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On the Approximate Solution of Non-Linear Stochastic Diffusion Equation Using Symbolic WHEP

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Abstract: In this paper, a stochastic perturbed nonlinear diffusion equation is studied under a stochastic nonlinear non-homogeneity. The Pickard approximation method is used to introduce a reference first order approximate solution. Under different correction levels, the WHEP technique is used to obtain approximate solutions. Using Mathematica-5, the solution algorithm is operated and several comparisons among correction levels together with error curves have been demonstrated. The method of solution is illustrated through case studies and figures.

Keywords: Stochastic nonlinear diffusion equation, Eigenfunction expansion, Pickard approximation, WHEP technique.

1. INTRODUCTION

The study of random solutions of partial differential equations was initiated by Kampe de Feriet in 1955 [1]. In his valuable survey on the theory of random equations, Bharucha-Reid showed how a stochastic heat equation of Cauchy type can be solved using the stochastic integrals theory [2]. In 1973, Lo Dato V. [3] considered the stochastic velocity field and the Navier-Stokes equation and discussed the mathematical problems associated with it. Becus A. Georges [4] introduced a general solution for the heat conduction problem with a random source term and random initial and boundary conditions. Many authors investigated the stochastic diffusion equation under different views, see [5-11].

El-Tawil M. used the Wiener-Hermite expansion together with perturbation theory (WHEP technique) to solve a perturbed nonlinear stochastic diffusion equation [12]. The technique has been then developed to be applied on non-perturbed differential equations using the homotopy perturbation method and is called Homotopy WHEP [13].

In this paper, the diffusion equations with linear and nonlinear losses and stochastic nonhomogeneity are solved using WHEP technique under different correction levels. The main goal of the paper is to show that the correction levels are true corrections for the solution where the errors are decreased whenever the corrections are updated.

Two techniques are used to obtain the ensemble average, covariance and variance of the solution process, mainly the WHEP technique and the Pickard approximation which is used to get a reference solution. In section 2, the linear case is solved. The nonlinear case is analysed in section 3 using Pickard approximation to get a reference solution. The WHEP technique is processed in section 4 using different correction levels where the results of the execution of the algorithms and comments on figures are included. Comparisons are concluded in section 5.

2. THE DIFFUSION EQUATION UNDER LINEAR LOSSES

Considering the following stochastic diffusion equation with $f(t,x;\omega)$ as random nonhomogeneity

$$\frac{\partial u(t,x;\omega)}{\partial t}=\frac{\partial^2 u}{\partial x^2}-\lambda u+f(t,x;\omega);\ (t,x)\in(0,\infty)\times(0,L),\qquad (1)$$
$$u(t,0)=0,u(t,L)=0 \text{ and } u(0,x)=g(x),$$

where ω is a random outcome of a triple probability space (Ω,χ,P) in which Ω is a sample space, χ is σ- field associated with Ω and P is a probability measure and λu is a linear losses term.

Applying the eigenfunction expansion technique [14], the following general expressions for the ensemble average, covariance and variance of the solution process are obtained as follows:

$$\mu_u(t,x)=e^{-\lambda t}\Big(\sum_{n=0}^{\infty}T_n e^{-(\frac{n\pi\alpha}{L})^2 t}\sin\frac{n\pi}{L}x+\sum_{n=0}^{\infty}E\,I_n(t)\sin\frac{n\pi}{L}x\Big),$$
$$T_n=\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx,$$
$$E\,I_n=e^{-(\frac{n\pi\alpha}{L})^2 t}\int_0^t e^{(\frac{n\pi\alpha}{L})^2\tau}E\,F_n(\tau)\,d\tau,\qquad (2)$$
$$E\,F_n(\tau)=\frac{2e^{\lambda t}}{L}\int_0^L E\,f(t,x;\omega)\sin\frac{n\pi}{L}x\,dx.$$

where E denotes the average operator,

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$$
\begin{aligned}
&Cov(u(t_1,x),u(t_2,x)) = e^{-\lambda(t_1+t_2)}\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}V_{nm}\sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\\
&V_{nm} = e^{-(\frac{n\pi\alpha}{L})^2 t_1}e^{-(\frac{m\pi\alpha}{L})^2 t_2}\int_0^{t_1}\int_0^{t_2}e^{(\frac{n\pi\alpha}{L})^2\tau_1}e^{(\frac{m\pi\alpha}{L})^2\tau_2}MFnFm\,d\tau_2\,d\tau_1\\
&MFnFm = \frac{4}{L^2}e^{\lambda(\tau_1+\tau_2)}\int_0^L\int_0^L E\,f(\tau_1,x_1;\omega)f(\tau_2,x_2;\omega)\\
&\qquad\sin\frac{n\pi}{L}x_1\sin\frac{m\pi}{L}x_2\,dx_2\,dx_1.
\end{aligned}
\tag{3}
$$

$$Varu(t,x) = Cov(u(t_1,x),u(t_2,x))\Big|_{t_1=t_2=t}. \tag{4}$$

Considering white noise, $\sigma n(x;\omega)$ *, or simply* $\sigma n(x)$ as a random nonhomogeneity with a deterministic scale σ, the following results are obtained using the statistical properties of white noise

$$\mu_u(t,x) = e^{-\lambda t}\Big(\sum_{n=0}^{\infty}T_n e^{-(\frac{n\pi\alpha}{L})^2 t}\sin\frac{n\pi}{L}x\Big). \tag{5}$$

$$
\begin{aligned}
&Cov(u(t_1,x),u(t_2,x)) = e^{-\lambda(t_1+t_2)}\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}V_{nm}\sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\\
&V_{nm} = e^{-(\frac{n\pi\alpha}{L})^2 t_1}e^{-(\frac{m\pi\alpha}{L})^2 t_2}\int_0^{t_1}\int_0^{t_2}e^{(\frac{n\pi\alpha}{L})^2\tau_1}e^{(\frac{m\pi\alpha}{L})^2\tau_2}MFnFm\,d\tau_2\,d\tau_1\\
&MFnFm = \frac{4\sigma^2}{L^2}e^{\lambda(\tau_1+\tau_2)}\int_0^L\sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\,dx
\end{aligned}
\tag{6}
$$

Expressions (5) and (6) are the exact solutions of the linear problem. For the sake of validating the WHEP technique [12], the technique was applied on the linear problem where the Gaussian solution can be expanded, for a first order approximation, as follows:

$$u(t,x;\omega) = u^{(0)}(t,x) + \int_0^L u^{(1)}(t,x;x_1)H^{(1)}(x_1;\omega)dx_1. \tag{7}$$

where $u^{(0)}(t,x)$ and $u^{(1)}(t,x;x_1)$ are deterministic kernels to be evaluated and $H^{(1)}(x_1;\omega)$ is the Wiener-Hermite polynomial of order one; mainly is the same as white noise $n(x_1)$. In this case, the average takes the form

$$\mu_u = u^{(0)}(t,x), \tag{8}$$

while the covariance takes the form

$$Cov(u(t_1,x),u(t_2,x)) = \int_0^L u^{(1)}(t_1,x;x_1)u^{(1)}(t_2,x;x_1)dx_1. \tag{9}$$

