main topic of research in many scientific publications. Nowadays, they are used as the substantial parts of nanoelectronics, nanodevices, and nanocomposites. One of these materials attracted great attention due to its high mechanical strength is carbon nanotube (CNT). CNTs were discovered by Iijima [1] in 1991. In spite of being too small and having light weight, they have very large Young's modulus in axial direction (nearly 1TPa). Undoubtedly, CNTs have the eligibility to be the new and most popular nanomaterial of this early part of the 21st century. Since the vibration of CNTs are of considerable importance in a number of nanomechanical devices such as oscillators, charge detectors, field emission devices and sensors, Many researches have been so far devoted to the problem of the vibration of these Nanomaterials [2-5]. However, most of the investigations conducted on the vibration of multiwalled carbon nanotubes (MWNTs) have been restricted to the linear regime and fewer works were done on the nonlinear vibration of these materials. Recently Fu [6] studied the nonlinear vibrations of embedded nanotubes using the incremental harmonic balanced method (IHBM). In that work, single-walled nanotubes (SWNTs) and double-walled nanotubes (DWNTs) considered for the study. The present work is an extension of the work of [6] for TWNTs using the variational iteration method (VIM). The aim of this investigation is to feature the capability of VIM for finding approximate solutions of many nonlinear vibrating systems. The VIM was first proposed as a general Lagrange multiplier method to solve nonlinear problems by Inokuti *et al*. [7] in 1978. This method has so far been shown to be effective, simple and accurate for solving a large variety of nonlinear problems [8-13] and also coupled system of differential equations [14, 15] with approximations converging rapidly to the accurate solutions. To illustrate the basic ideas of VIM, consider the following general nonlinear system

$$Lu(t) + Nu(t) = f(t) \tag{1}$$

where L is a linear operator, N is a nonlinear operator, and $f(t)$ is a known analytic function. According to the variational iteration method, we can construct the following iteration formulation

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(Lu_n(\zeta) + N\tilde{u}_n(\zeta) - f(\zeta)\right)d\zeta \tag{2}$$

so called a correction functional with initial approximation of $u_0(t)$. Here λ is called a general Lagrange multiplier, which can be determined optimally *via* the variational theory, and $\tilde{u}_n$ is considered as a restricted variation [16], i.e. $\delta\tilde{u}_n = 0$. Now we adopt VIM to the problem of the nonlinear vibrations of CNTs.

2. SOLUTION PROCEDURE

2.1. Applying VIM for Nonlinear Vibration of a SWNT

Consider a SWNT of length l, Young's modulus E, density ρ, cross-sectional area A, and cross-sectional inertia moment I, embedded in an elastic medium with material

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1874-1142/10

constant k. The nonlinear vibration equation for this CNT is in the following form [6]

$$\frac{d^2W}{dt^2}+\left(\frac{\pi^4 EI}{\rho A l^4}+\frac{k}{\rho A}\right)W+\frac{\pi^4 E}{4\rho l^4}W^3=0 \tag{3}$$

Under the transformations $r=\sqrt{\frac{I}{A}}, a=\frac{W}{r}, \omega_l=\frac{\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}}, \omega_k=\sqrt{\frac{k}{\rho A}}, \tau=\omega t,$ the above equation can be transformed to the following dimensionless nonlinear vibration equation

$$\omega^2\frac{d^2a}{d\tau^2}+\omega_b^2 a+\alpha\omega_l^2 a^3=0, \tag{4}$$

in which $\alpha=0.25$ and $\omega_b=\sqrt{\omega_l^2+\omega_k^2}$, is the linear, free vibration frequency. Applying VIM constructs the following correction functional on Eq. (4)

$$a_{n+1}=a_n+\int_0^t \lambda(\zeta)\left(\frac{d^2a_n(\zeta)}{d\zeta^2}+\omega_b^2 a_n+\alpha\omega_l^2\tilde{a}_n^3\right)d\zeta \tag{5}$$

herein $\tilde{a}_n$ is considered as a restricted variation. Making the above correction functional stationary, together with considering $\delta a(0)=0,$ we arrive at

$$\delta a_{n+1}(t)=\delta a_n(t)+\lambda(\delta a_n)'\,|_0^t-\lambda'\delta a_n\,|_0^t+\int_0^t\left(\lambda''+\lambda\omega_b^2\right)\delta a_n d\zeta=0 \tag{6}$$

