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Comparisons Between WHEP and Homotopy Perturbation Techniques in Solving Stochastic Cubic Oscillatory Problems

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Abstract: In this paper, nonlinear oscillators under cubic nonlinearity with stochastic inputs are considered. Two techniques are used to introduce approximate solutions, mainly; the Wiener-Hermite expansion and perturbation (WHEP) technique and the homotopy perturbation method (HPM). Some statistical moments are computed using the two methods with the big aid of symbolic computations of mathematica-5. Comparisons are illustrated through figures.

Keywords: Nonlinear stochastic differential equations, Wiener-Hermite expansion, WHEP technique, homotopy perturbation method, HPM, Mathematica.

1. INTRODUCTION

Cubic oscillation arises through many applied models in science and engineering when studying oscillatory systems [1]. These systems can be exposed to a lot of uncertainties through the external forces, the damping coefficient, the frequency and/or the initial or boundary conditions. These input uncertainties cause the output solution process to be also uncertain. For most of the cases, getting the probability density function (p.d.f.) of the solution process may be impossible. So, developing approximate techniques through which approximate statistical moments can be obtained, is an important and necessary work. There are many techniques which can be used to obtain statistical moments of such problems, among of which are the proposed techniques; the WHEP and HPM. The main goal of this paper is to compare these two methods when applied to a cubic nonlinearity problem.

The problem is formulated in section 2. The WHEP and HPM techniques are illustrated and used to solve the general problem and then applied for an illustrative case study in sections 3 and 4 respectively. Comparisons are also illustrated in these sections between the two methods.

2. PROBLEM FORMULATION

In this paper, the following cubic nonlinear oscillatory equation is considered

$$\ddot{x}+2w\zeta\dot{x}+w^2x+\varepsilon w^2x^3=F(t;\omega)\ ,\quad t\in[0,T] \tag{1}$$

under stochastic excitation $F(t;\omega)$ with deterministic initial conditions

$$x(0)=x_0,\quad \dot{x}(0)=\dot{x}_0\,,$$

where w: frequency of oscillation,

ζ : damping coefficient

ε : deterministic nonlinearity scale

$\omega\in(\Omega,\sigma,p)$: a triple probability space with Ω as the sample space , σ is a σ -algebra on events in Ω and P is a probability measure.

3. THE WHEP TECHNIQUE

Since Meecham and his co-workers [2] developed a theory of turbulence involving a truncated Wiener-Hermite expansion (WHE) of the velocity field, many authors studied problems concerning turbulence [3-8]. The nonlinear oscillators were considered as an opened area for the applications of WHE as can be found in [9-15]. There are a lot of applications in boundary value problems [16,17] and generally in different mathematical studies [18, 19].

The application of the WHE aims at finding a truncated series solution to the solution process of differential equations. The truncated series composes of two major parts; the first is the Gaussian part which consists of the first two terms, while the rest of the series constitute the non-Gaussian part. In nonlinear cases, there exists always difficulties of solving the resultant set of deterministic integro-differential equations got from the applications of a set of comprehensive averages on the stochastic integro-differential equation obtained after the direct application of WHE. Many authors introduced different methods to face these obstacles. Among them, the WHEP technique was introduced in [14] using the perturbation technique to solve perturbed nonlinear problems.

The WHE method utilizes the Wiener-Hermite polynomials which are the elements of a complete set of statistically orthogonal random functions [20]. The Wiener-Hermite polynomial $H^{(i)}(t_1,t_2,...t_i)$ satisfies the following recurrence relation:

$$\begin{aligned}H^{(i)}(t_1,t_2,...t_i)&=H^{(i-1)}(t_1,t_2,...t_{i-1}).H^{(1)}(t_i)\\&-\sum_{m=1}^{i-1}H^{(i-2)}(t_{i_1},t_{i_2},...t_{i_{i-2}}).\delta(t_{i-m}-t_i),\ i\geq 2\end{aligned} \tag{2}$$

