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Nonlinear Cubic Homogeneous Schrodinger Equations with Complex Initial Conditions, Limited Time Response

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Abstract: In this paper, a perturbing nonlinear homogeneous Schrodinger equation is studied under limited time interval, complex initial conditions and zero Neumann conditions. The perturbation and Picard approximation methods together with the eigenfunction expansion and variational parameters methods are used to introduce an approximate solution for the perturbative nonlinear case for which a power series solution is proved to exist. Using Mathematica, the solution algorithm is tested through computing the possible orders of approximations. The method of solution is illustrated through case studies and figures.

Keywords: Nonlinear Schrodinger equation, Perturbation, Eigenfunction expansion, Mathematica, Picard Approximation.

1. INTRODUCTION

The nonlinear Schrodinger equation (NLS) is the principal equation to be analyzed and solved in many fields, see [1-5] for examples. Nonlinear Schrodinger Equations (NLS) are one of the most important models of mathematical physics arising in a great array of contexts as for conductor electronics, optics in nonlinear media, photonics, plasmas, fundamentation of quantum mechanics, dynamics of accelerators, mean-field theory of Bose-Einstein condensates or in biomolecule dynamics. It was also the second nonlinear partial differential equation (PDE) whose initial value problem was discovered to be solvable *via* the inverse scattering transform (IST) method. In the last ten decades, there are a lot of NLS problems depending on additive or multiplicative noise in the random case [6, 7] or a lot of solution methodologies in the deterministic case. Wang M. and *et al.* [8] obtained the exact solutions to NLS using what they called the sub-equation method. They got four kinds of exact solutions for which no sign to the initial or boundary conditions type is made. Xu L. and Zhang J. [9] followed the same previous technique in solving a higher order NLS. Sweilam N. [10] solved a nonlinear cubic Schrodinger equation which gives rise to solitary solutions using variational iteration method. Zhu S. [11] used the extended hyperbolic auxiliary equation method in getting exact explicit solutions to a more complicated NLS without any conditions. Sun J. and *et al.* [12] solved an NLS with an initial condition using Lie group method. By using coupled amplitude phase formulation, Parsezian K. and Kalithasan B. [13] constructed the quartic anharmonic oscillator equation from the coupled higher order NLS. Two-dimensional grey solitons to the NLS were numerically analyzed by Sakaguchi H. and Higashiuchi T. [14]. The generalized derivative NLS was studied by Huang D. *et al.* [15] introducing a new auxiliary equation expansion method. The NLS equation arises in many application areas [16-20] such as wave propagation in nonlinear media, surface wave in sufficiently deep waters and signal propagation in optical fibers.

In this paper, a straight forward solution algorithm is introduced using the transformation from a complex solution to a coupled equations in two real solutions, eliminating one of the solutions to get separate independent and higher order equations, and finally introducing a perturbative approximate solution to the system.

2. THE GENERAL LINEAR CASE

Consider the non homogeneous linear Schrodinger equation:

$$i\,\frac{\partial u(t,z)}{\partial z} + \alpha\,\frac{\partial^2 u(t,z)}{\partial t^2} = F_1(t,z) + i\,F_2(t,z),\ (t,z) \in (0,T)\ x\ (0,\infty) \quad (1)$$

where $u(t,z)$ is a complex valued function which is subjected to:

$I.C.: u(t,0) = f_1(t) + i\,f_2(t)$, a complex valued function, -- (2)

$$B.C.: u(0,z) = u(T,z) = 0. \quad (3)$$

Let

$$u(t,z) = \psi(t,z) + i\,\phi(t,z),\ \psi,\phi$$

are real valued functions. The following coupled equations are got as follows:

$$\frac{\partial\phi(t,z)}{\partial z} = \alpha\,\frac{\partial^2\psi(t,z)}{\partial t^2} + G_1(t,z), \quad (4)$$

$$\frac{\partial\psi(t,z)}{\partial z} = \alpha\,\frac{\partial^2\phi(t,z)}{\partial t^2} + G_2(t,z), \quad (5)$$

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Where

$\psi(t,0) = f_1(t), \phi(t,0) = f_2(t), G_1(t,z) = -F_1(t,z),\ G_1(t,z) = F_2(t,z),$ and all corresponding other I.C. and B.C. are zeros.

Eliminating one of the variables in equations (4) and (5), one can get the following independent equations:

$$\frac{\partial^4\psi(t,z)}{\partial t^4} + \frac{1}{\alpha^2}\frac{\partial^2\psi(t,z)}{\partial t^2} = \frac{1}{\alpha^2}\tilde{\psi}_1(t,z), \quad (6)$$

$$\frac{\partial^4\phi(t,z)}{\partial t^4} + \frac{1}{\alpha^2}\frac{\partial^2\phi(t,z)}{\partial t^2} = \frac{1}{\alpha^2}\tilde{\psi}_2(t,z), \quad (7)$$

Where

$$\tilde{\psi}_1(t,z) = \frac{\partial G_2(t,z)}{\partial z} - \alpha\frac{\partial^2 G_1(t,z)}{\partial t^2}, \quad (8)$$

$$\tilde{\psi}_2(t,z) = \alpha\frac{\partial^2 G_2(t,z)}{\partial t^2} + \frac{\partial G_1(t,z)}{\partial z}, \quad (9)$$

Using the eigenfunction expansion technique [23], the following solution expressions are obtained:

$$\psi(t,z) = \sum_{n=0}^{\infty} T_n(z)\sin(\frac{n\pi}{T})t, \quad (10)$$

$$\phi(t,z) = \sum_{n=0}^{\infty} \tau_n(z)\sin(\frac{n\pi}{T})t, \quad (11)$$

Where $T_n(z)$ and $\tau_n(z)$ can be got through the applications of initial conditions and then solving the resultant second order differential equations using the method of variational of parameter [24]. The final expressions can be got as the following :

$$T_n(z) = (C_1 + A_1(z))\sin\beta_n z + (C_2 + B_1(z))\cos\beta_n z, \quad (12)$$

$$\tau_n(z) = (C_3 + A_2(z))\sin\beta_n z + ((C_4 + B_2(z))\cos\beta_n z, \quad (13)$$

Where

$$\beta_n = \alpha(\frac{n\pi}{T})^2 \quad (14)$$

$$A_1(z) = \frac{1}{\beta_n}\int \tilde{\psi}_{1n}(z;n)\sin(\beta_n z)\,dz, \quad (15)$$

$$B_1(z) = \frac{-1}{\beta_n}\int \tilde{\psi}_{1n}(z;n)\sin(\beta_n z)\,dz, \quad (16)$$

$$A_2(z) = \frac{1}{\beta_n}\int \tilde{\psi}_{2n}(z;n)\cos(\beta_n z)\,dz, \quad (17)$$

$$B_2(z) = \frac{-1}{\beta_n}\int \tilde{\psi}_{2n}(z;n)\cos(\beta_n z)\,dz, \quad (18)$$

In which

$$\tilde{\psi}_{1n}(z;n) = \frac{2}{T}\int \tilde{\psi}_1(t,z)\sin(\frac{n\pi}{T}t)dt, \quad (19)$$

$$\tilde{\psi}_{2n}(z;n) = \frac{2}{T}\int \tilde{\psi}_2(t,z)\sin(\frac{n\pi}{T}t)dt, \quad (20)$$