In the WHEP technique, the deterministic kernels are expanded as a power series in the parameter λ as follows:

$$u^{(j)}(t,x) = u_0^{(j)}(t,x) + \lambda u_1^{(j)}(t,x) + \lambda^2 u_2^{(j)}(t,x) + \cdots,\ j = 0,1. \tag{10}$$

Following the technique algorithm [12], the following results are obtained for the first correction (up to λ) :

$$u_0^{(0)}(t,x) = \sum_{n=0}^{\infty}T_n e^{-(\frac{n\pi\alpha}{L})^2 t}\sin\frac{n\pi}{L}x, \tag{11}$$

$$
\begin{aligned}
&u_1^{(0)}(t,x) = \sum_{n=0}^{\infty}I_0\sin\frac{n\pi}{L}x,\\
&I_0 = \int_0^t e^{-(\frac{n\pi\alpha}{L})^2(t-\tau)}F_n(\tau)d\tau,\\
&F_n(t) = \frac{-2}{L}\int_0^L u_0^{(0)}(t,x)\sin\frac{n\pi}{L}x\,dx,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&u_0^{(1)}(t,x;x_1) = \sum_{n=0}^{\infty}I_1\sin\frac{n\pi}{L}x,\\
&I_1 = \int_0^t e^{-(\frac{n\pi\alpha}{L})^2(t-\tau)}F_{n1}(\tau)d\tau,\\
&F_{n1}(t) = \frac{2}{L}\int_0^L\delta(x-x_1)\sin\frac{n\pi}{L}x\,dx\\
&\qquad = \frac{2}{L}\sin\frac{n\pi}{L}x_1,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&u_1^{(1)}(t,x;x_1) = \sum_{n=0}^{\infty}I_{11}\sin\frac{n\pi}{L}x,\\
&I_{11} = \int_0^t e^{-(\frac{n\pi\alpha}{L})^2(t-\tau)}F_{n11}(\tau)d\tau,\\
&F_{n11}(t) = \frac{-2}{L}\int_0^L u_0^{(1)}(t,x;x_1)\sin\frac{n\pi}{L}x\,dx.
\end{aligned}
\tag{14}
$$

The first order, first correction solution is

$$\mu_{u1} = u_0^{(0)}(t,x) + \lambda u_1^{(0)}(t,x) \tag{15}$$

$$
\begin{aligned}
Cov(u(t_1,x),u(t_2,x)) = &\int_0^L u_0^{(1)}(t_1,x;y)u_0^{(1)}(t_2,x;y)dy\\
&+\lambda\left[\begin{aligned}&\int_0^L u_1^{(1)}(t_1,x;y)u_0^{(1)}(t_2,x;y)dy +\\ &\int_0^L u_1^{(1)}(t_2,x;y)u_0^{(1)}(t_1,x;y)dy\end{aligned}\right]\\
&+\lambda^2\left[\int_0^L u_1^{(1)}(t_1,x;y)u_1^{(1)}(t_2,x;y)dy\right]
\end{aligned}
\tag{16}
$$

The results for the second correction (up to λ^2) are

$$
\begin{aligned}
&u_2^{(0)}(t,x) = \sum_{n=0}^{\infty}I_{02}\sin\frac{n\pi}{L}x,\\
&I_{02} = \int_0^t e^{-(\frac{n\pi\alpha}{L})^2(t-\tau)}F_{n02}(\tau)d\tau,\\
&F_{n02}(t) = \frac{-2}{L}\int_0^L u_1^{(0)}(t,x)\sin\frac{n\pi}{L}x\,dx,
\end{aligned}
\tag{17}
$$

$$u_2^{(1)}(t,x;x_1)=\sum_{n=0}^{\infty} I_{12}\sin\frac{n\pi}{L}x,$$
$$I_{12}=\int_0^t e^{-(\frac{n\pi\alpha}{L})^2(t-\tau)}F_{n12}(\tau)d\tau, \quad (18)$$
$$F_{n12}(t)=\frac{-2}{L}\int_0^L u_1^{(1)}(t,x;x_1)\sin\frac{n\pi}{L}xdx.$$

The first order, second correction solution takes the following form

$$\mu_{u1}=u_0^{(0)}(t,x)+\lambda u_1^{(0)}(t,x)+\lambda^2 u_2^{(0)}(t,x)\,, \quad (19)$$

$$\begin{aligned}Cov(u(t_1,x),u(t_2,x)) &= \int_0^L u_0^{(1)}(t_1,x;y)u_0^{(1)}(t_2,x;y)dy \\ &+\lambda\left[\begin{array}{l}\int_0^L u_1^{(1)}(t_1,x;y)u_0^{(1)}(t_2,x;y)dy+ \\ \int_0^L u_1^{(1)}(t_2,x;y)u_0^{(1)}(t_1,x;y)dy\end{array}\right] \\ &+\lambda^2\left[\begin{array}{l}\int_0^L u_0^{(1)}(t_1,x;y)u_2^{(1)}(t_2,x;y)dy+ \\ \int_0^L u_1^{(1)}(t_1,x;y)u_1^{(1)}(t_2,x;y)dy+ \\ \int_0^L u_2^{(1)}(t_1,x;y)u_0^{(1)}(t_2,x;y)dy\end{array}\right] \\ &+\lambda^3\left[\begin{array}{l}\int_0^L u_1^{(1)}(t_1,x;y)u_2^{(1)}(t_2,x;y)dy+ \\ \int_0^L u_2^{(1)}(t_1,x;y)u_1^{(1)}(t_2,x;y)dy\end{array}\right] \\ &+\lambda^4\left[\int_0^L u_1^{(1)}(t_1,x;y)u_2^{(1)}(t_2,x;y)dy\right]\end{aligned} \quad (20)$$

Following the same way, one can obtain corrections for any required order. Figs. (**1-4**) illustrate the change of the average and variance errors, compared to the exact solutions, with time at different space and correction levels.

One can notice that the higher correction is better than the preceding ones which means that we have true corrections in this case. We can also notice that the approximate solution approaches the exact one, i.e. convergent to the exact solution.

3. THE DIFFUSION EQUATION UNDER NONLINEAR LOSSES, A REFERENCE SOLUTION

Let us consider the following stochastic nonlinear-diffusion equation with nonlinear losses, $\lambda.u^2$:

$$\frac{\partial u(t,x;\omega)}{\partial t}=\frac{\partial^2 u}{\partial x^2}-\lambda.u^2+\sigma.n(x;\omega);\ (t,x)\in(0,\infty)\times(0,L), \quad (21)$$
$$u(t,0)=0,u(t,L)=0 \text{ and } u(0,x)=g(x).$$