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where

$$H^{(0)} = 1,$$
$$H^{(1)}(t) = n(t),$$
$$H^{(2)}(t_1,t_2) = H^{(1)}(t_1).H^{(1)}(t_2) - \delta(t_1 - t_2),$$
$$H^{(3)}(t_1,t_2,t_3) = H^{(2)}(t_1,t_2).H^{(1)}(t_3) - H^{(1)}(t_1).\delta(t_2 - t_3) - H^{(1)}(t_2).\delta(t_1 - t_3),$$
$$H^{(4)}(t_1,t_2,t_3,t_4) = H^{(3)}(t_1,t_2,t_3).H^{(1)}(t_4) - H^{(2)}(t_1,t_2).\delta(t_3 - t_4) - H^{(2)}(t_1,t_3).\delta(t_2 - t_4) - H^{(2)}(t_2,t_3).\delta(t_1 - t_4), \quad (3)$$

in which n(t) is the white noise with the following statistical properties

$$E\ \mathrm{n(t)}=0, \qquad E\ \mathrm{n(t_1)}.n(t_2) = \delta(t_1 - t_2), \quad (4)$$

where δ(-) is the Dirac delta function and E denotes the ensemble average operator.

The Wiener-Hermite set is a statistically orthogonal set, i.e.

$$E\ H^{(i)}.H^{(j)} = 0 \quad \forall\ \mathrm{i} \neq \mathrm{j}. \quad (5)$$

The average of almost all H functions vanishes, particularly,

$$E\ H^{(i)} = 0 \quad \text{for i} \geq 1. \quad (6)$$

Due to the completeness of the Wiener-Hermite set, any random function $G(t;\omega)$ can be expanded as

$$G(t;\omega) = G^{(0)}(t) + \int_{-\infty}^{\infty} G^{(1)}(t;t_1)H^{(1)}(t_1)dt_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G^{(2)}(t;t_1,t_2)H^{(2)}(t_1,t_2)dt_1dt_2 + \ldots.. \quad (7)$$

where the first two terms are the Gaussian part of $G(t;\omega)$. The rest of the terms in the expansion represent the non-Gaussian part of $G(t;\omega)$. The average of $G(t;\omega)$ is

$$\mu_G = E\,G(t;\omega) = G^{(0)}(t) \quad (8)$$

The covariance of $G(t;\omega)$ is

$$Cov(G(t;\omega),G(\tau;\omega)) = E(G(t;\omega) - \mu_G(t))(G(\tau;\omega) - \mu_G(\tau))$$
$$= \int_{-\infty}^{\infty} G^{(1)}(t;t_1)G^{(1)}(\tau,t_1)dt_1 + 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G^{(2)}(t;t_1,t_2)G^{(2)}(\tau,t_1,t_2)dt_1dt_2 \quad (9)$$

The variance of $G(t;\omega)$ is

$$Var\,G(t;\omega) = E(G(t;\omega) - \mu_G(t))^2$$
$$= \int_{-\infty}^{\infty} [G^{(1)}(t;t_1)]^2 dt_1 + 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [G^{(2)}(t;t_1,t_2)]^2 dt_1dt_2 \quad (10)$$

In WHEP technique, every deterministic kernel of the expansion of the unknown $x(t;\omega)$, mainly $x^{(j)}(t,-)$ is expanded in the problem small parameter, say ε. In this case the kernel is assumed as follows;

$$x^{(j)}(t,-) = x_0^{(j)}(t,-) + \varepsilon x_1^{(j)}(t,-) + \varepsilon^2 x_2^{(j)}(t,-) + \cdots.$$

Substituting in the original equations, after taking the necessary set of averages, one always has number of equations as the number of unknowns $x_k^{(j)}(t,-)$.

The WHEP technique can be applied on linear or nonlinear perturbed systems described by ordinary or partial differential equations. The solution can be modified in the sense that additional parts of the Wiener-Hermite expansion can always be taken into considerations and the required order of approximations can always be made depending on the computing tool. It can be even run through a package if it is coded in some sort of symbolic languages. The technique was successfully applied to several nonlinear stochastic equations, see [20,22,23,25].