The following conditions should also be satisfied:

$$C_2 = \frac{2}{T}\int_0^T f_1(t)\sin\left(\frac{n\pi}{T}t\right)dt - B_1(0), \quad (21)$$

$$C_4 = \frac{2}{T}\int_0^T f_2(t)\sin\left(\frac{n\pi}{T}t\right)dt - B_2(0). \quad (22)$$

Finally the following solution is obtained:

$$u(t,z) = \psi(t,z) + i\,\phi(t,z), \quad (23)$$

Or

$$|u(t,z)|^2 = \psi^2(t,z) + \phi^2(t,z). \quad (24)$$

3. THE NON- LINEAR CASE

Consider the non-homogeneous non-linear Schrodinger equation:

$$i\frac{\partial u(t,z)}{\partial z} + \alpha\frac{\partial^2 u(t,z)}{\partial t^2} + \varepsilon|u(t,z)|^2 u(t,z) + i\gamma u(t,z) = 0, (t,z) \in (0,T)\,x\,(0,\infty) \quad (25)$$

Where $u(t,z)$ is a complex valued function which is subjected to the initial and boundary conditions mentioned before in equations (2), (3) respectively.

Lemma [21,22]

The solution of equation (25) with the constraints (2), (3) is a power series in ε if exists.

Proof

The proof can be found in [21] or [22].

According to the previous lemma, one can assume the solution of equation (25) as the following:

$$u(t,z) = \sum_{n=0}^{\infty} \varepsilon^n u_n(t,z) \quad (26)$$

Let $u(t,z) = \psi(t,z) + i\,\phi(t,z), \psi, \phi$: are real valued functions. The following coupled equations are got:

$$\frac{\partial\phi(t,z)}{\partial z} = \alpha\frac{\partial^2\psi(t,z)}{\partial t^2} + \varepsilon(\psi^2 + \phi^2)\psi - \gamma\phi, \quad (27)$$

$$\frac{\partial\psi(t,z)}{\partial z} = -\alpha\frac{\partial^2\phi(t,z)}{\partial t^2} - \varepsilon(\psi^2 + \phi^2)\phi - \gamma\psi, \quad (28)$$

Where $\psi(t,0) = f_1(t),\ \phi(t,0) = f_2(t)$, and all corresponding other I.C. and B.C. are zeros.

As a perturbation solution, one can assume that:

$$\psi(t,z) = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3, \tag{29}$$

$$\phi(t,z) = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3, \tag{30}$$

Where $\psi_0(t,0) = f_1(t),\ \phi_0(t,0) = f_2(t)$, and all corresponding other I.C. and B.C. are zeros.

Substituting from equations (29) and (30) into equations (27) and (28) and then equating the equal powers of ε, one can get the following set of coupled equations:

$$\frac{\partial\phi_0(t,z)}{\partial z} = \alpha\frac{\partial^2\psi_0(t,z)}{\partial t^2} - \gamma\phi_0, \tag{31}$$

$$\frac{\partial\psi_0(t,z)}{\partial z} = -\alpha\frac{\partial^2\phi_0(t,z)}{\partial t^2} - \gamma\psi_0, \tag{32}$$

$$\frac{\partial\phi_1(t,z)}{\partial z} = \alpha\frac{\partial^2\psi_1(t,z)}{\partial t^2} - \gamma\phi_1 + e^{-2\gamma z}\left(\psi_0^{\ 3} + \psi_0\phi_0^{\ 2}\right), \tag{33}$$

$$\frac{\partial\psi_1(t,z)}{\partial z} = -\alpha\frac{\partial^2\phi_1(t,z)}{\partial t^2} - \gamma\psi_1 - e^{-2\gamma z}\left(\phi_0^{\ 3} + \phi_0\psi_0^{\ 2}\right), \tag{34}$$

$$\frac{\partial\phi_2(t,z)}{\partial z} = \alpha\frac{\partial^2\psi_2(t,z)}{\partial t^2} - \gamma\phi_2 + e^{-2\gamma z}\left(3\psi_0^{\ 2}\psi_1 + 2\psi_0\phi_0\phi_1 + \psi_1\phi_0^{\ 2}\right), \tag{35}$$

$$\frac{\partial\psi_2(t,z)}{\partial z} = -\alpha\frac{\partial^2\phi_2(t,z)}{\partial t^2} - \gamma\psi_2 - e^{-2\gamma z}\left(3\phi_0^{\ 2}\phi_1 + 2\phi_0\psi_0\psi_1 + \phi_1\psi_0^{\ 2}\right), \tag{36}$$

$$\frac{\partial\phi_3(t,z)}{\partial z} = \alpha\frac{\partial^2\psi_3(t,z)}{\partial t^2} - \gamma\phi_3 + e^{-2\gamma z}\left(3\psi_1^{\ 2}\psi_0 + 3\psi_0^{\ 2}\psi_2 + \psi_2\phi_0^{\ 2} + 2\psi_1\phi_0\phi_1 + \psi_0\phi_1^{\ 2} + 2\psi_0\phi_0\phi_2\right) \tag{37}$$

$$\frac{\partial\psi_3(t,z)}{\partial z} = -\alpha\frac{\partial^2\phi_3(t,z)}{\partial t^2} - \gamma\psi_3 - e^{-2\gamma z}\left(3\phi_1^{\ 2}\phi_0 + 3\phi_0^{\ 2}\phi_2 + \phi_2\psi_0^{\ 2} + 2\phi_1\psi_0\psi_1 + \phi_0\psi_1^{\ 2} + 2\phi_0\psi_0\psi_2\right) \tag{38}$$

and so on. The prototype equations to be solved are:

$$\frac{\partial\phi_i(t,z)}{\partial z} = \alpha\frac{\partial^2\psi_i(t,z)}{\partial t^2} + G_i^{(1)}, \quad i \geq 1 \tag{39}$$

$$\frac{\partial\psi_i(t,z)}{\partial z} = \alpha\frac{\partial^2\phi_i(t,z)}{\partial t^2} + G_i^{(2)}, \quad i \geq 1 \tag{40}$$

Where $\psi_i(t,0) = \delta_{i,0}f_1(t),\ \phi_i(t,0) = \delta_{i,0}f_2(t)$, and all other all corresponding conditions are zeros. $G_i^{(1)}, G_i^{(2)}$ are functions to be computed from previous steps.

3.1. The Order of Approximations

The following final expressions can be used to obtain different order of approximations.