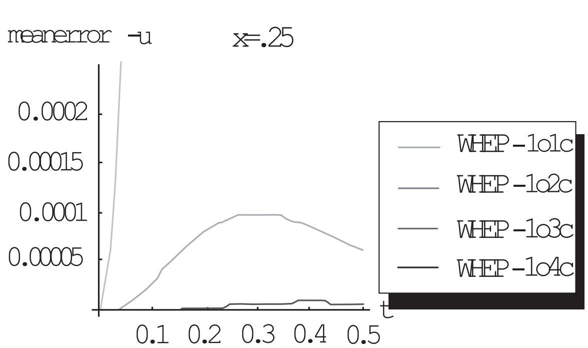
where λ is a deterministic scale for the nonlinear term. The physical meaning of the nonlinear term is that there exists a loss proportional to u^2. The Pickard approximation is used to get a reference approximate solution to which the WHEP corrections solutions are referred. In this technique, the linear part of the differential operator is kept in the left hand side of the equation whereas the rest of the nonlinear terms are moved to the right part. The successive Pickard approximations are processed according to letting the L.H.S. as the $n+1$ approximation for the solution process depending on the n^{th} approximation in the R.H.S, $n\geq 0$. Following this routine, we get the following iterative equations:

$$\frac{\partial u_0}{\partial t}-\frac{\partial^2 u_0}{\partial x^2}-\sigma n(x;\omega)=0, \quad (22)$$
$$u_0(0,x)=g(x),u_0(t,0)=u_0(t,L)=0$$

$$\frac{\partial u_{n+1}}{\partial t}-\frac{\partial^2 u_{n+1}}{\partial x^2}-\sigma n(x;\omega)=-\varepsilon u_n,\ n\geq 0, \quad (23)$$
$$u_{n+1}(0,x)=g(x),u_{n+1}(t,0)=u_{n+1}(t,L)=0$$

Using eigenfunction expansion, the following general solutions are got.

a



b

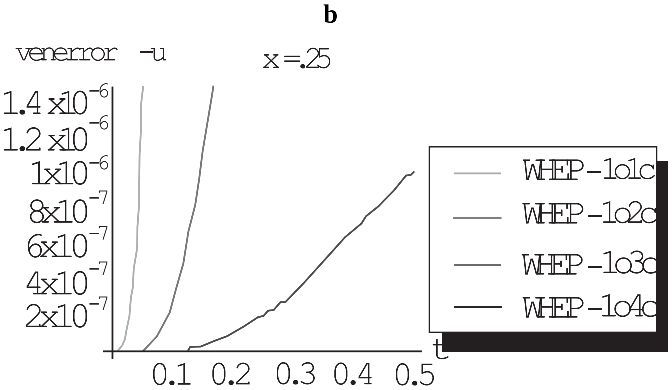


Fig. (1). a. The change of the mean error of u with time t at x =.25 for different corrections levels ($L=1, M=1, \sigma=1, g(x)=x$).
b. The change of the variance error of u with time t at x =.25 for different corrections levels ($L=1, M=1, \sigma=1, g(x)=x$).

a **b**

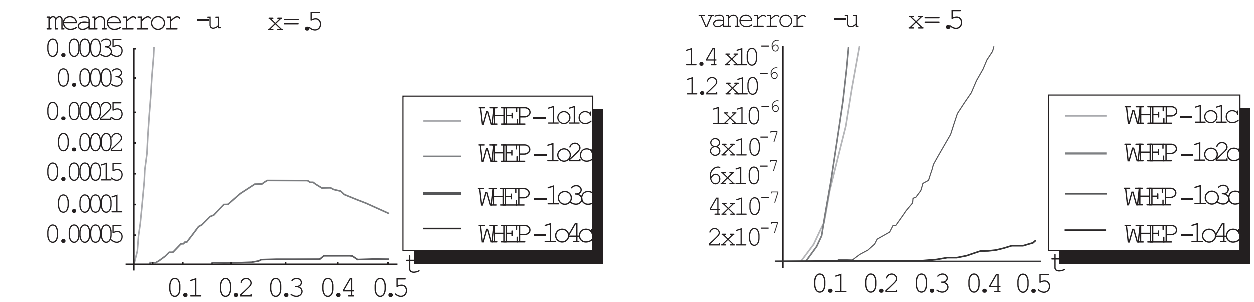


Fig. (2). a. The change of the mean error of u with time t at x =.5 for different corrections levels ($L=1, M=1, \sigma=1, g(x)=x$).
b. The change of the variance error of u with time t at x =.5 for different corrections levels ($L=1, M=1, \sigma=1, g(x)=x$).

a **b**

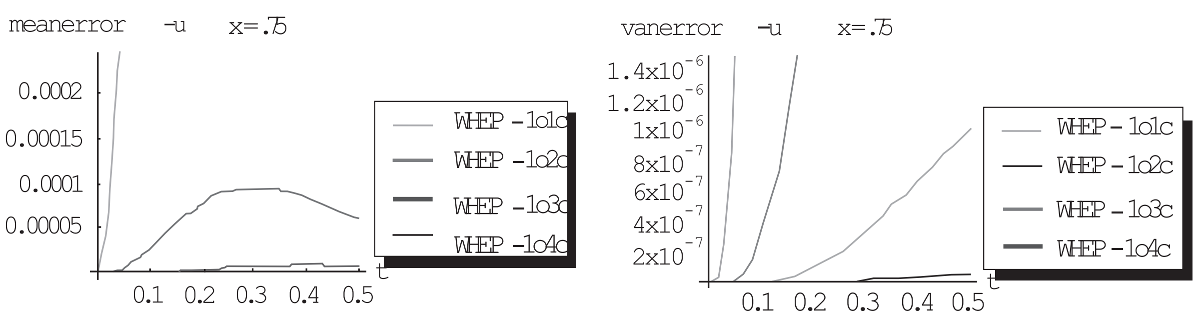


Fig. (3). a. The change of the mean error of u with time t at x =.75 for different corrections levels ($L=1, M=1, \sigma=1, g(x)=x$).
b. The change of the variance error of u with time t at x =.75 for different corrections levels ($L=1, M=1, \sigma=1, g(x)=x$).

a **b**

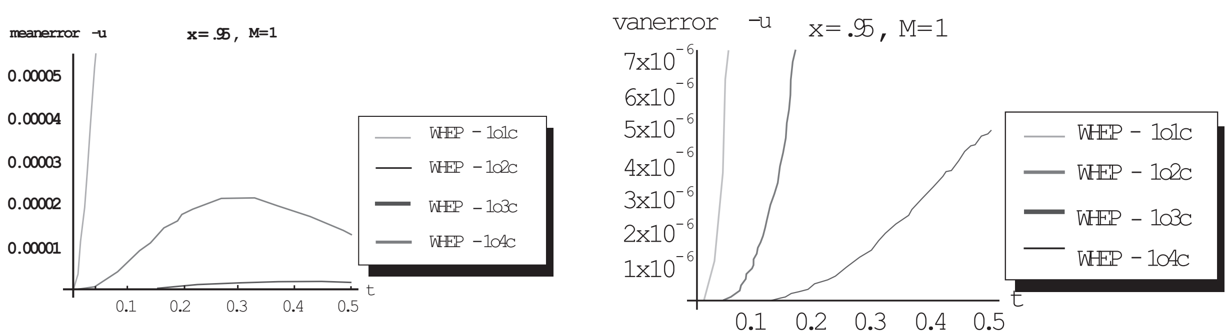


Fig. (4). a. The change of the mean error of u with time t at x =.95 for different corrections levels ($L=1, M=1, \sigma=1, \varphi(x)=x$).
b. The change of the variance error of u with time t at x =.95 for different corrections levels ($L=1, M=1, \sigma=1, \varphi(x)=x$).