Its stationary conditions can be written down as

$$\begin{aligned}&1-\lambda'(\zeta)\,|_{\zeta=t}=0,\\&\lambda(\zeta)\,|_{\zeta=t}=0,\\&\lambda''(\zeta)+\omega_b^2\lambda(\zeta)=0,\end{aligned} \tag{7}$$

Solving this, a Lagrange multiplier can be readily identified as $\lambda(\zeta)=\frac{1}{\omega_b}\sin\omega_b(\zeta-t)$. Therefore the iteration formula of Eq. (5) becomes

$$a_{n+1}=a_n+\frac{1}{\omega_b}\int_0^t\sin\omega_b(\zeta-t)\left(\frac{d^2a_n(\zeta)}{d\zeta^2}+\omega_b^2a_n+\alpha\omega_l^2a_n^3\right)d\zeta \tag{8}$$

In order to seek the periodic solution of Eq. (4) assume the initial approximation to be the linear solution of Eq. (4) as $a_0=X\cos(\psi\omega_b t)$. This initial approximation is a trial function and it is used to obtain more accurate approximate solution of Eq. (4). Here ψ, is the ratio of the nonlinear frequency, ω, to the linear frequency, ω_b. Substituting the initial approximation into Eq. (4) results in the following residual

$$R_0(\zeta)=(-X\psi^2\omega_b^2+\omega_b^2X+0.75\alpha\omega_l^2X^3)\cos(\psi\omega_b\zeta)+0.25\alpha\omega_l^2X^3\cos(3\psi\omega_b\zeta) \tag{9}$$

In order to ensure that no secular terms appear in the next iteration, the coefficient of $\cos(\psi\omega_b\zeta)$ must vanish. Therefore

$$\psi=\sqrt{1+\frac{3}{4}\alpha(\frac{\omega_l}{\omega_b})^2X^2} \tag{10}$$

Using Eqs. (8) and (9), we have

$$\begin{aligned}a_1&=a_0+\int_0^t\lambda(\zeta)R_0(\zeta)d\zeta\\&=\frac{1}{(9\psi^2-1)}\left[\begin{array}{l}-X\cos(\psi\omega_b t)+9X\psi^2\cos(\psi\omega_b t)-\\0.0625X^3\cos(\omega_b t)+0.0625X^3\cos(3\psi\omega_b t)\end{array}\right]\end{aligned} \tag{11}$$

with ψ defined as in Eq. (10). The amplitude-frequency response curves for a SWNT for different spring k constants are shown in Fig. (**1**). The material and geometric parameters taken here are $E=1.1\text{TPa}, \rho=1300kg/m^3, l=45\text{nm},$ the outer diameter $d_1=3\text{nm},$ and the inner diameter $d_0=2.32\text{nm}.$ In Fig. (**1**), ψ is the ratio of nonlinear frequency to linear frequency as discussed earlier and X is the maximum vibration amplitude. It can be seen that as the spring constant k increases, the nonlinear frequencies tend to approach the linear ones especially when k exceeds the value $10^7\ \text{n/m}^2$. It should be noted that Fig. (**1**) is exactly the same as the figure obtained *via* incremental harmonic balance method (IHBM) [6].

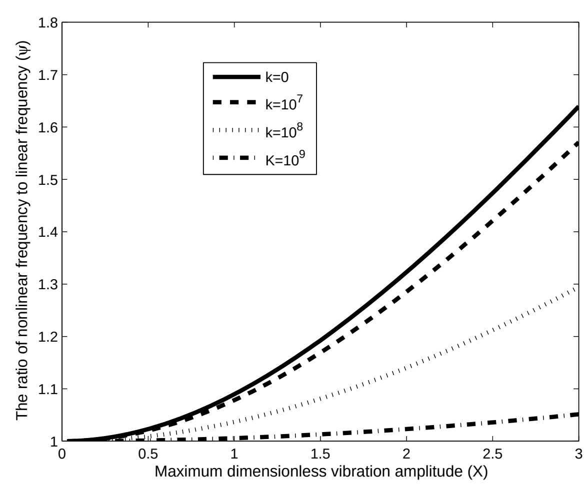

Fig. (1). Effect of spring constant *k* on nonlinear amplitude-frequency response curves of SWNT.