1. The absolute value of the zero order approximation is:

$$\left|u^{(0)}(t,z)\right|^2 = \psi_0^{\ 2} + \phi_0^{\ 2} \tag{41}$$

2. The absolute value of the first order approximation is:

$$u^{(1)}(t,z) = u^{(0)} + \varepsilon(\psi_1 + i\,\phi_1) \tag{42}$$

or

$$\left|u^{(1)}(t,z)\right|^2 = \left|u^{(0)}(t,z)\right|^2 + 2\varepsilon(\psi_0\psi_1 + \phi_0\phi_1) + \varepsilon^2(\psi_1^{\ 2} + \phi_1^{\ 2}) \tag{43}$$

3. The absolute value of the second order approximation is:

$$u^{(2)}(t,z) = u^{(1)} + \varepsilon^2(\psi_2 + i\,\phi_2) \tag{44}$$

or

$$\left|u^{(2)}(t,z)\right|^2 = \left|u^{(1)}(t,z)\right|^2 + 2\varepsilon^2(\psi_0\psi_2 + \phi_0\phi_2) + 2\varepsilon^3(\psi_1\psi_2 + \phi_1\phi_2) + \varepsilon^4(\psi_2^{\ 2} + \phi_2^{\ 2}) \tag{45}$$

4. The absolute value of the third order approximation is:

$$u^{(3)}(t,z) = u^{(2)} + \varepsilon^3(\psi_3 + i\,\phi_3) \tag{46}$$

or

$$\left|u^{(3)}(t,z)\right|^2 = \left|u^{(2)}(t,z)\right|^2 + 2\varepsilon^3(\psi_0\psi_3 + \phi_0\phi_3) + 2\varepsilon^4(\psi_1\psi_3 + \phi_1\phi_3) + 2\varepsilon^5(\psi_2\psi_3 + \phi_2\phi_3) + \varepsilon^6(\psi_3^{\ 2} + \phi_3^{\ 2}) \tag{47}$$

4. CASE STUDIES

To examine the proposed solution algorithm, some case studies are illustrated.

4.1. Case Study 1

Taking the case $f_1(t) = \rho_1, f_2(t) = \rho_2, \gamma = 0$ and following the algorithm, the following selective results for the first, second and third order approximations Figs. (**1-9**) are obtained:

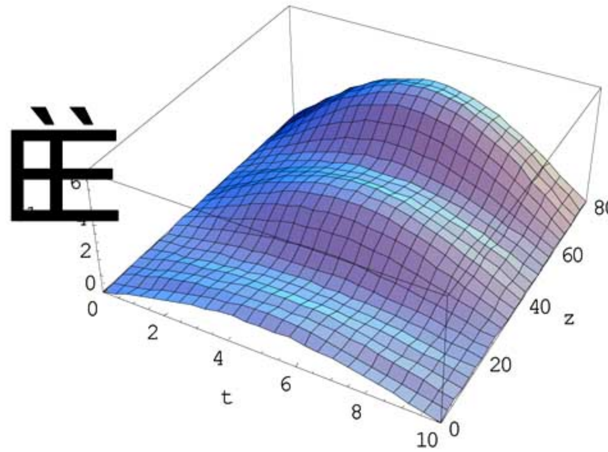

Fig. (1). The first order approximation of $\left|u^{(1)}\right|$ at $\varepsilon = 0.2, \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10$ with considering only ten terms of the series.

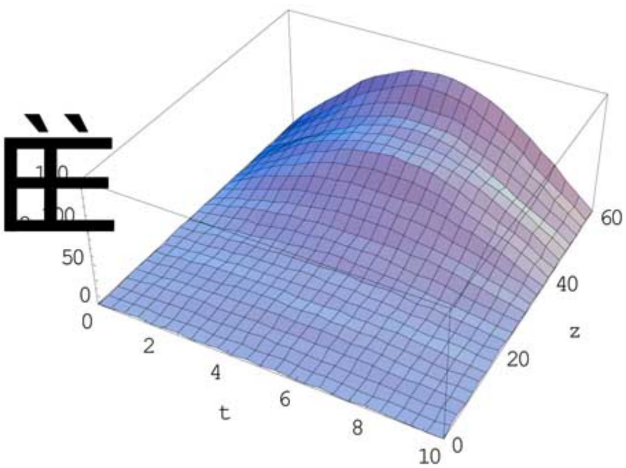

Fig. (2). The second order approximation of $\left|u^{(2)}\right|$ at $\varepsilon = 0.2$, $\gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10$ with considering only ten terms on the series (M=10).

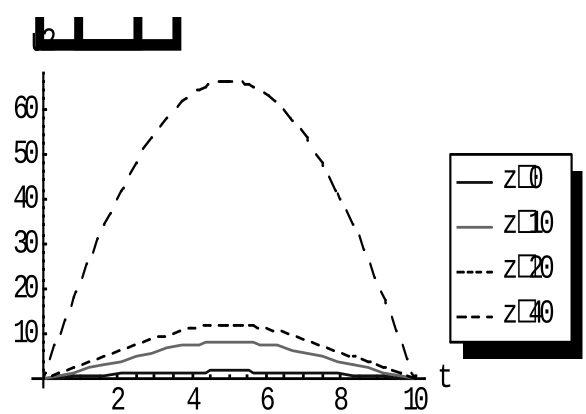

Fig. (3). The second order approximation of $\left|u^{(2)}\right|$ at $\varepsilon = 0.2$, $\gamma = 0, \alpha, \rho_1, \rho_2 = 1, T = 10, M = 10$ for different values of z.

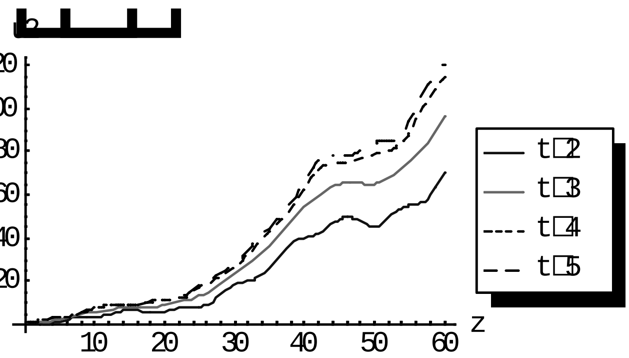

Fig. (4). The second order approximation of $\left|u^{(2)}\right|$ at $\varepsilon = 0.2$, $\gamma = 0$ $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10$ for different values of t.

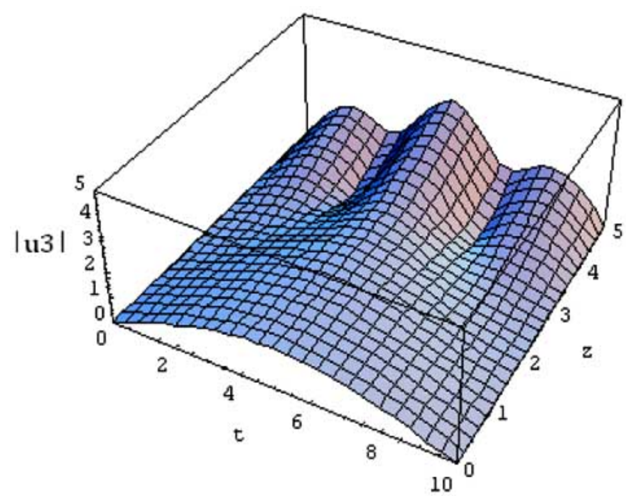

Fig. (5). The third order approximation of $\left|u^{(3)}\right|$ at $\varepsilon = 0.2,\ \gamma = 0$ $\alpha, \rho_1, \rho_2 = 1, T = 10$ with considering only ten terms of the series (M=10).