3.1. Case-Study

The cubic nonlinear oscillatory problem, equation (1), is solved using WHEP technique. The second order approximation of the solution process takes the following form:

$$x(t;\omega) = x^{(0)}(t)H^{(0)}(t) + \int_{D_1} x^{(1)}(t,t_1)H^{(1)}(t_1)dt_1 + \iint_{D_2} x^{(2)}(t,t_1,t_2)H^{(2)}(t_1,t_2)dt_1dt_2 + \ldots\ldots.. \quad (11)$$

Applying the WHEP technique , the following equations in the deterministic kernels are obtained:

$$Lx^{(0)}(t) + \varepsilon\omega^2\left\{ \left[x^{(0)}\right]^3 + 8\iiint_D x^{(2)}(t,t_1,t_2)x^{(2)}(t,t_1,t_3)x^{(2)}(t,t_2,t_3)dt_1dt_2dt_3 + 3x^{(0)}(t)\int_D\left[x^{(0)}(t,t_1)\right]^2 dt_1 + 6x^{(0)}(t)\iint_D\left[x^{(2)}(t,t_1,t_2)\right]^2 dt_1dt_2 + 6\iint_D x^{(2)}(t,t_1,t_2)x^{(1)}(t,t_1)x^{(1)}(t,t_2)dt_1dt_2 \right\} = F^{(0)}(t) \quad (12)$$

$$Lx^{(1)}(t,t_1) + \varepsilon\omega^2\left\{ 3\left[x^{(0)}(t)\right]^2 x^{(1)}(t,t_1) + 3x^{(1)}(t,t_1)\int_D\left[x^{(1)}(t,t_1)\right]^2 dt_1 + 12x^{(0)}(t)\int_D x^{(2)}(t,t_1,t_2)x^{(1)}(t,t_2)dt_2 + 6x^{(1)}(t,t_1)\iint_D\left[x^{(2)}(t,t_1,t_2)\right]^2 dt_1dt_2 + 24\iint_D x^{(1)}(t,t_2)x^{(2)}(t,t_1,t_3)x^{(3)}(t,t_2,t_3)dt_2dt_3 \right\} = F^{(1)}(t,t_1) \quad (13)$$

$$\begin{aligned}Lx^{(2)}(t,t_1,t_2)+\varepsilon\omega^2\Big\{&3\left[x^{(0)}(t)\right]^2x^{(2)}(t,t_1,t_2)+3x^{(0)}(t)\int_D x^{(2)}(t,t_1,t_3)x^{(2)}(t,t_2,t_3)dt_3\\&+3x^{(2)}(t,t_1,t_2)\int_D\left[x^{(1)}(t,t_3)\right]^2dt_3+6x^{(1)}(t,t_1)\int_D x^{(1)}(t,t_3)x^{(2)}(t,t_2,t_3)dt_3\\&+6x^{(2)}(t,t_2)\int_D x^{(2)}(t,t_3)x^{(2)}(t,t_1,t_3)dt_3\\&+24\iint_D x^{(2)}(t,t_1,t_3)x^{(2)}(t,t_2,t_4)x^{(2)}(t,t_3,t_4)dt_3dt_4\\&+6x^{(2)}(t,t_1,t_2)\iint_D\left[x^{(2)}(t,t_3,t_4)\right]^2dt_3dt_4\\&=F^{(2)}(t,t_1,t_2)\end{aligned} \tag{14}$$

Let us take the simple case of evaluating the only Gaussian part (first order approximation) of the solution process. In this case, the ensemble average is

$$\mu_x(t)=x^{(0)}(t)\,, \tag{15}$$

and the variance is

$$\sigma_x^2(t)=\int_{-\infty}^{\infty}\left[x^{(1)}(t;t_1)\right]^2dt_1\ . \tag{16}$$

In this case, the WHEP technique uses the following expansion for its deterministic kernels,

$$x^{(i)}(t)=x_0^{(i)}+\varepsilon x_1^{(i)}+\varepsilon^2x_2^{(i)}+\varepsilon^3x_3^{(i)}+---,\ i=0,1, \tag{17}$$

where the first two terms consider the first correction (up to ε), the first three terms represent the second correction (up to ε^2) and so on. This means that we have a lot of corrections possibilities within each order of approximation.