2.2. Applying VIM for Nonlinear Vibration of a DWNT

The nonlinear vibration governing equation for a DWNT is in the following form [6].

$$\begin{aligned}&\frac{d^2W_1}{dt^2}+\left(\frac{\pi^4EI_1}{\rho A_1l^4}+\frac{c_1}{\rho A_1}\right)W_1+\frac{\pi^4E}{4\rho l^4}W_1^3-\frac{c_1}{\rho A_1}W_2=0,\\&\frac{d^2W_2}{dt^2}+\left(\frac{\pi^4EI_2}{\rho A_2l^4}+\frac{c_1}{\rho A_2}+\frac{k}{\rho A_2}\right)W_2+\frac{\pi^4E}{4\rho l^4}W_2^3-\frac{c_1}{\rho A_2}W_1=0,\end{aligned} \tag{12}$$

where c_i is the coefficient of the van der Waals force between the *i*th tube and the *i*-1th tube. By substituting the following dimensionless parameters

$$r=\sqrt{\frac{I_1}{A_1}},\ a_1=\frac{W_1}{r},\ a_2=\frac{W_2}{r},\ \omega_l=\frac{\pi^2}{l^2}\sqrt{\frac{EI_1}{\rho A_1}},\ \omega_k=\sqrt{\frac{k}{\rho A_1}},\ \omega_c=\sqrt{\frac{c}{\rho A_1}},\tau=\omega t,\ \beta=\frac{A_1}{A_2},\ \gamma=\frac{I_1}{I_2},\ \alpha=0.25,$$

Eq. (12) can be transformed to the following dimensionless nonlinear system

$$\begin{aligned}&\left(\frac{\omega}{\omega_l}\right)^2\frac{d^2a_1}{d\tau^2}+B_1a_1+\alpha a_1^{\ 3}-B_2a_2=0,\\&\left(\frac{\omega}{\omega_l}\right)^2\frac{d^2a_2}{d\tau^2}+B_3a_2+\alpha a_2^{\ 3}-B_4a_1=0,\end{aligned} \tag{13}$$

with B_1 to B_4 defined as

$$B_1=1+\left(\frac{\omega_c}{\omega_l}\right)^2,\ B_2=\left(\frac{\omega_c}{\omega_l}\right)^2,\ B_3=\beta\left[\frac{1}{\gamma}+\left(\frac{\omega_c}{\omega_l}\right)^2+\left(\frac{\omega_k}{\omega_l}\right)^2\right],$$

$$B_4=\beta\left(\frac{\omega_c}{\omega_l}\right)^2 \tag{14}$$

Applying VIM yields the correction functional of Eq. (13) as follows

$$\begin{aligned}&a_{1n+1}=a_{1n}+\int_0^t\lambda_1(\zeta)\left(\frac{d^2a_{1n}(\zeta)}{d\zeta^2}+B_1\omega_l^2a_{1n}+\alpha\omega_l^2\tilde a_{1n}^{\ 3}-B_2\omega_l^2\tilde a_{2n}\right)d\zeta\\&a_{2n+1}=a_{2n}+\int_0^t\lambda_2(\zeta)\left(\frac{d^2a_{2n}(\zeta)}{d\zeta^2}+B_3\omega_l^2a_{2n}+\alpha\omega_l^2\tilde a_{2n}^{\ 3}-B_4\omega_l^2\tilde a_{1n}\right)d\zeta\end{aligned} \tag{15}$$

herein $\tilde a_{1n}$ and $\tilde a_{2n}$, are considered as the restricted variations. Considering $\delta a_1(0)=0,$ and $\delta a_2(0)=0,$ the stationary conditions of the above correction functionals can be expressed as follows

$$\begin{aligned}&\delta a_{1n}:\ 1-\lambda_1'(\zeta)\,|_{\zeta=t}=0,\\&\delta a_{1n}':\ \lambda_1(\zeta)\,|_{\zeta=t}=0,\\&\delta a_{1n}:\ \lambda_1''(\zeta)+\omega_l^2s_1^{\ 2}\lambda_1(\zeta)=0,\\&\delta a_{2n}:\ 1-\lambda_2'(\zeta)\,|_{\zeta=t}=0,\\&\delta a_{2n}':\ \lambda_2(\zeta)\,|_{\zeta=t}=0,\\&\delta a_{2n}:\ \lambda_2''(\zeta)+\omega_l^2s_2^{\ 2}\lambda_2(\zeta)=0,\end{aligned} \tag{16}$$