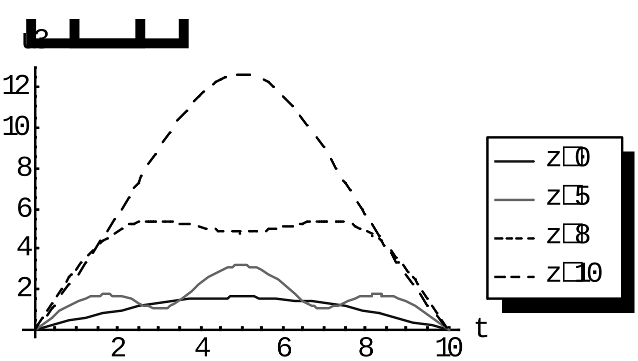

Fig. (6). The third order approximation of $\left|u^{(3)}\right|$ at $\varepsilon = 0.2,\ \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10$ for different values of z.

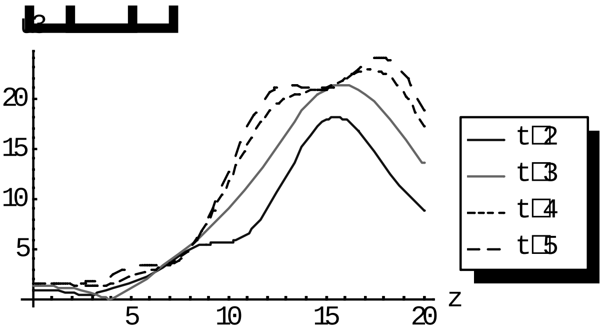

Fig. (7). The third order approximation of $\left|u^{(3)}\right|$ at $\varepsilon = 0.2,\ \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10$ for different values of t.

We can notice the increase of the absolute value of u with the increase of z which can be considered a case of instability.

Note: with constant initial conditions we calculated till third order which takes around 2 days continuously and we cannot calculate more since the machine gives “MATHEMATICA KERNEL OUT OF MEMORY”.

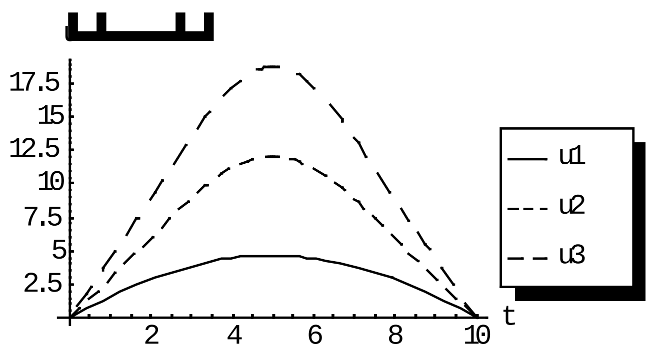

Fig. (8). Comparison between first, second and third order approximation at $\varepsilon = 0.2,\ \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10, z = 20$.

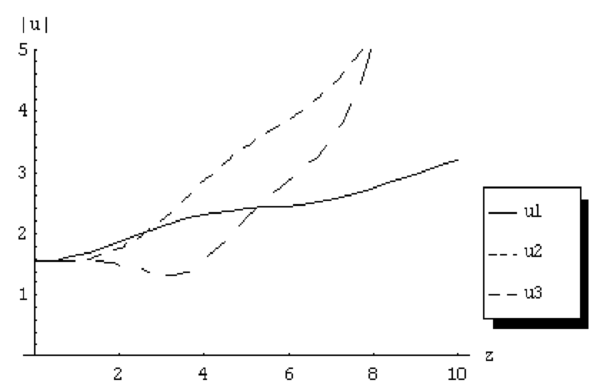

Fig. (9). Comparison between first, second and third order approximation at $\varepsilon = 0.2,\ \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10, t = 4$.

4.2. Case Study 2

Taking the case $f_1(t) = \rho_1, f_2(t) = \rho_2, \gamma = 1$ and following the algorithm, the following selective results for the first and second order of approximations Figs. (**10-16**) are obtained:

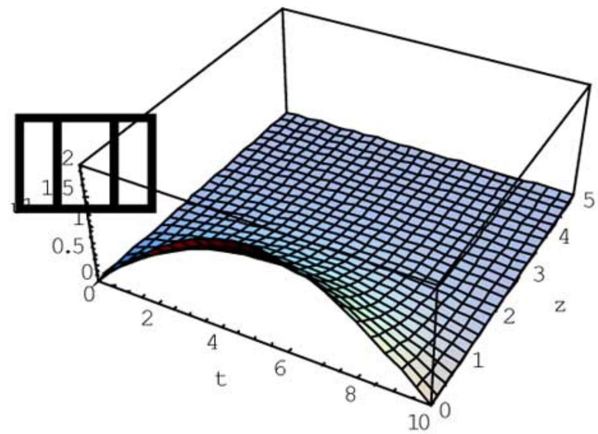

Fig. (10). The first order approximation of $\left|u^{(1)}\right|$ at $\varepsilon = 0.2$, $\alpha, \rho_1, \rho_2 = 1, T = 10, \gamma = 1$ with considering only ten terms of the series (M=10).

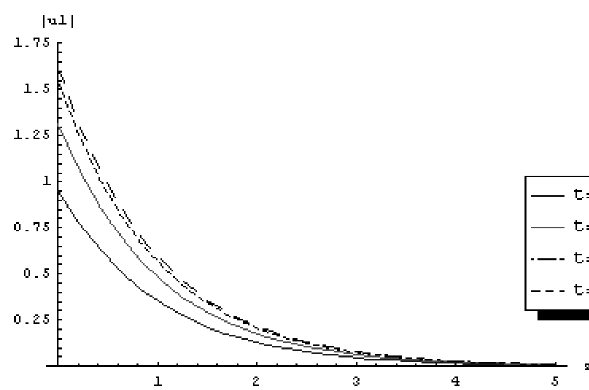

Fig. (11). The first order approximation of $\left|u^{(1)}\right|$ at $\varepsilon = 0.2$, $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10, \gamma = 1$ for different values of t.

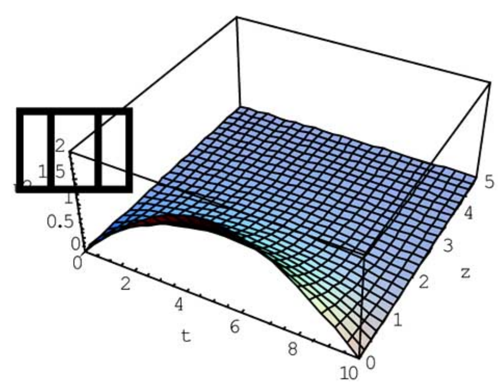

Fig. (12). The second order approximation of $\left|u^{(2)}\right|$ at $\varepsilon = 1$, $\alpha, \rho_1, \rho_2 = 1, T = 10, \gamma = 1$ with considering only ten terms of the series (M=10).

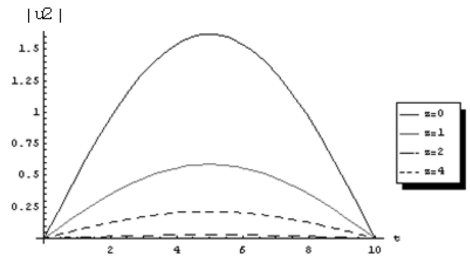

Fig. (13). The second order approximation of $\left|u^{(2)}\right|$ at $\varepsilon = 0.2$, $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10, \gamma = 1$ for different values of z.