$$u_0(t,x)=\sum_{n=0}^{\infty} T_n e^{-(\frac{n\pi}{L})^2 t} \sin\frac{n\pi}{L}x+\sum_{n=0}^{\infty} I_{n_0}(t)\sin\frac{n\pi}{L}x \tag{24}$$

where

$$I_{n_0}(t)=\int_0^t e^{-(\frac{n\pi}{L})^2(t-\tau)} F_{n_0}(\tau)\,d\tau\,, \tag{25}$$

$$F_{n_0}(t)=\frac{2\sigma}{L}\int_0^L n(x)\sin\frac{n\pi}{L}x\,dx\,. \tag{26}$$

Also,

$$u_{n+1}(t,x)=\sum_{m=0}^{\infty} T_{n_{n+1}} e^{-(\frac{m\pi}{L})^2 t} \sin\frac{m\pi}{L}x+\sum_{m=0}^{\infty} I_{n_{n+1}}(t)\sin\frac{m\pi}{L}x \tag{27}$$

where

$$T_{n_{n+1}}=\frac{2}{L}\int_0^L \varphi(x)\sin\frac{m\pi}{L}x\,dx\,, \tag{28}$$

$$I_{n_{n+1}}(t)=\int_0^t e^{-(\frac{m\pi}{L})^2(t-\tau)} F_{n_{n+1}}(\tau)\,d\tau \tag{29}$$

$$F_{n_{n+1}}(t)=\frac{2\sigma}{L}\int_0^L n(x)\sin\frac{m\pi}{L}x\,dx-\frac{2\varepsilon}{L}\int_0^L u_n^2(t,x)\sin\frac{m\pi}{L}x\,dx \tag{30}$$

If the convergence of the process is insured, one can obtain the solution as

$$u(t,x)=\lim_{n\to\infty}u_n(t,x). \tag{31}$$

One can notice that all order of approximations are stochastic processes. The ensemble average of the zero order approximation is obtained as

$$\mu_{u_0}(t,x)=\sum_{n=0}^{\infty}T_n e^{-(\frac{n\pi}{L})^2 t}\sin\frac{n\pi}{L}x. \tag{32}$$

The covariance of u_0 is given by

$$Cov(u_0(t,x_1),u_0(t,x_2))=\frac{4\sigma^2}{L^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}I_{n,m}(\int_0^t e^{-(\frac{n\pi}{L})^2(t-\tau)}d\tau)(\int_0^t e^{-(\frac{m\pi}{L})^2(t-\tau)}d\tau)\sin\frac{n\pi}{L}x_1\sin\frac{m\pi}{L}x_2 \tag{33}$$

where

$$I_{n,m}=\int_0^L\sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\,dx. \tag{34}$$

The variance is

$$\sigma_{u_0}^2(t,x)=\frac{4\sigma^2}{L^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}I_{n,m}(\int_0^t e^{-(\frac{n\pi}{L})^2(t-\tau)}d\tau)(\int_0^t e^{-(\frac{m\pi}{L})^2(t-\tau)}d\tau)\sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x. \tag{35}$$

The following results for the first order approximation are obtained:

$$\mu_{u_1}(t,x)=\sum_{n=0}^{\infty}T_n e^{-(\frac{n\pi}{L})^2 t}\sin\frac{n\pi}{L}x+\sum_{n=0}^{\infty}(\int_0^t e^{-(\frac{n\pi}{L})^2(t-\tau)}E\,F_{n_1}(\tau)\,d\tau)\sin\frac{n\pi}{L}x \tag{36}$$

where

$$E\,F_{n_1}(t)=\frac{-2\lambda}{L}\int_0^L(\sigma_{u_0}^2+\mu_{u_0}^2)\sin\frac{n\pi}{L}x\,dx, \tag{37}$$

$$\begin{aligned}Cov(u_1(t,x_1),u_1(t,x_2))&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}EI_{n_1}(t)I_{m_1}(t)\sin\frac{n\pi}{L}x_1\sin\frac{m\pi}{L}x_2\\&-\sum_{n=0}^{\infty}(\int_0^t e^{-(\frac{n\pi}{L})^2(t-\tau)}EF_{n_1}(\tau)d\tau)\sin\frac{n\pi}{L}x_2.\sum_{n=0}^{\infty}EI_{n_1}(t)\sin\frac{n\pi}{L}x_1\\&-\sum_{n=0}^{\infty}(\int_0^t e^{-(\frac{n\pi}{L})^2(t-\tau)}EF_{n_1}(\tau)d\tau)\sin\frac{n\pi}{L}x_1.\sum_{n=0}^{\infty}EI_{n_1}(t)\sin\frac{n\pi}{L}x_2\\&+\sum_{n=0}^{\infty}EI_{n_1}(t)\sin\frac{n\pi}{L}x_1.\sum_{n=0}^{\infty}EI_{n_1}(t)\sin\frac{n\pi}{L}x_2,\end{aligned} \tag{38}$$

$$EI_{n_1}(t)=\int_0^t e^{-(\frac{n\pi}{L})^2(t-\tau)}EF_{n_1}(\tau)\,d\tau, \tag{39}$$

$$EI_{n_1}(t)I_{m_1}(t)=\int_0^t\int_0^t e^{-(\frac{n\pi}{L})^2(t-\tau_1)}e^{-(\frac{m\pi}{L})^2(t-\tau_2)}EF_{n_1}(\tau_1)F_{m_1}(\tau_2)\,d\tau_1 d\tau_2 \tag{40}$$

in which

$$\begin{aligned}EF_{n_1}(\tau_1)F_{m_1}(\tau_1)&=\frac{4\sigma^2}{L^2}\int_0^L\sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\,dx\\&-\frac{4\lambda\sigma}{L^2}\int_0^L\int_0^L\sin\frac{n\pi}{L}x_3\sin\frac{m\pi}{L}x_4\;En(x_3)u_0^2(\tau_2,x_4)\,dx_3dx_4\\&-\frac{4\lambda\sigma}{L^2}\int_0^L\int_0^L\sin\frac{n\pi}{L}x_3\sin\frac{m\pi}{L}x_4\;En(x_4)u_0^2(\tau_1,x_3)\,dx_3dx_4\\&+\frac{4\lambda^2}{L^2}\int_0^L\int_0^L\sin\frac{n\pi}{L}x_3\sin\frac{m\pi}{L}x_4\;Eu_0^2(\tau_1,x_3)u_0^2(\tau_2,x_4)\,dx_3dx_4\end{aligned} \tag{41}$$

where

$$En(x)u_0^2(t,y)=2(\sum_{n=0}^{\infty}T_n e^{-(\frac{n\pi}{L})^2 t}\sin\frac{n\pi}{L}y)\sum_{n=0}^{\infty}\sin\frac{n\pi}{L}y\;En(x)I_{n_0}(t) \tag{42}$$

in which

$$En(x)I_{n_0}(t)=\frac{2\sigma}{L}\sin\frac{n\pi}{L}x\int_0^t e^{-(\frac{n\pi}{L})^2(t-\tau)}d\tau, \tag{43}$$