3.1.1. Example

Let us take

$$F(t;\omega)=e^{-t}+\varepsilon\, n(t;\omega)\ , \tag{18}$$

Where $n(t;\omega)$ is white noise. Solving using the WHEP technique, the following results are obtained, see Figs. (**1**, **2** and **3**).

4. THE HOMOTOPY PERTURBATION METHOD (HPM)

In this technique [21-25], a parameter $p\in[0,1]$ is embedded in a homotopy function $v(r,p):\phi\times[0,1]\to\Re$ which satisfies

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(r)]=0 \tag{19}$$

where u_0 is an initial approximation to the solution of the equation

$$A(u)-f(r)=0,\ r\in\phi \tag{20}$$

with boundary conditions

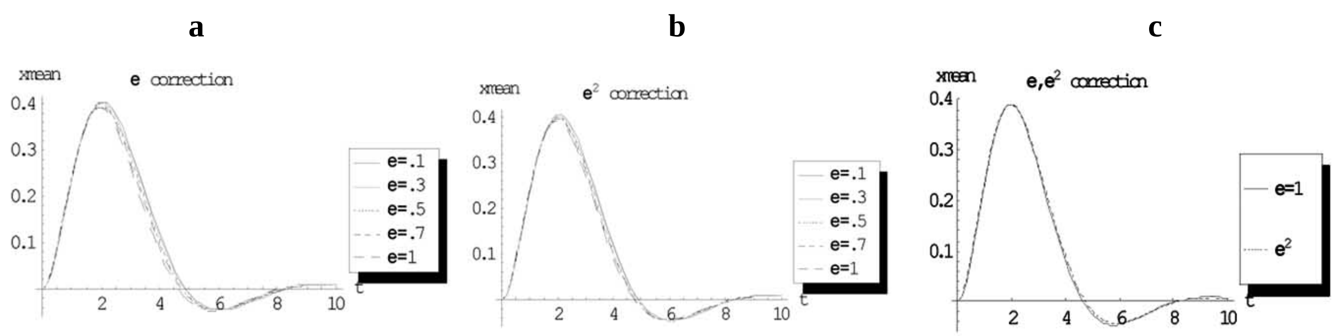

Fig. (1). a: The first order approximation and first correction of the mean at different ε values (WHEP).
b: The first order approximation and second correction level of the mean at different ε values (WHEP).
c: Comparison between the two correction levels of the mean at $\varepsilon=1$ (WHEP).