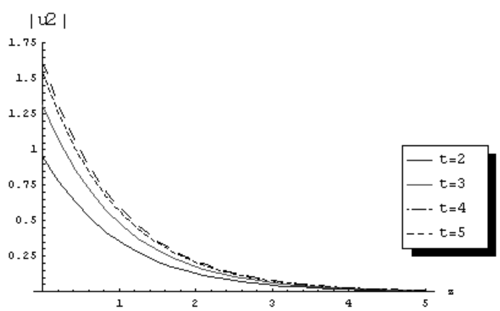

Fig. (14). The second order approximation of $\left|u^{(2)}\right|$ at $\varepsilon = 0.2$, $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10, \gamma = 1$ for different values of t.

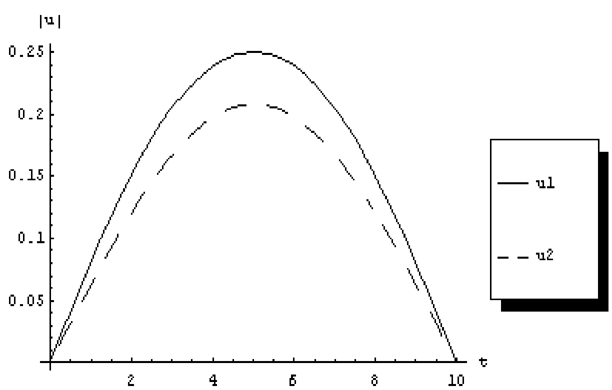

Fig. (15). Comparison between first and second order approximation at $\varepsilon = 1$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10, \gamma = 1, z = 2$.

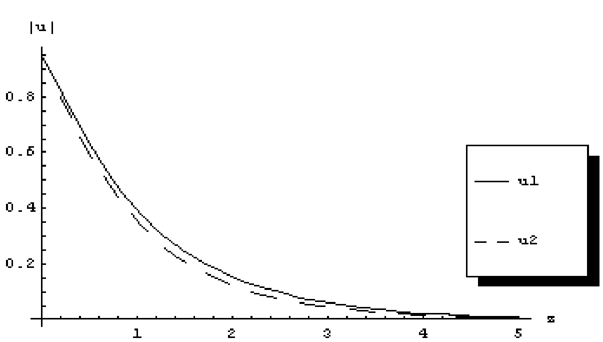

Fig. (16). Comparison between first and second order approximation at $\varepsilon = 1$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10, \gamma = 1, t = 2$.

We can notice the tremendous effect of the presence of gamma factor on the stability of the solution, even for high values of epsilon.

Note: with constant initial conditions and γ exist we calculated till second order which takes around 3 days continuously and we cannot calculate more since the machine gives "MATHEMATICA KERNEL OUT OF MEMORY" .

4.3. Case Study3

Taking the case $F_1(t,z) = 0,\ F_2(t,z) = 0\ f_1(t) = \rho_1 e^{-t}$, $f_2(t) = \rho_2 e^{-t}, \gamma = 1$ and following the algorithm, the following selective results for the first and second order of approximations Figs. (**17-18**) are obtained:

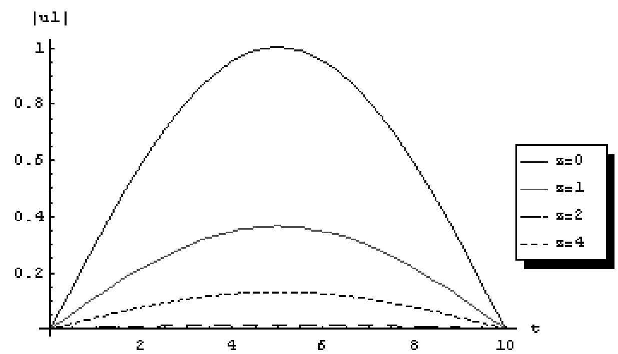

Fig. (17). The first order approximation of $\left|u^{(1)}\right|$ at $\varepsilon = 0.2, \gamma = 1$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10$ for different values of z.

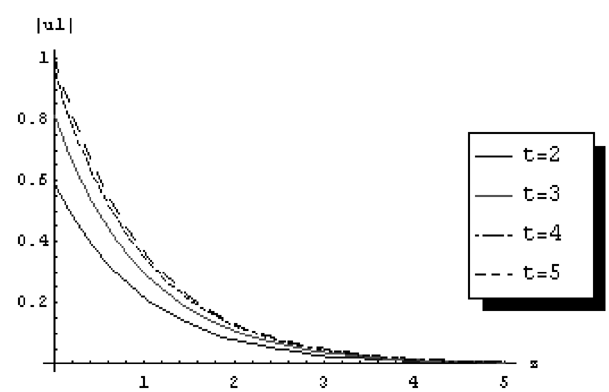

Fig. (18). The first order approximation of $\left|u^{(1)}\right|$ at $\varepsilon = 0.2$, $\gamma = 1$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10$ for different values of t.

Note: the calculations for first order takes 3 days and we can not calculate more orders since the machine gives "MATHEMATICA KERNEL OUT OF MEMORY" .

4.4. Case Study 4

Taking the case $F_1(t,z) = 0,\ F_2(t,z) = 0\ f_1(t) = \rho_1$, $f_2(t) = \rho_2 \sin\left(\frac{m\pi}{T}\right) t$ and following the algorithm, the following selective results for the first, second order of approximations Figs. (**19-23**) are obtained:

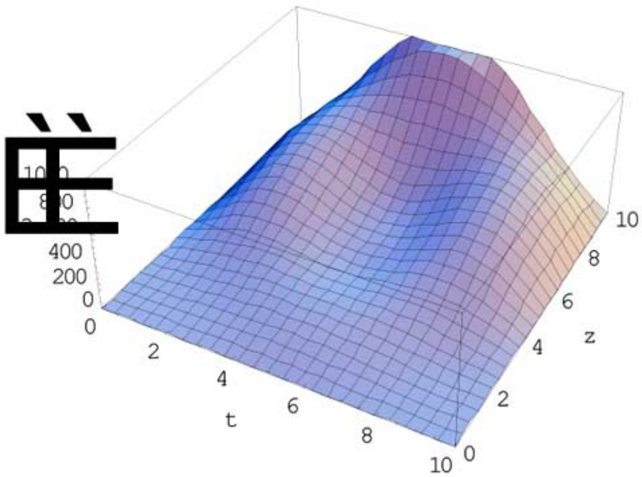

Fig. (19). The third order approximation of $\left|u^{(3)}\right|$ at $\varepsilon = 1$, $\alpha, \rho_1, \rho_2 = 1, T = 10$ with considering only ten terms of the series (M=10).

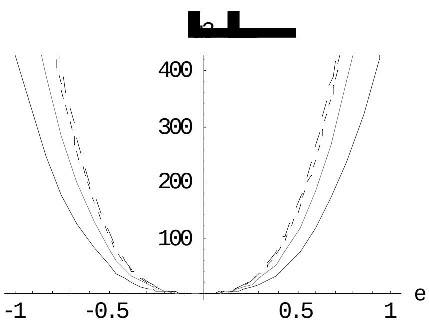

Fig. (20). The third order approximation of $\left|u^{(3)}\right|$ at $z = 10, \gamma = 0$, $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10$ for different values of t.