and

$$\begin{aligned}Eu_0^2(\tau_1,x_3)u_0^2(\tau_2,x_4)&=\phi^2(\tau_1,x_3)\phi^2(\tau_2,x_4)\\&+\phi^2(\tau_1,x_3)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}EI_{n_0}(\tau_2)I_{m_0}(\tau_2).\sin\frac{n\pi}{L}x_4\sin\frac{m\pi}{L}x_4\\&+4\phi(\tau_1,x_3)\phi(\tau_2,x_4)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}EI_{n_0}(\tau_1)I_{m_0}(\tau_2).\sin\frac{n\pi}{L}x_3\sin\frac{m\pi}{L}x_4\\&+\phi^2(\tau_2,x_4)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}EI_{n_0}(\tau_1)I_{m_0}(\tau_1).\sin\frac{n\pi}{L}x_3\sin\frac{m\pi}{L}x_3\\&+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}EI_{n_0}(\tau_1)I_{m_0}(\tau_1)I_{l_0}(\tau_2)I_{k_0}(\tau_2).\\&\sin\frac{n\pi}{L}x_3\sin\frac{m\pi}{L}x_3\sin\frac{l\pi}{L}x_4\sin\frac{k\pi}{L}x_4\end{aligned} \tag{44}$$

in which

$$\phi(t,x)=\sum_{n=0}^{\infty}T_n e^{-(\frac{n\pi}{L})^2 t}\sin\frac{n\pi}{L}x, \tag{45}$$

$$EI_{n_0}(\tau_1)I_{m_0}(\tau_1)=\frac{4\sigma^2}{L^2}I_{n,m}\int_0^{\tau_1}\int_0^{\tau_1}e^{-(\frac{n\pi}{L})^2(\tau_1-\tau_3)}e^{-(\frac{m\pi}{L})^2(\tau_1-\tau_4)}d\tau_3 d\tau_4 \tag{46}$$

$$EI_{n_0}(\tau_2)I_{m_0}(\tau_2)=\frac{4\sigma^2}{L^2}I_{n,m}\int_0^{\tau_2}\int_0^{\tau_2}e^{-(\frac{n\pi}{L})^2(\tau_2-\tau_3)}e^{-(\frac{m\pi}{L})^2(\tau_2-\tau_4)}d\tau_3 d\tau_4 \tag{47}$$

$$EI_{n_0}(\tau_1)I_{m_0}(\tau_2)=\frac{4\sigma^2}{L^2}I_{n,m}\int_0^{\tau_2}\int_0^{\tau_1}e^{-(\frac{n\pi}{L})^2(\tau_1-\tau_3)}e^{-(\frac{m\pi}{L})^2(\tau_2-\tau_4)}d\tau_3 d\tau_4 \tag{48}$$

$$EI_{n_0}(\tau_1)I_{m_0}(\tau_1)I_{l_0}(\tau_2)I_{k_0}(\tau_2)=\int_0^{\tau_1}\int_0^{\tau_1}\int_0^{\tau_2}\int_0^{\tau_2} e^{-(\frac{n\pi}{L})^2(\tau_1-\tau_3)}e^{-(\frac{m\pi}{L})^2(\tau_1-\tau_4)} e^{-(\frac{l\pi}{L})^2(\tau_2-\tau_5)}e^{-(\frac{k\pi}{L})^2(\tau_2-\tau_6)}EF_{n_0}(\tau_3)F_{m_0}(\tau_4)F_{l_0}(\tau_5) F_{k_0}(\tau_6)\,d\tau_3 d\tau_4 d\tau_5 d\tau_6, \tag{49}$$

where

$$\begin{aligned} EF_{n_0}(\tau_3)F_{m_0}(\tau_4)F_{l_0}(\tau_5)F_{k_0}(\tau_6) = &\frac{16\sigma^4}{L^4}\int_0^L\int_0^L \sin\frac{n\pi}{L}x_1 \sin\frac{m\pi}{L}x_1 \sin\frac{l\pi}{L}x_2 \sin\frac{k\pi}{L}x_2\, dx_1 dx_2 \\ &+\frac{16\sigma^4}{L^4}\int_0^L\int_0^L \sin\frac{n\pi}{L}x_1 \sin\frac{l\pi}{L}x_1 \sin\frac{m\pi}{L}x_2 \sin\frac{k\pi}{L}x_2\, dx_1 dx_2 \\ &+\frac{16\sigma^4}{L^4}\int_0^L\int_0^L \sin\frac{n\pi}{L}x_1 \sin\frac{k\pi}{L}x_1 \sin\frac{m\pi}{L}x_2 \sin\frac{l\pi}{L}x_2\, dx_1 dx_2. \end{aligned} \tag{50}$$

Using mathematica-5, the previous huge computations were performed and the following sample results are obtained; Figs. (**5-10**).

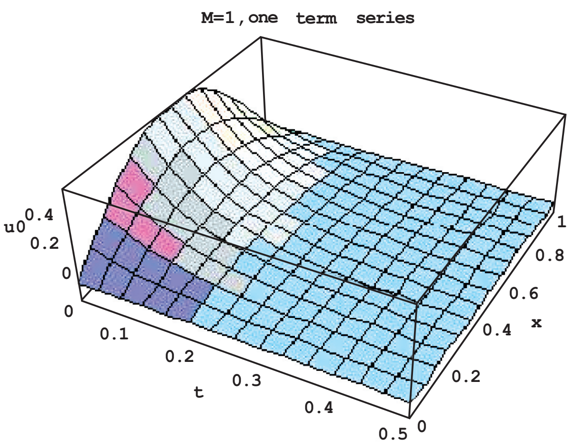


Fig. (5). The change of the mean of the zero order approximation u_0 with time t and space variable x. ($L=1,\ M=1,\ \sigma=1,\ g(x)=x$).

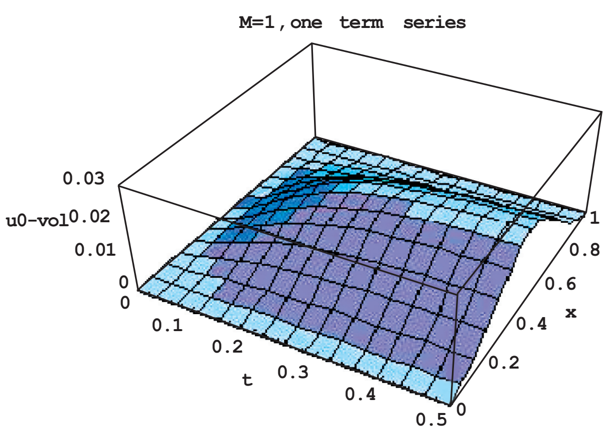


Fig. (6). The change of the variance of the zero order approximation u_0 with space variable x and time t. ($L=1,\ M=1,\ \sigma=1,\ g(x)=x$).