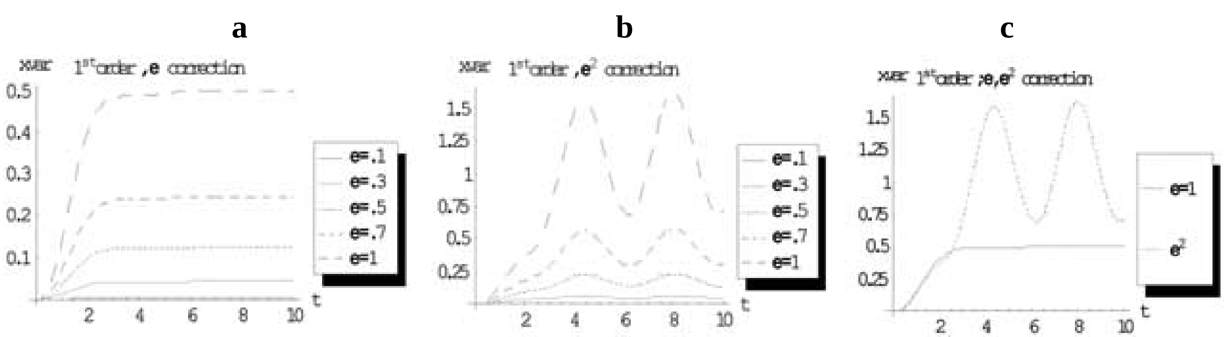

Fig. (2). a: The first order approximation and first correction of the variance at different ε values (WHEP).
b: The first order approximation and second correction level of the variance at ε values (WHEP).
c: Comparison between the two correction levels of the variance at $\varepsilon=1$ (WHEP).

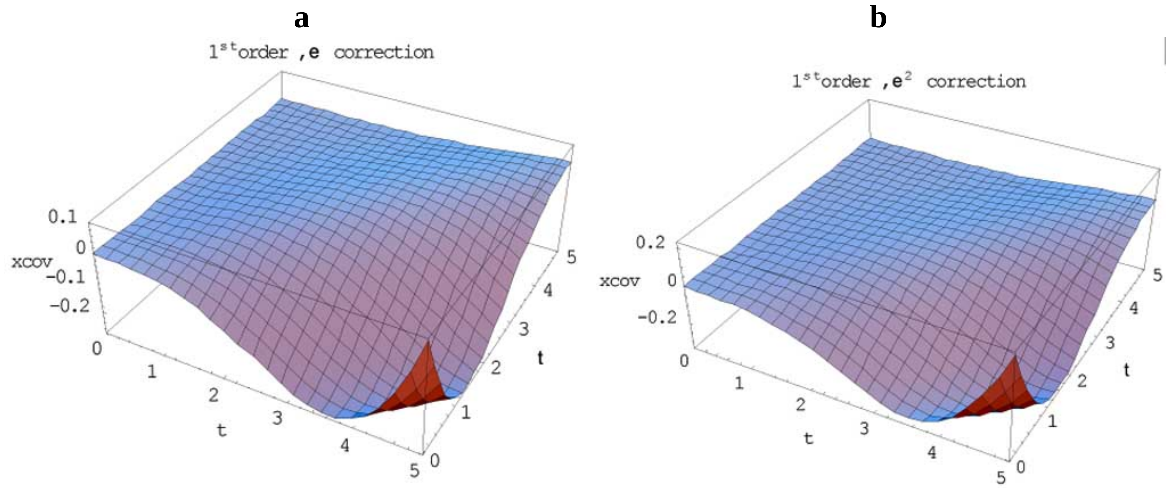

Fig. (3). a: The first order approximation and first correction of the covariance at $\varepsilon = .5$ (WHEP).
b: The first order approximation and second correction of the covariance at $\varepsilon = .5$ (WHEP).

$$B(u, \frac{\partial u}{\partial n}) = 0, r \in \Gamma \tag{21}$$

in which A is a nonlinear differential operator which can be decomposed into a linear operator L and a nonlinear operator N, B is a boundary operator, f(r) is a known analytic function and Γ is the boundary of ϕ. The homotopy introduces a continuously deformed solution for the case of p=0 , $L(v) - L(u_0) = 0$, to the case of p=1, $A(v) - f(r) = 0$, which is the original equation (19). This is the basic idea of the homotopy method which is to continuously deform from a simple problem (and easy to solve) into the difficult problem under study [26].

The basic assumption of the HPM method is that the solution of the original equation (19) can be expanded as a power series in p as:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + ... \tag{22}$$

Now, setting p=1, the approximate solution of equation (22) is obtained as:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + ... \tag{23}$$

The rate of convergence of the method depends greatly on the initial approximation v_0 which is considered as the main disadvantage of HPM.