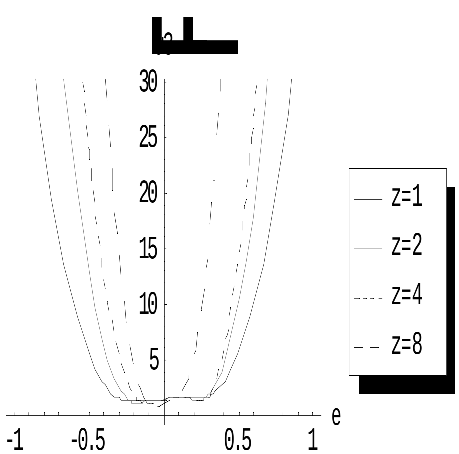

Fig. (21). The third order approximation of $\left|u^{(3)}\right|$ at $t = 4, \gamma = 0$, $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10$ for different values of z.

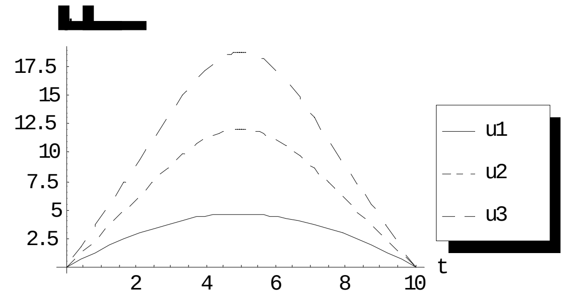

Fig. (22). Comparison between first, second and third order approximation at $\varepsilon = 0.2$, $\gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1$, $T = 10, M = 10, z = 20$.

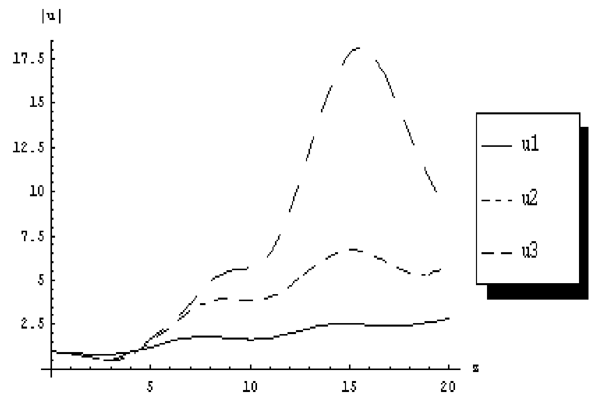

Fig. (23). Comparison between first, second and third order approximation at $\varepsilon = 0.2$, $\gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1$, $T = 10, M = 10, t = 2$.

We still notice the instability of the solution with epsilon or in the case of zero gamma.

4.5. Case Study 5

Taking the case $F_1(t,z) = 0,\ F_2(t,z) = 0\ f_1(t) = \rho_1$, $f_2(t) = \rho_2 \sin\left(\frac{m\pi}{T}\right)t, \gamma = 1$ and following the algorithm, the following selective result for the first and second order approximations Figs. (**24-27**) are got:

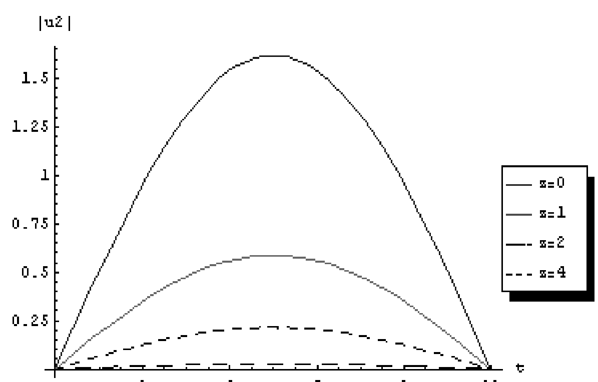

Fig. (24). The second order approximation of $\left|u^{(2)}\right|$ at $\varepsilon = 0.2$, $\gamma = 1$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10$ for different values of z.

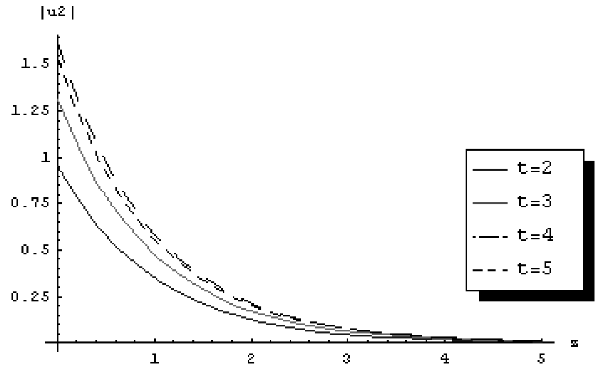

Fig. (25). The second order approximation of $|u^{(2)}|$ at $\varepsilon = 0.2$, $\gamma = 1$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10$ for different values of t.

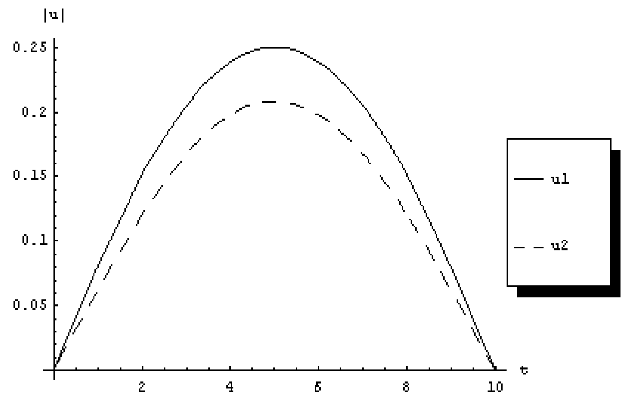

Fig. (26). Comparison between first and second order approximation at $\varepsilon = 1, \gamma = 1$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10, z = 2$.

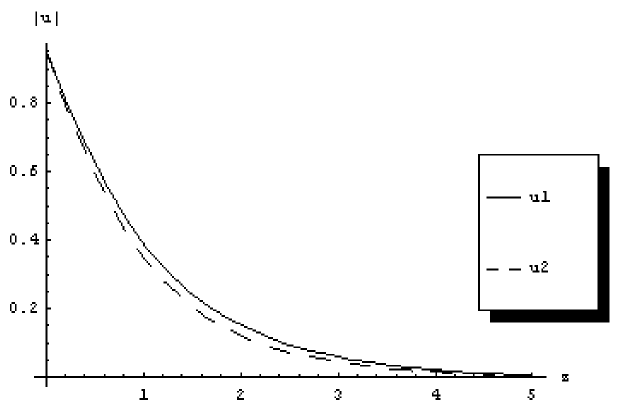

Fig. (27). Comparison between first and second order approximation at $\varepsilon = 1, \gamma = 1$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 10, t = 2$.

We still notice the effect of gamma on the stability of the solution.

5. PICARD APPROXIMATION

To validate our previous results, in the absence of the exact solution, let us follow another approximation technique. The Picard approximation is considered in this section.

Solving equation (25) with the same conditions (2) and (3) and following the Picard algorithm which puts the nonlinear terms in the right hand side of the equation evaluated at the previous step, which means that we solve the linear case iteratively [25].