Defining $s_1=\sqrt{B_1}$ and $s_2=\sqrt{B_3}$, the Lagrange multipliers can be readily identified as

$$\begin{aligned}&\lambda_1(\zeta)=\frac{1}{\omega_ls_1}\sin\omega_ls_1(\zeta-t)\\&\lambda_2(\zeta)=\frac{1}{\omega_ls_2}\sin\omega_ls_2(\zeta-t)\end{aligned} \tag{17}$$

and the following iteration formulas can be obtained

$$\begin{aligned}&a_{1n+1}=a_{1n}+\frac{1}{\omega_ls_1}\int_0^t\sin\omega_ls_1(\zeta-t)\left(\frac{d^2a_{1n}(\zeta)}{d\zeta^2}+B_1\omega_l^{\ 2}a_{1n}+\alpha\omega_l^{\ 2}\tilde a_{1n}^{\ 3}-B_2\omega_l^{\ 2}\tilde a_{2n}\right)d\zeta\\&a_{2n+1}=a_{2n}+\frac{1}{\omega_ls_2}\int_0^t\sin\omega_ls_2(\zeta-t)\left(\frac{d^2a_{2n}(\zeta)}{d\zeta^2}+B_3\omega_l^{\ 2}a_{1n}+\alpha\omega_l^{\ 2}\tilde a_{2n}^{\ 3}-B_4\omega_l^{\ 2}\tilde a_{1n}\right)d\zeta\end{aligned} \tag{18}$$

Substituting $a_{10}=X_1\cos(\psi\omega_bt)$ and $a_{20}=X_2\cos(\psi\omega_bt)$, as the initial approximations of a_1 and a_2 into the Eq. (13) results in the following residuals

$$\begin{aligned}&R_{10}(\zeta)=(-X_1\psi^2\omega_b^{\ 2}+B_1\omega_l^{\ 2}X_1+0.75\alpha\omega_l^{\ 2}X_1^{\ 3}-B_2\omega_l^{\ 2}X_2)\cos(\psi\omega_b\zeta)+0.25\alpha\omega_l^{\ 2}X_1^{\ 3}\cos(3\psi\omega_b\zeta)\\&R_{20}(\zeta)=(-X_2\psi^2\omega_b^{\ 2}+B_3\omega_l^{\ 2}X_2+0.75\alpha\omega_l^{\ 2}X_2^{\ 3}-B_4\omega_l^{\ 2}X_1)\cos(\psi\omega_b\zeta)+0.25\alpha\omega_l^{\ 2}X_2^{\ 3}\cos(3\psi\omega_b\zeta)\end{aligned} \tag{19}$$

Herein ψ, the ratio of the nonlinear frequency ω to the linear frequency ω_b, is the unknown constant. Following the same approach as above and also eliminating the coefficient of $\cos\psi\omega_bt$ in the above system due to avoiding the secular terms, results in the following nonlinear system which can be easily solved using a simple mathematical algorithm such as Newton-Raphson technique.

$$\begin{cases}-\left(\dfrac{\psi}{\omega_l}\right)^2X_1\omega_b^{\ 2}+B_1X_1+\dfrac{3}{4}\alpha X_1^{\ 3}-B_2X_2=0\\-\left(\dfrac{\psi}{\omega_l}\right)^2X_2\omega_b^{\ 2}+B_3X_2+\dfrac{3}{4}\alpha X_2^{\ 3}-B_4X_1=0\end{cases} \tag{20}$$

Also the solutions a_1 and a_2 can be achieved from the following equations

$$a_{11}=a_{10}+\int_0^t\lambda_1(\zeta)R_{10}(\zeta)\ ,\ a_{21}=a_{20}+\int_0^t\lambda_2(\zeta)R_{20}(\zeta) \tag{21}$$

To calculate the linear vibration frequencies for DWNT, we shall first substitute $a_1 = X_1 \cos\omega t$ and $a_2 = X_2 \cos\omega t$ into Eq. (13) without considering the nonlinear terms in Eq. (13), so that

$$\begin{bmatrix} {\omega_l}^2 + {\omega_c}^2 - \omega^2 & -{\omega_c}^2 \\ -\beta{\omega_c}^2 & \beta\left(\dfrac{{\omega_l}^2}{\gamma} + {\omega_c}^2 + {\omega_k}^2\right) - \omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (22)$$