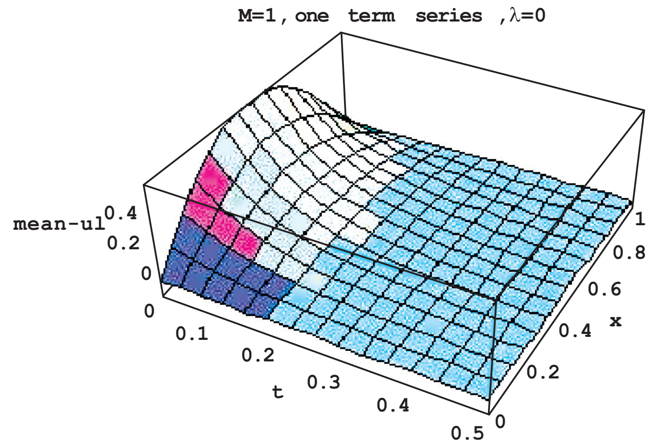


Fig. (7). The change of the mean of the first order approximation u_1 with space x and time t at $\lambda=0$. ($L=1,\ M=1,\ \sigma=1,\ g(x)=x$).

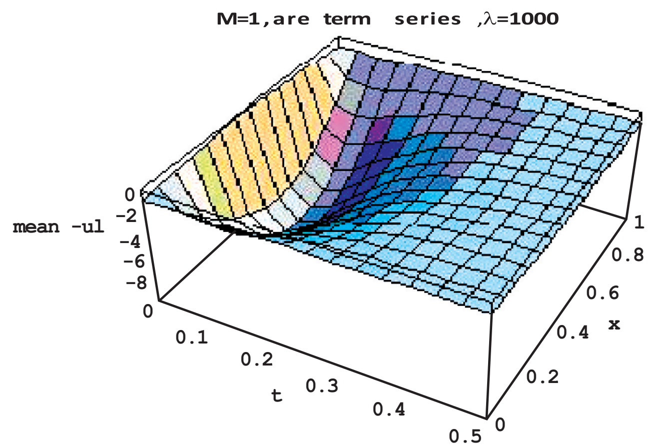


Fig. (8). The change of the mean of the first order approximation u_1 with space x and time t at $\lambda=1000$. ($L=1,\ M=1,\ \sigma=1,\ g(x)=x$).

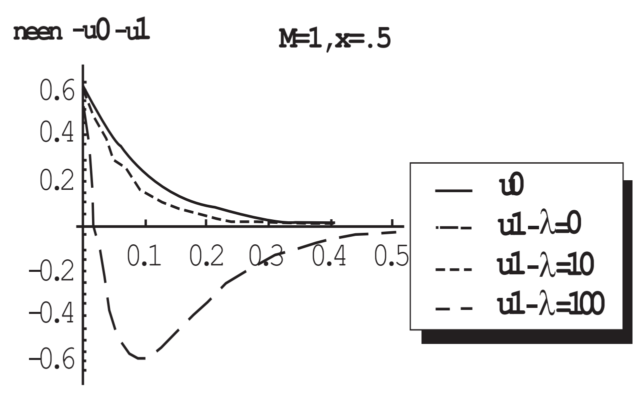


Fig. (9). Comparisons between the mean of the zero and first orders approximation with time t at different λ levels, x=.5 ($L=1,\ M=1,\ \sigma=1,\ g(x)=x$).

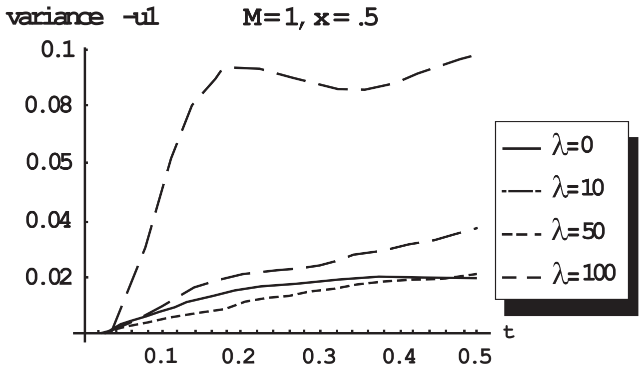


Fig. (10). Comparisons between the mean of the zero and first orders approximation with time t at different λ levels, x=.5 ($L=1,\ M=1, \sigma=1, g(x)=x$).

4. THE DIFFUSION EQUATION UNDER NONLINEAR LOSSES, THE WHEP TECHNIQUE

Since Meecham and his co-workers [15] developed a theory of turbulence involving a truncated Wiener-Hermite expansion (WHE) of the velocity field, many authors studied problems concerning turbulence [16-21] and others [22-31].

The application of the WHE aims at finding a truncated series solution to the solution process of differential equations. The truncated series composes of two major parts; the first is the Gaussian part which consists of the first two terms, while the rest of the series constitute the non-Gaussian part. In nonlinear cases, there exists always difficulties of solving the resultant set of deterministic integro-differential equations got from the applications of a set of comprehensive averages on the stochastic integro-differential equation obtained after the direct application of WHE. Many authors introduced different methods to face these obstacles. Among them, the WHEP technique was introduced in [28] using the perturbation technique to solve perturbed nonlinear problems. The WHE method utilizes the Wiener-Hermite polynomials which are the elements of a complete set of statistically orthogonal random functions [13].

Due to the completeness of the Wiener-Hermite set ,any random function $G(t;\omega)$ can be expanded as

$$G(t;\omega) = G^{(0)}(t) + \int_{-\infty}^{\infty} G^{(1)}(t;t_1)H^{(1)}(t_1)dt_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G^{(2)}(t;t_1,t_2)H^{(2)}(t_1,t_2)dt_1dt_2 + ... \tag{51}$$

where the first two terms are the Gaussian part of $G(t;\omega)$. The rest of the terms in the expansion represent the non-Gaussian part of $G(t;\omega)$. The average of $G(t;\omega)$ is

$$\mu_G = E\,G(t;\omega) = G^{(0)}(t) \tag{52}$$

The covariance of $G(t;\omega)$ is

$$Cov(G(t;\omega),G(\tau;\omega)) = E(G(t;\omega)-\mu_G(t))(G(\tau;\omega)-\mu_G(\tau)) = \int_{-\infty}^{\infty} G^{(1)}(t;t_1)G^{(1)}(\tau,t_1)dt_1 + 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G^{(2)}(t;t_1,t_2)G^{(2)}(\tau,t_1,t_2)dt_1dt_2 \tag{53}$$

The variance of $G(t;\omega)$ is

$$Var\,G(t;\omega) = E(G(t;\omega)-\mu_G(t))^2 = \int_{-\infty}^{\infty}[G^{(1)}(t;t_1)]^2dt_1 + 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[G^{(2)}(t;t_1,t_2)]^2dt_1dt_2 \tag{54}$$

The WHEP technique can be applied on linear or nonlinear perturbed systems described by ordinary or partial differential equations. The solution can be modified in the sense that additional parts of the Wiener-Hermite expansion can always be taken into considerations and the required order of approximations can always be made depending on the computing tool. It can be even run through a package if it is coded in some sort of symbolic languages. The technique was successfully applied to several nonlinear stochastic equations [32,33], the symbolic WHEP algorithm is charted in appendix-A.