Applying the HPM technique on the proposed example in 3.1.1, one can get the following results when getting the fourth approximation:

$$\mu_{x(t)} = I(t) - \varepsilon w^2 \int_0^t h(t-s)\left[I^3(s) + 3\varepsilon^2 I(s) \int_0^s h^2(s-v)dv \right] ds \tag{24}$$

$$\sigma^2_{x(t)} = 9\varepsilon^4 w^4 \int_0^t h(t-s_1) \int_0^t h(t-s_2)$$

$$\left[I^2(s_1) I^2(s_2) + 3\varepsilon^2 I(s) \int_0^{s_1} h(s_1 - v) h(s_2 - v) dv \right] ds_2 ds_1 \tag{25}$$

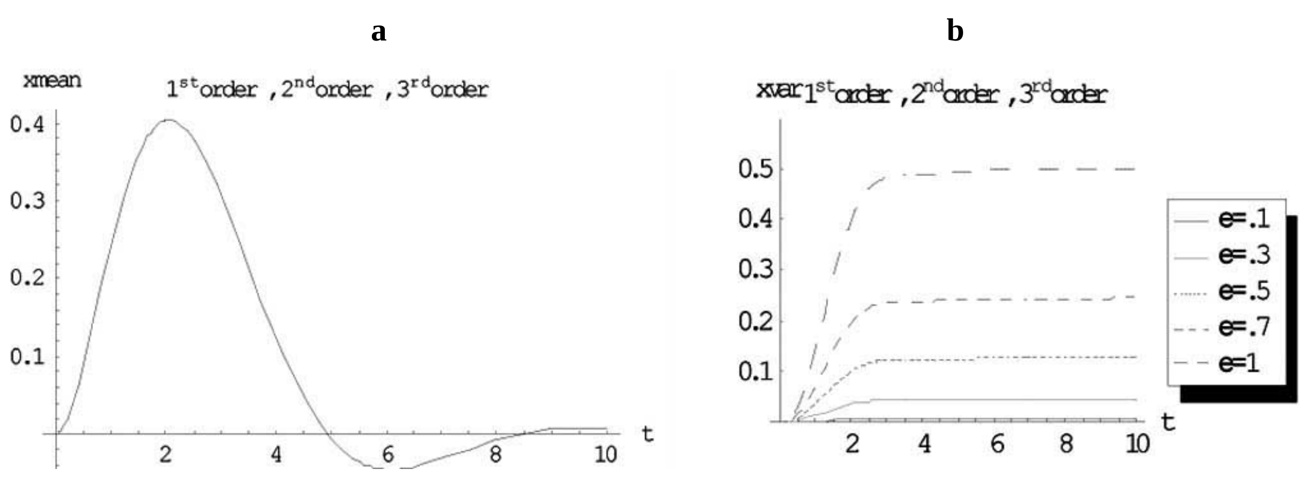

Fig. (4). a: The first, second and third order approximations of the mean (HPM).
b: The first ,second and third order approximations of the variance at different ε (HPM).

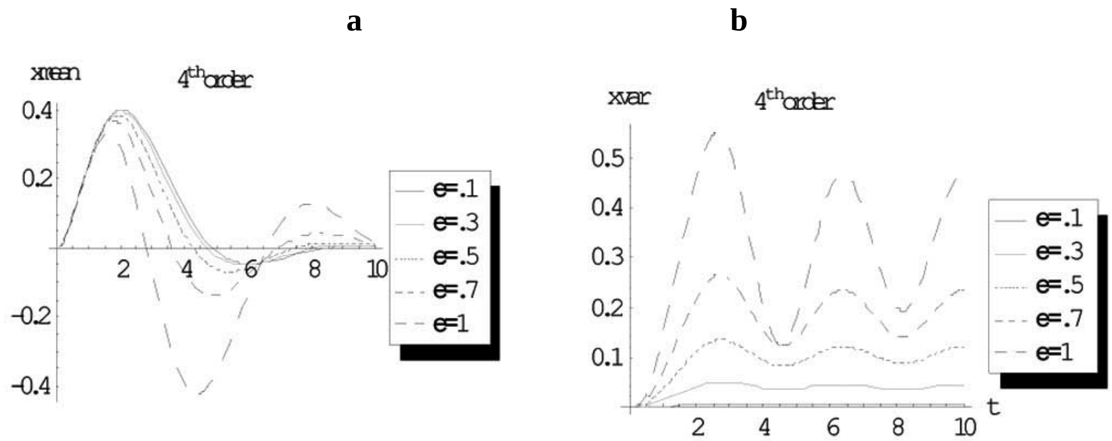

Fig. (5). a: The fourth order approximation of the mean for different ε (HPM).
b: The fourth order approximation of the variance for different ε (HPM).