Then by setting the determinant of the matrix in Eq. (22) equal to zero, the frequency characteristic equation will be obtained. The fundamental linear vibration frequency of DWNT is the lowest root of the resulting equation. Fig. (**2**) shows the variation of the nonlinear amplitude-frequency response curves of DWNT against the maximum vibration amplitude for different spring constants *k*. The material and geometric parameters used to obtain this figure are $E = 1.1\,\mathrm{TPa}, \rho = 1300\,\mathrm{kg/m^3}, c = 0.3\times10^{12}\,\mathrm{N/m^2}$, $l = 45\,\mathrm{nm}, d_0 = 1.64\,\mathrm{nm}, d_1 = 2.32\,\mathrm{nm}$ and $d_2 = 3\,\mathrm{nm}$. It can be seen that the effect of spring constant on nonlinear vibration of DWNT is similar to that emerged in the case of SWNT (Fig. **1**) and this figure is exactly the same figure as that obtained *via* (IHBM) [6].

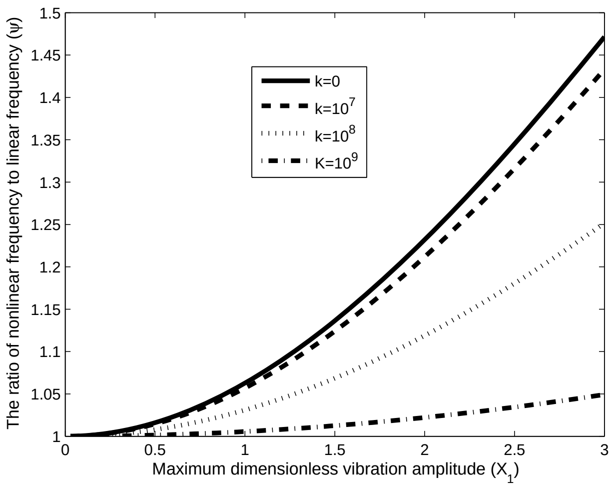

Fig. (2). Effect of spring constant *k* on nonlinear amplitude-frequency response curves of DWNT.

2.3. Applying VIM for Nonlinear Vibration of a TWNT

The nonlinear vibration governing equations for TWNTs are in the following form

$$\frac{d^2W_1}{dt^2} + \left(\frac{\pi^4 EI_1}{\rho A_1 l^4} + \frac{c_1}{\rho A_1}\right)W_1 + \frac{\pi^4 E}{4\rho l^4}W_1^3 - \frac{c_1}{\rho A_1}W_2 = 0,$$
$$\frac{d^2W_2}{dt^2} + \left(\frac{\pi^4 EI_2}{\rho A_2 l^4} + \frac{c_1}{\rho A_2} + \frac{c_2}{\rho A_2}\right)W_2 + \frac{\pi^4 E}{4\rho l^4}W_2^3 - \frac{c_1}{\rho A_2}W_1 - \frac{c_2}{\rho A_2}W_3 = 0,$$
$$\frac{d^2W_3}{dt^2} + \left(\frac{\pi^4 EI_3}{\rho A_3 l^4} + \frac{c_2}{\rho A_3} + \frac{k}{\rho A_3}\right)W_3 + \frac{\pi^4 E}{4\rho l^4}W_3^3 - \frac{c_2}{\rho A_3}W_2 = 0, \qquad (23)$$

In a similar manner, introducing the following dimensionless parameters

$$r = \sqrt{\frac{I_1}{A_1}},\ a_1 = \frac{W_1}{r},\ a_2 = \frac{W_2}{r},\ a_3 = \frac{W_3}{r},\ \omega_l = \frac{\pi^2}{l^2}\sqrt{\frac{EI_1}{\rho A_1}},\ \omega_k = \sqrt{\frac{k}{\rho A_1}},\ \omega_c = \sqrt{\frac{c}{\rho A_1}},$$
$$\tau = \omega t,\ \beta = \frac{A_1}{A_2},\ \gamma = \frac{I_1}{I_2}, \eta = \frac{A_1}{A_3},\ \zeta = \frac{I_1}{I_3}, \alpha = 0.25,$$