Considering equation (21) and searching for the Gaussian part of the solution process, equation (7), with taking the necessary averages, we get the following four sets of deterministic equations:

$$\frac{\partial u_0^{(0)}(t,x)}{\partial t} = \frac{\partial^2 u_0^{(0)}}{\partial x^2} \tag{55}$$
$$u_0^{(0)}(t,0)=0, u_0^{(0)}(t,L)=0 \text{ and } u_0^{(0)}(0,x)=g(x),$$

$$\frac{\partial u_1^{(0)}(t,x)}{\partial t} = \frac{\partial^2 u_1^{(0)}}{\partial x^2} - [u_0^{(0)}]^2 - \int_0^L [u_0^{(1)}]^2 dx_1, \tag{56}$$
$$u_1^{(0)}(t,0)=0, u_1^{(0)}(t,L)=0 \text{ and } u_1^{(0)}(0,x)=0,$$

$$\frac{\partial u_0^{(1)}(t,x;x_1)}{\partial t} = \frac{\partial^2 u_0^{(1)}}{\partial x^2} + \sigma.\delta(x-x_1) \tag{57}$$
$$u_0^{(1)}(t,0;x_1)=0, u_0^{(1)}(t,L;x_1)=0 \text{ and } u_0^{(1)}(0,x;x_1)=0,$$

$$\frac{\partial u_1^{(1)}(t,x;x_1)}{\partial t} = \frac{\partial^2 u_1^{(1)}}{\partial x^2} - 2u_0^{(0)}.u_0^{(1)} \tag{58}$$
$$u_1^{(1)}(t,0;x_1)=0, u_1^{(1)}(t,L;x_1)=0 \text{ and } u_1^{(1)}(0,x;x_1)=0.$$

The algorithm of solution is evaluating $u_0^{(0)}$ and $u_0^{(1)}$ first using the separation of variables and the eigenfunction expansion respectively and then computing the other two kernels independently using the eigenfunction expansion. The final results of the first order first correction mean and variance respectively are:

$$\mu_u(x,t) = \mathrm{u}_0^{(0)}(x,t) + \lambda \mathrm{u}_1^{(0)}(x,t), \tag{59}$$

$$Var\ u(t,x)=\int_0^L [u_0^{(1)}(t,x;x_1)]^2 dx_1 + 2\lambda\int_0^L u_0^{(1)}.u_1^{(1)} dx_1. \tag{60}$$

The following are some sample results Figs. (**11-18**).

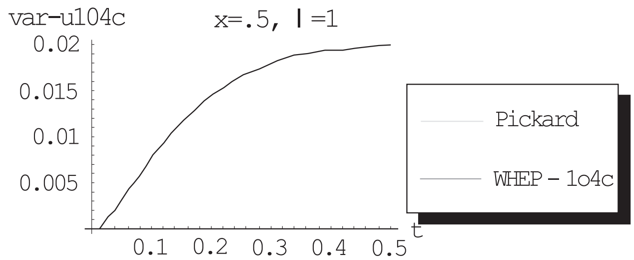


Fig. (11). The change of the first order approximation and fourth order correction of the mean with time t at x=.5 and $\lambda=1$, ($L=1, M=1, \sigma=1, g(x)=x$).

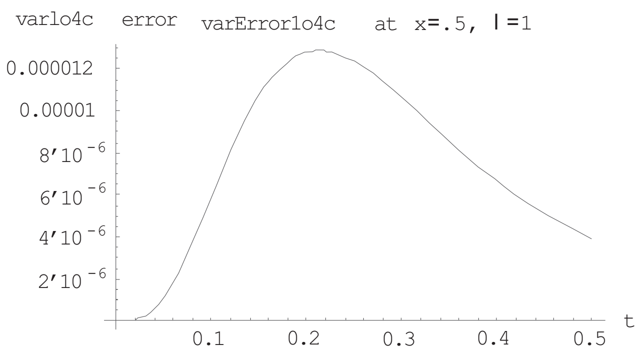


Fig. (12). The change of the first order approximation and fourth order correction of the variance with time t at x=.5 and $\lambda=1$, ($L=1, M=1, \sigma=1, g(x)=x$).

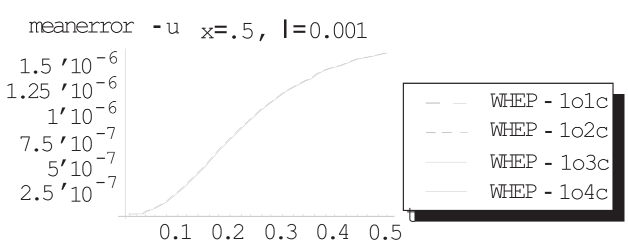


Fig. (13). The change of the first order approximation and fourth order correction of the mean with time t at x=.5 and λ=0.001. ($L=1, M=1, \sigma=1, g(x)=x$).

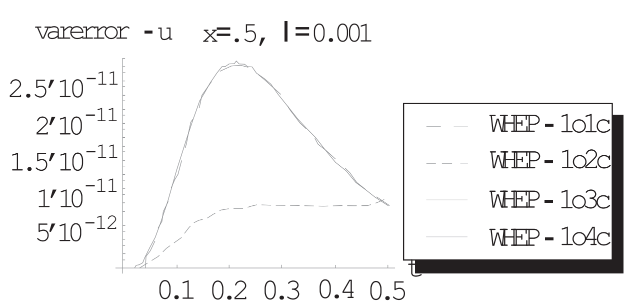


Fig. (14). The change of the first order approximation and fourth order correction of the variance with time t at x=.5 and λ=0.001. ($L=1, M=1, \sigma=1, g(x)=x$).

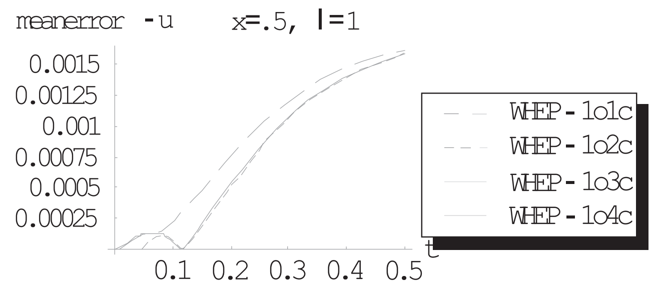


Fig. (15). The change of the first order approximation and fourth order correction of the mean-error with time t at x=.5 , $\lambda=1$ ($L=1, M=1, \sigma=1, g(x)=x$).

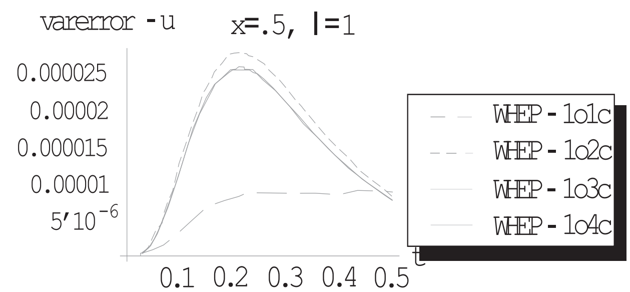


Fig. (16). The change of the first order approximation and fourth order correction of the variance-error with time t at x=.5 , $\lambda=1$ ($L=1, M=1, \sigma=1, g(x)=x$).