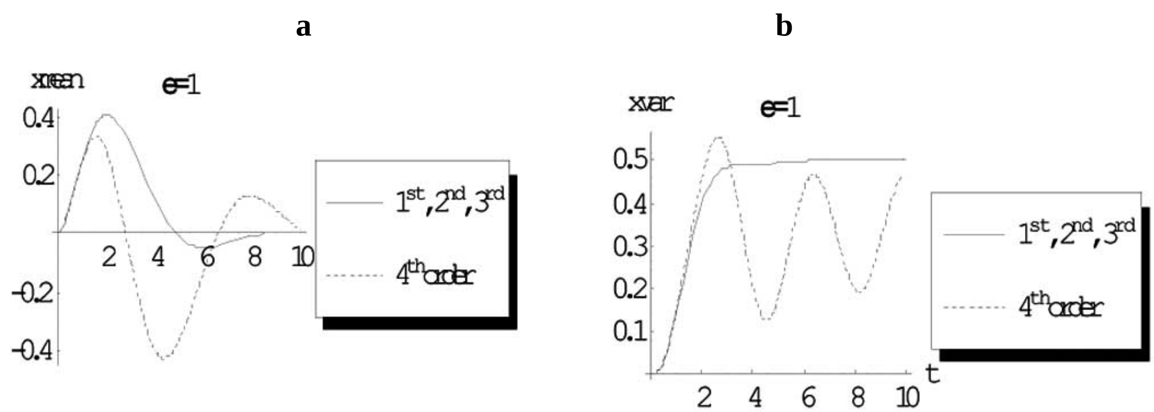

Fig. (6). a: A comparison between first , second , third order and the fourth order of the mean (HPM). $\varepsilon = 1$ at.
b: A comparison between first , second , third order and the fourth order of the (HPM). $\varepsilon = 1$ variance at.

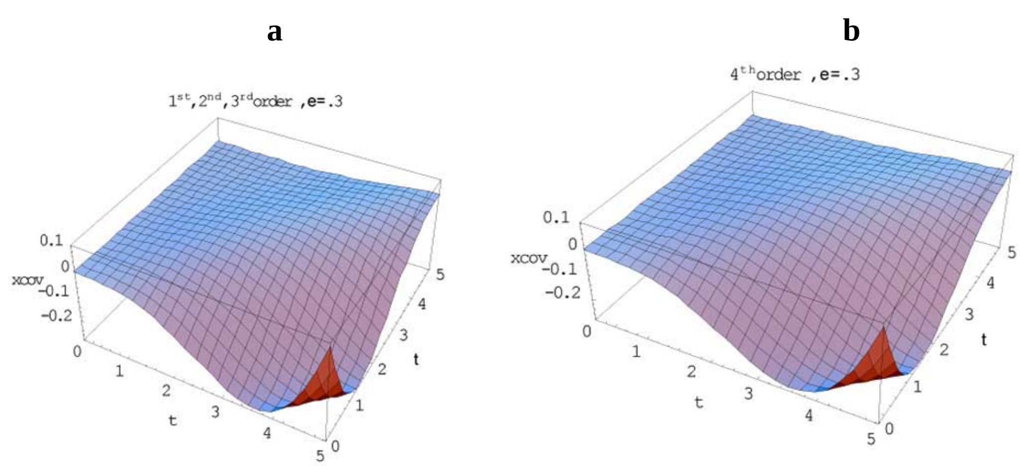

Fig. (7). a: The first, second and third order approximation of the covariance at $\varepsilon = .3$ (HPM).
b: The fourth order approximation of (HPM). $\varepsilon = .3$ the covariance at.