to the Eqs. (23) leads to the dimensionless nonlinear vibration equations as

$$\left(\frac{\omega}{\omega_l}\right)^2 \frac{d^2a_1}{d\tau^2} + \left[1 + \left(\frac{\omega c}{\omega_l}\right)^2\right]a_1 + \alpha {a_1}^3 - \left(\frac{\omega c}{\omega_l}\right)^2 a_2 = 0,$$
$$\left(\frac{\omega}{\omega_l}\right)^2 \frac{d^2a_2}{d\tau^2} + \beta\left[\frac{1}{\gamma} + 2\left(\frac{\omega c}{\omega_l}\right)^2\right]a_2 + \alpha {a_2}^3 - \beta\left(\frac{\omega c}{\omega_l}\right)^2 a_1 - \beta\left(\frac{\omega c}{\omega_l}\right)^2 a_3 = 0,$$
$$\left(\frac{\omega}{\omega_l}\right)^2 \frac{d^2a_3}{d\tau^2} + \eta\left[\frac{1}{\zeta} + \left(\frac{\omega c}{\omega_l}\right)^2 + \left(\frac{\omega_k}{\omega_l}\right)^2\right]a_3 + \alpha {a_3}^3 - \eta\left(\frac{\omega c}{\omega_l}\right)^2 a_2 = 0 \qquad (24)$$

Applying VIM, yields the following correction functionals

$$a_{1n+1} = a_{1n} + \int_0^t \lambda_1(\zeta) \left(\frac{d^2a_{1n}(\zeta)}{d\zeta^2} + B_1\omega_l^2 a_{1n} + \alpha\omega_l^2 {\tilde{a}_{1n}}^3 - B_2\omega_l^2 \tilde{a}_{2n}\right)d\zeta$$
$$a_{2n+1} = a_{2n} + \int_0^t \lambda_2(\zeta) \left(\frac{d^2a_{2n}(\zeta)}{d\zeta^2} + B_3\omega_l^2 a_{1n} + \alpha\omega_l^2 {\tilde{a}_{2n}}^3 - \beta B_2\omega_l^2 \tilde{a}_{1n} - \beta B_2\omega_l^2 \tilde{a}_{3n}\right)d\zeta \qquad (25)$$
$$a_{3n+1} = a_{3n} + \int_0^t \lambda_3(\zeta) \left(\frac{d^2a_{3n}(\zeta)}{d\zeta^2} + B_4\omega_l^2 a_{3n} + \alpha\omega_l^2 {\tilde{a}_{3n}}^3 - \eta B_2\omega_l^2 \tilde{a}_{2n}\right)d\zeta$$

In a similar manner as above, making the above correction functionals stationary, together with considering $\delta a_1(0) = 0$, $\delta a_2(0) = 0$, and $\delta a_3(0) = 0$, solving their stationary conditions, The Lagrange multipliers can be obtained as

$$\lambda_1(\zeta) = \frac{1}{\omega_l s_1}\sin\omega_l s_1(\zeta - t),\ \lambda_2(\zeta) = \frac{1}{\omega_l s_2}\sin\omega_l s_2(\zeta - t),\ \lambda_3(\zeta) = \frac{1}{\omega_l s_3}\sin\omega_l s_3(\zeta - t) \qquad (26)$$

where $s_3 = \sqrt{B_4}$, s_1 and s_2 are similar as above. Substituting $a_{10} = X_1\cos(\psi\omega_b t)$, $a_{20} = X_2\cos(\psi\omega_b t)$ and $a_{30} = X_3\cos(\psi\omega_b t)$, as the initial approximations of a_1, a_2 and a_3 into the Eq. (24) results in the following residuals

$$R_{10}(\zeta) = (-X_1\psi^2\omega_b{}^2 + B_1\omega_l{}^2X_1 + 0.75\alpha\omega_l{}^2X_1{}^3 - B_2\omega_l{}^2X_2)\cos(\psi\omega_b\zeta) + 0.25\alpha\omega_l{}^2X_1{}^3\cos(3\psi\omega_b\zeta)$$

$$R_{20}(\zeta) = (-X_2\psi^2\omega_b{}^2 + B_3\omega_l{}^2X_2 + 0.75\alpha\omega_l{}^2X_2{}^3 - \beta B_2\omega_l{}^2X_1 - \beta B_2\omega_l{}^2X_3)\cos(\psi\omega_b\zeta) + 0.25\alpha\omega_l{}^2X_2{}^3\cos(3\psi\omega_b\zeta) \quad (27)$$

$$R_{30}(\zeta) = (-X_3\psi^2\omega_b{}^2 + B_4\omega_l{}^2X_3 + 0.75\alpha\omega_l{}^2X_3{}^3 - \eta B_2\omega_l{}^2X_2)\cos(\psi\omega_b\zeta) + 0.25\alpha\omega_l{}^2X_1{}^3\cos(3\psi\omega_b\zeta)$$