Let $u(t,z) = \psi(t,z) + i\,\phi(t,z), \psi, \phi$: are real valued functions. The following coupled equations are got:

$$\frac{\partial \phi(t,z)}{\partial z} = \alpha \frac{\partial^2 \psi(t,z)}{\partial t^2} + \varepsilon(\psi^2 + \phi^2)\psi - \gamma\phi. \tag{48}$$

$$\frac{\partial \psi(t,z)}{\partial z} = -\alpha \frac{\partial^2 \phi(t,z)}{\partial t^2} - \varepsilon(\psi^2 + \phi^2)\phi - \gamma\phi. \tag{49}$$

Where $\psi(t,0) = f_1(t),\ \phi(t,0) = f_2(t)$, and all corresponding other I.C. and B.C. are zeros. Applying the Picard algorithm on equations (48) and (49), we obtain the following iterative coupled equations

$$\frac{\partial \phi_i(t,z)}{\partial z} = \alpha \frac{\partial^2 \psi_i(t,z)}{\partial t^2} + G_i^{(1)}, \quad i \geq 1 \tag{50}$$

$$\frac{\partial \psi_i(t,z)}{\partial z} = \alpha \frac{\partial^2 \phi_i(t,z)}{\partial t^2} + G_i^{(2)}, \quad i \geq 1 \tag{51}$$

Where $\psi_i(t,0) = \delta_{i,0} f_1(t),\ \phi_i(t,0) = \delta_{i,0} f_2(t)$, and all other all corresponding conditions are zeros. $G_i^{(1)}, G_i^{(2)}$ are functions to be computed from previous steps. Computing some iterations, the following order of approximations are obtained.

$$u^{(0)}(t,z) = \psi_0 + i\,\phi_0\,, \tag{52}$$

where ψ_j and ϕ_j are evaluated using the linear case algorithm.

6. CASE STUDIES, PICARD

To examine the proposed solution algorithm, some case studies are illustrated.

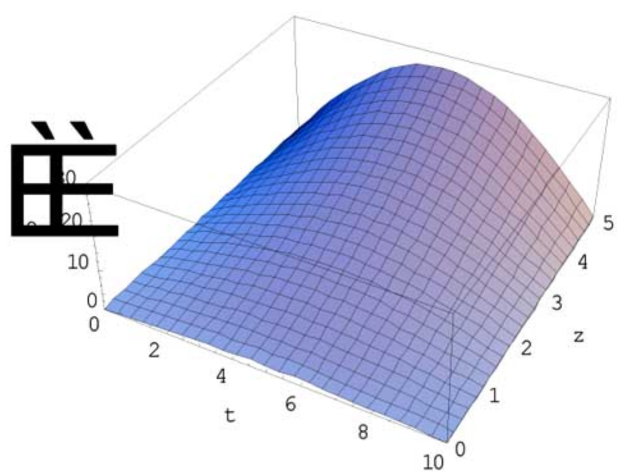

Fig. (28). The second order approximation of $|u^{(2)}|$ at $\varepsilon = 0.2$, $\gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10$ with considering only ten terms of the series (M=1).

6.1. Case Study 1

Taking the case $F_1(t,z) = 0,\ F_2(t,z) = 0\ f_1(t) = \rho_1$, $f_2(t) = \ \rho_2 \sin\left(\frac{\pi}{T}\, t\right), \gamma = 0$ and following the algorithm, the following selective result for the first, second and third order approximations Figs. (**28-30**) are got:

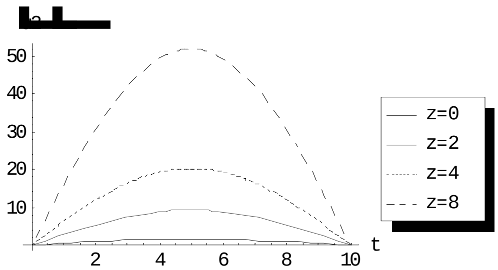

Fig. (29). The second order approximation of $|u^{(2)}|$ at $\varepsilon = 0.2$, $\gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 1$ for different values of z.

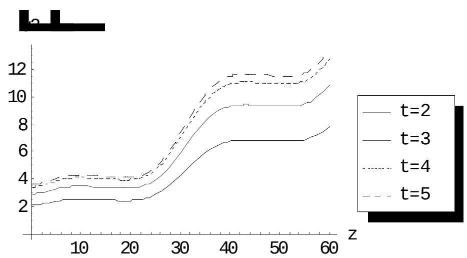

Fig. (30). The second order approximation of $|u^{(2)}|$ at $\varepsilon = 0.2$, $\gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 1$ for different values of t.

We can notice instability with the zero gamma.

6.2. Case Study 2

Taking the case $F_1(t,z) = 0, F_2(t,z) = 0\ f_1(t) = \rho_1$, $f_2(t) = \rho_2 \sin\left(\frac{\pi}{T}\, t\right), \gamma = 1$ and following the algorithm, the following selective result for the first, second and third order approximations Figs. (**31-35**) are got:

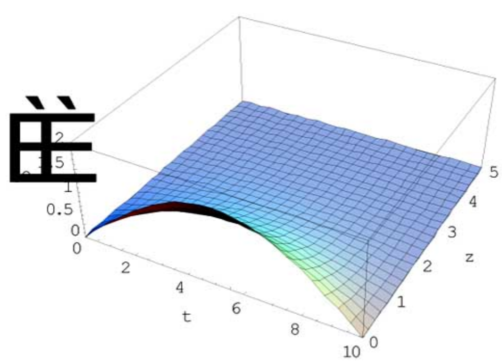

Fig. (31). The second order approximation of $|u^{(2)}|$ at $\varepsilon = 0.05$ and $\alpha, \rho_1, \rho_2, \gamma = 1, T = 10$ with considering only ten terms on the series (M=1).

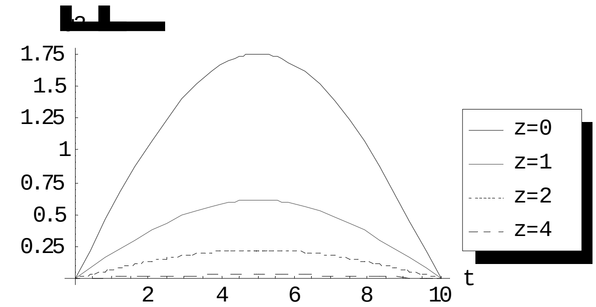

Fig. (32). The second order approximation of $|u^{(2)}|$ at $\varepsilon = 0.2$ and $\alpha, \rho_1, \rho_2, \gamma = 1, T = 10, M = 1$ for different values of z.

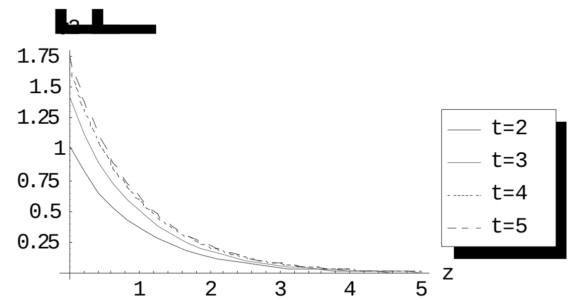

Fig. (33). The second order approximation of $|u^{(2)}|$ at $\varepsilon = 0.2$, $\alpha, \rho_1, \rho_2, \gamma = 1, T = 10, M = 1$ for different values of t.