where

$$I(t) = \int_0^t e^{-v} h(t-v) dv \qquad (26)$$

Figs. (**4** and **5**) are some illustrations for the previous results:

5. COMPARISONS BETWEEN WHEP AND HMP

Now let us compare between some results of the different two methods, mainly the WHEP technique and the homotopy perturbation method (HPM), see Figs. (**6-9**):

a **b**

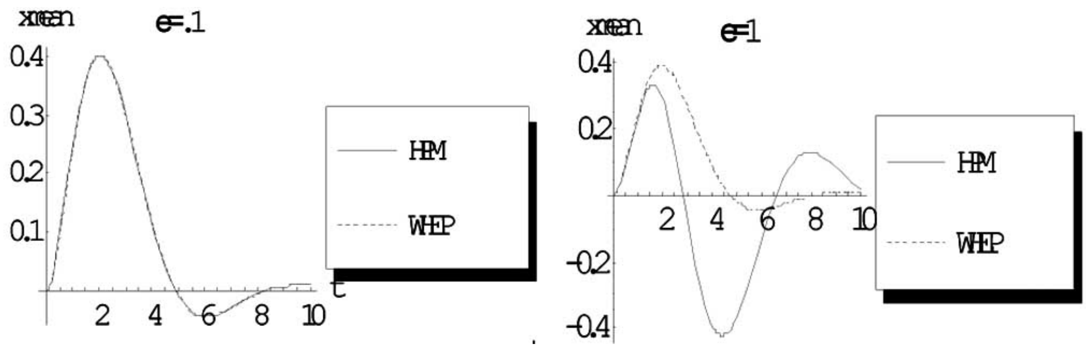

Fig. (8). a: A comparison between HPM and WEEP of the mean at $\varepsilon = .1$.

b: A comparison between HPM and WHEP of the mean at $\varepsilon = 1$.

a **b**

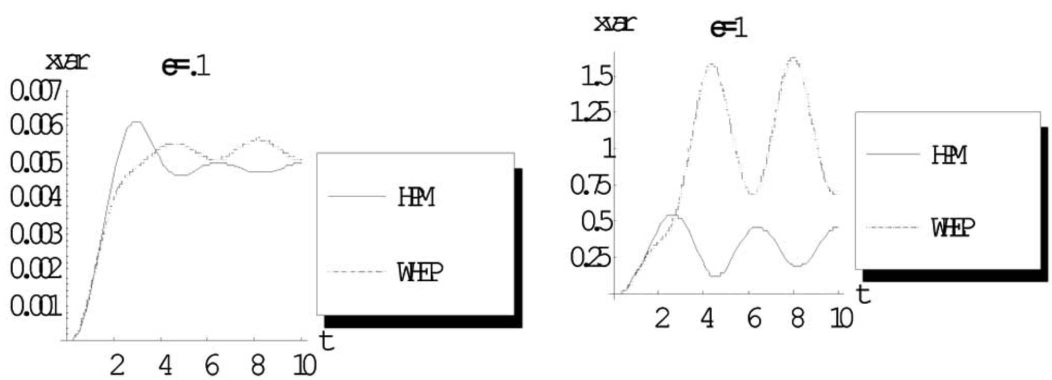

Fig. (9). a: Comparison between HPM and WHEP of the variance at $\varepsilon = .1$.

b: A comparison between HPM and WHEP of the variance at $\varepsilon = 1$.

5. CONCLUSIONS

Concerning the cubic nonlinearity problem and the prototype example used for illustrating the efficiency of the processed approximation techniques, the WHEP technique can be corrected for each order of approximation while the HPM method is very sensitive to the choice of the initial condition. The comparisons between WHEP technique and HMP are illustrated through figures and showed near results according to the order of approximation.

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