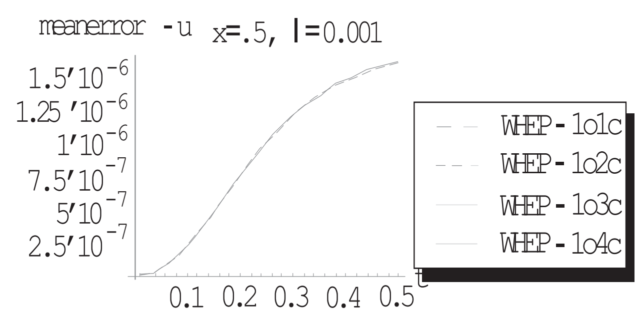


Fig. (17). The change of the first order approximation and fourth order correction of the mean-error with time t at x=.5 ,λ=0.001 ($L=1, M=1, \sigma=1, g(x)=x$).

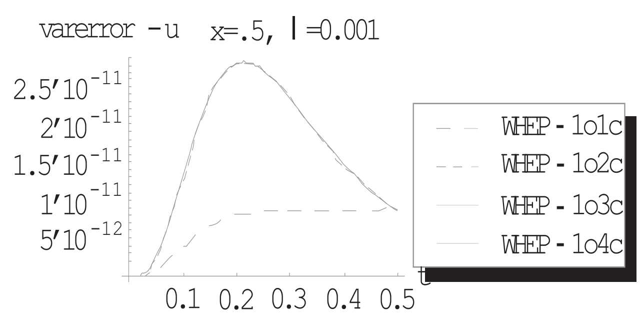


Fig. (18). The change of the first order approximation and fourth order correction of the variance-error with time t at x=.5, λ=0.001 ($L=1, M=1, \sigma=1, g(x)=x$).

One can notice that the approximations are very close, especially second and higher corrections as shown in Tables **1** and **2**.

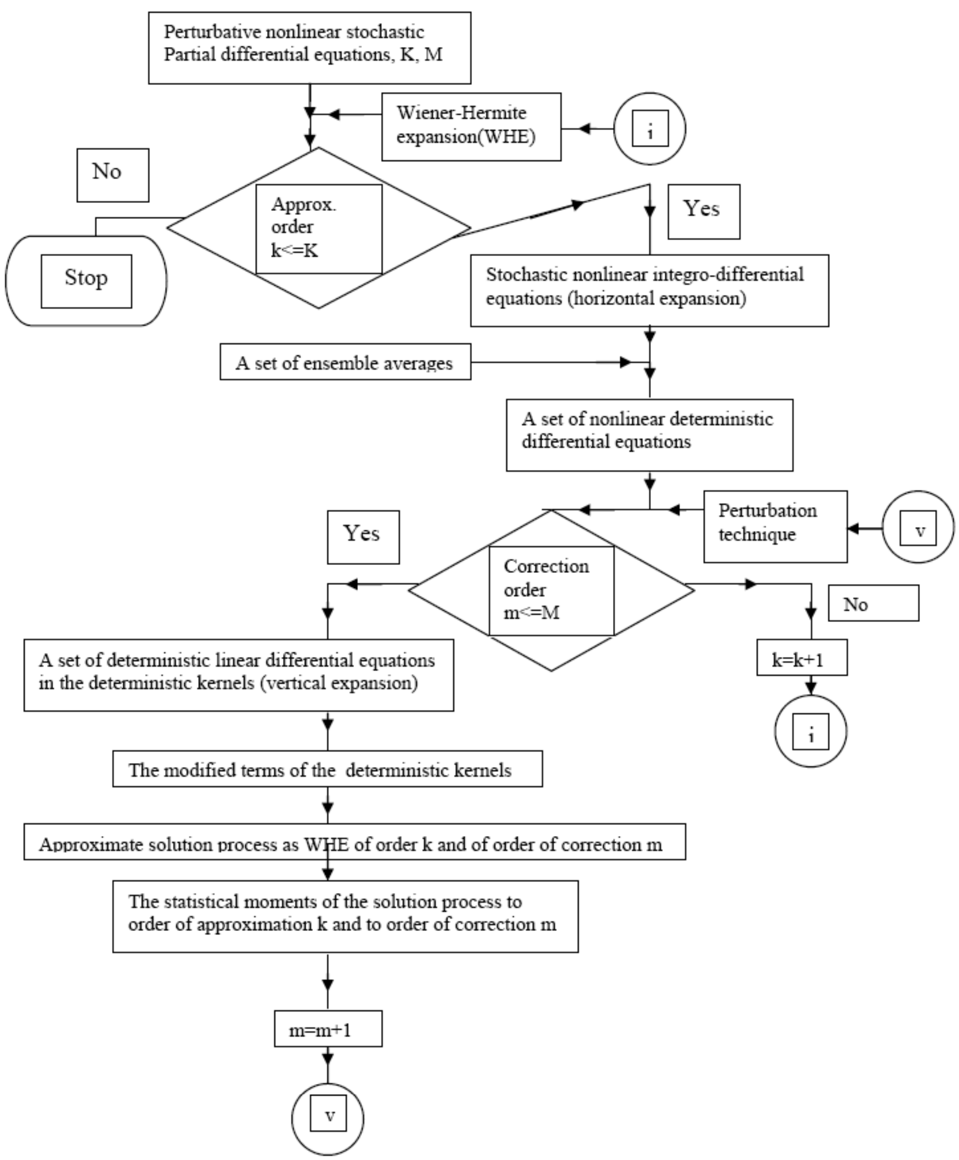
Table 1. Errors of the WHEP Corrections of the First Order Mean Solution

Time Mean	0.0	0.1	0.2	0.3	0.4	0.5
First correction	0	0.000220419	0.000755961	0.00120395	0.00147417	0.00161483
Second correction	0	0.0000714979	0.000517808	0.00106582	0.00139979	0.00157618
Third correction	4.06506^{-20}	0.0000612266	0.000530039	0.00107424	0.00140498	0.00157945
Fourth correction	3.17478^{-20}	0.0000615896	0.000529407	0.00107373	0.00140465	0.00157926

Table 2. Errors of the WHEP Corrections of the First Order Variance Solution

Time Variance	0.0	0.1	0.2	0.3	0.4	0.5
First correction	0	3.73958^{-12}	8.05185^{-12}	8.58321^{-12}	8.65956^{-12}	8.87587^{-12}
Second correction	0	1.29844^{-11}	2.82209^{-11}	2.33893^{-11}	1.45704^{-11}	8.63275^{-12}
Third correction	0	1.29837^{-11}	2.82188^{-11}	2.33873^{-11}	1.45691^{-11}	8.63197^{-12}
Fourth correction	0	1.29837^{-11}	2.82188^{-11}	2.33873^{-11}	1.45691^{-11}	8.63197^{-12}

APPENDIX-A: THE SYMBOLIC WHEP ALGORITHM



CONCLUSIONS

Concerning the presented case studies in this paper, the WHEP technique offers corrections for the approximate solution which are true solution developments in the case of taking the exact solution as a reference. We Still have relative developments when taking an approximate solution as a reference.

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Received: August 04, 2009 Revised: October 27, 2009 Accepted: November 28, 2009