Eliminating the coefficient of $\cos(\psi\omega_b\zeta)$, yields

$$\begin{cases} -\left(\dfrac{\psi}{\omega_l}\right)^2 X_1\omega_b{}^2 + B_1X_1 + \dfrac{3}{4}\alpha X_1{}^3 - B_2X_2 = 0, \\ -\left(\dfrac{\psi}{\omega_l}\right)^2 X_2\omega_b{}^2 + B_3X_2 + \dfrac{3}{4}\alpha X_2{}^3 - \beta B_2X_1 - \beta B_2X_3 = 0, \\ -\left(\dfrac{\psi}{\omega_l}\right)^2 X_3\omega_b{}^2 + B_4X_3 + \dfrac{3}{4}\alpha X_3{}^3 - \eta B_2X_2 = 0, \end{cases} \quad (28)$$

which can be solved using a simple mathematical approach to give the unknown constant ψ. Calculation of the linear vibration frequencies for TWNT can be performed in the same manner that mentioned earlier for DWNT. Substituting $a_1 = X_1\cos\omega t, a_2 = X_2\cos\omega t$ and $a_3 = X_3\cos\omega t$ into Eq. (24), simultaneously neglecting the nonlinear terms in Eq. (24) yields

$$\begin{bmatrix} \omega_l{}^2 + \omega_c{}^2 - \omega^2 & -\omega_c{}^2 & 0 \\ -\beta\omega_c{}^2 & \beta\left(\dfrac{\omega_l{}^2}{\gamma} + 2\omega_c{}^2\right) - \omega^2 & -\beta\omega_c{}^2 \\ 0 & -\eta\omega_c{}^2 & \eta\left(\dfrac{\omega_l{}^2}{\zeta} + \omega_c{}^2 + \omega_k{}^2\right) - \omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (29)$$

For a nontrivial solution to exist, the determinant of the above matrix must be vanished which leads to the frequency characteristic equation to be solved. The fundamental linear vibration frequency of TWNT is the lowest root of the resulting equation. The variation of the nonlinear amplitude-frequency response curves of TWNT against the maximum vibration amplitude for different spring constants is also illustrated in Fig. (**3**). The material and geometric parameters used are $c_1 = c_2 = 0.3\times10^{12}\,\mathrm{N/m^2}, l = 45\,\mathrm{nm}, d_0 = .$ $0.96\,\mathrm{nm}, d_1 = 1.64\,\mathrm{nm}, \mathrm{d}_2 = 2.32\,\mathrm{nm}, \mathrm{d}_3{=}3\,\mathrm{nm}$ Clearly the same behavior as above is indefeasible in the case of TWNT. A comparison between the amplitude of the nonlinear vibration of the first layer of TWNT with its linear vibration amplitude is shown in Fig. (**4**) for $X_1 = 3$ and $k = 0$ against the linear period of vibration ($\tau = \omega_b t$). Its worthwhile to say that the discrepancy between the linear and nonlinear amplitudes increases with the increment of the maximum amplitude. In Fig. (**5**), the parameters are $k = 10^7\,\mathrm{N/m^2}$, $c = 0.3\times10^{12}\,\mathrm{N/m^2}$ and $d_2 = 3\mathrm{nm}$. It is observed that with the increase of the aspect ratio of the nanotubes, the nonlinear vibration frequencies of MWNTs decrease. Due to convenience in calculating the nonlinear free vibration frequency ω, the linear vibration frequencies ω_b of SWNT, DWNT and TWNT for all cases are listed in Table **1**.

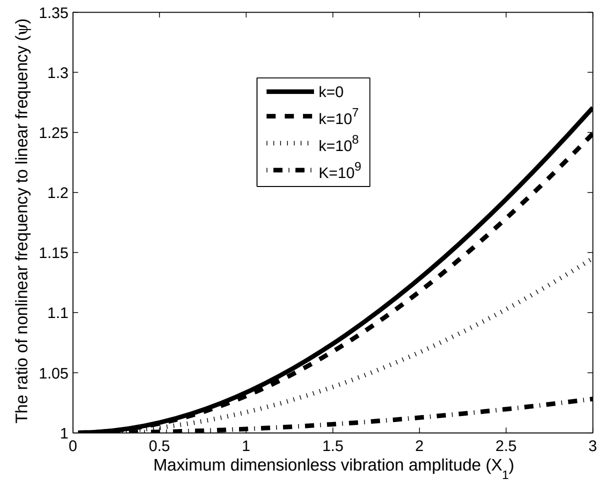

Fig. (3). Effect of spring constant *k* on nonlinear amplitude-frequency response curves of TWNT.

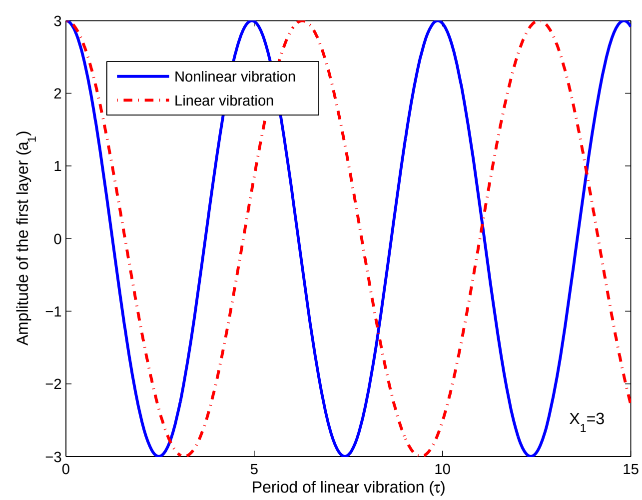

Fig. (4). The nonlinear amplitude of the vibration of the first layer of TWNT.

3. CONCLUSIONS

In this paper, we have studied the problem of the nonlinear vibrations of multiwalled carbon nanotubes with the variational iteration method. Using this technique, a correction functional can be constructed by a general Lagrange multiplier, and the multiplier can be readily obtained by va-

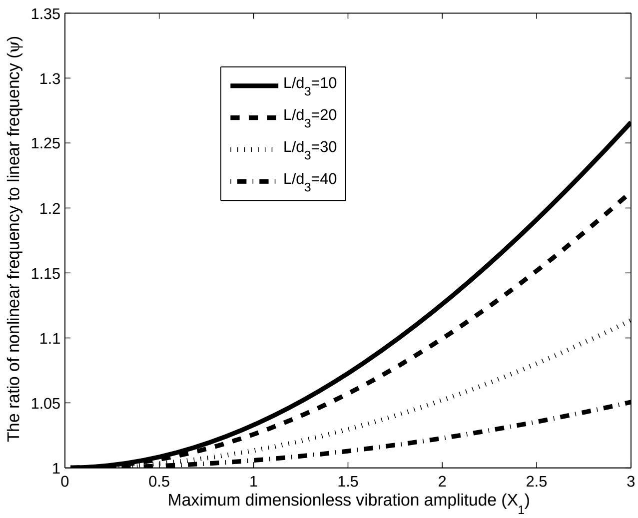

Fig. (5). Effect of aspect ratio L / d_2 on nonlinear amplitude-frequency response curves for TWNT.

Table 1. The Linear Free Vibration Frequencies ω_b of SWNT, DWNT and TWNT in Figs. (1-3)

$k(\mathrm{N} / \mathrm{m}^2)$	ω_b (THz)		
	SWNT	**DWNT**	**TWNT**
0	0.128	0.116	0.111
10^7	0.138	0.122	0.117
10^8	0.209	0.170	0.156
10^9	0.536	0.410	0.365

riational theory. Including the application of restricted variations in correction functional makes it much easier to determine the multiplier. Numerical solutions have been compared with the results obtained *via* IHBM and excellent correlation has been obtained. The results clarify the significance dependency of the nonlinear free vibration of nanotubes to the surrounding elastic medium. The nonlinear vibration frequency of nanotubes rises rapidly with increasing the amplitude especially when the stiffness of the medium is relatively small. For larger stiffnesses (say $k > 10^9 N / m^2$), the nonlinear vibration tends to the linear regime. This method can be easily extended to the multiwalled CNTs with number of walls more than three. It is worthwhile to mention that VIM is straightforward and it is a promising and powerful technique for solving many nonlinear equations arising in mathematical physics.

ACKNOWLEDGMENT

The authors would like to thank referees for their comments which improved the presentation of the paper.

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Received: June 18, 2009 Revised: November 01, 2009 Accepted: November 20, 2009