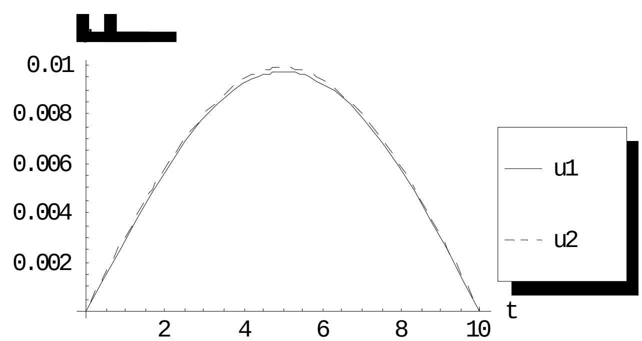

Fig. (34). Comparison between first and second order approximation at $\varepsilon = 0.2$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 1, z = 5$.

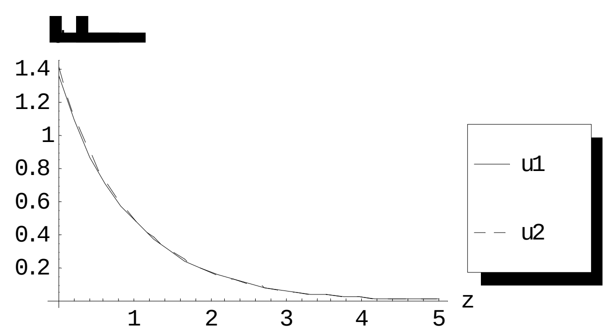

Fig. (35). Comparison between first and second order approximation at $\varepsilon = 0.2$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, M = 1, t = 3$.

7. COMPARISON BETWEEN PERTURBATION & PICARD APPROXIMATION

Let us compare between the two methods.

7.1. Case 1

Taking the case $f_1(t) = \rho_1, f_2(t) = \rho_2, \gamma = 0$.

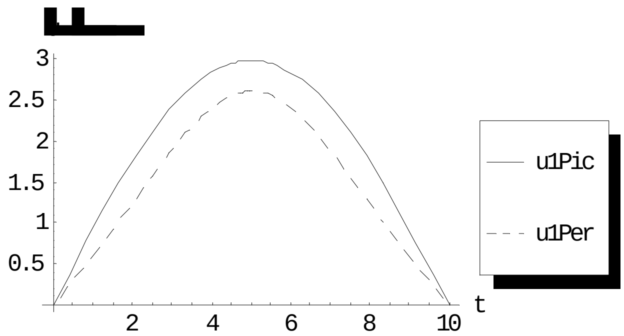

Fig. (36). Comparison between Picard approximation and Perturbation method for first order at $\varepsilon = 0.2, \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, z = 5$.

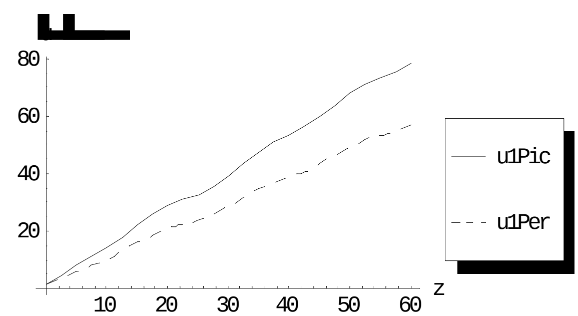

Fig. (37). Comparison between Picard approximation and Perturbation method for first order at $\varepsilon = 1, \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, t = 3$.

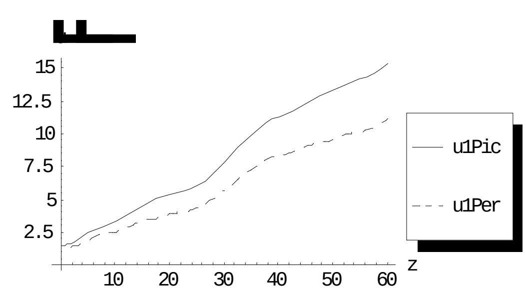

Fig. (38). Comparison between Picard approximation and Perturbation method for first order at $\varepsilon = 0.2, \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, t = 3$.

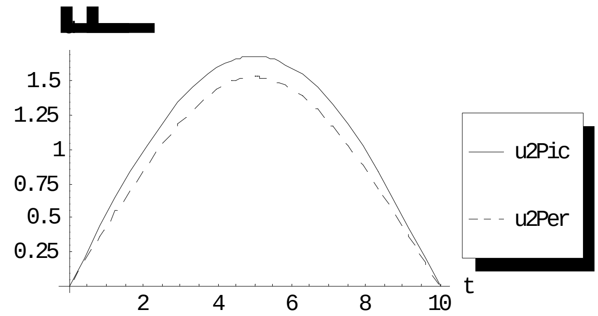

Fig. (39). Comparison between Picard approximation and Perturbation method for second order at $\varepsilon = 0.02, \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1$, $T = 10, z = 5$.

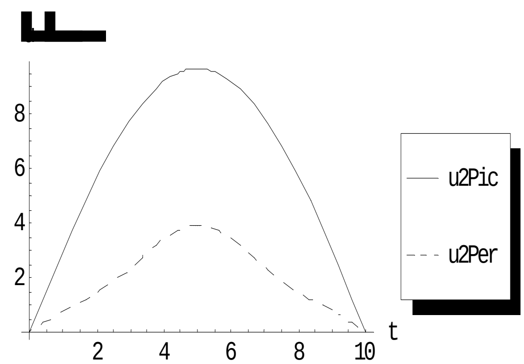

Fig. (40). Comparison between Picard approximation and Perturbation method for second order at $\varepsilon = 0.2, \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1$, $T = 10, z = 5$.

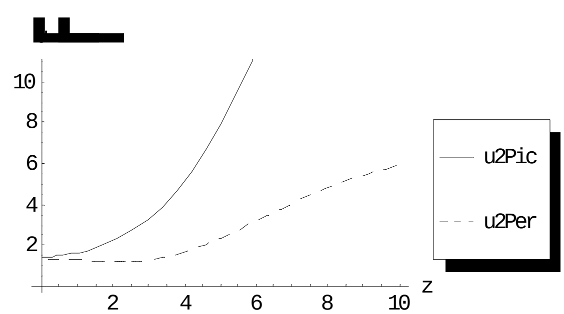

Fig. (41). Comparison between Picard approximation and Perturbation method for second order at $\varepsilon = 0.2, \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1$, $T = 10, t = 3$.

7.2. Case 2

Taking the case $f_1(t) = \rho_1, f_2(t) = \rho_2, \gamma = 1$, the following selective results, (Fig. (**36-41**)) are obtained:

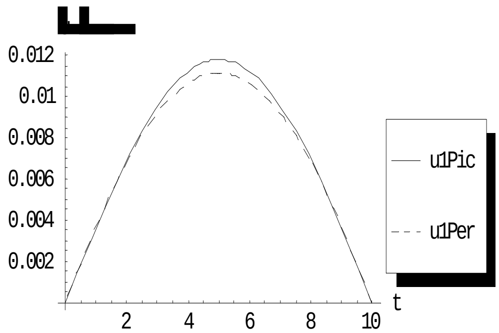

Fig. (42). Comparison between Picard approximation and Perturbation method for first order at $\varepsilon = 1, \gamma = 1$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, z = 5$.

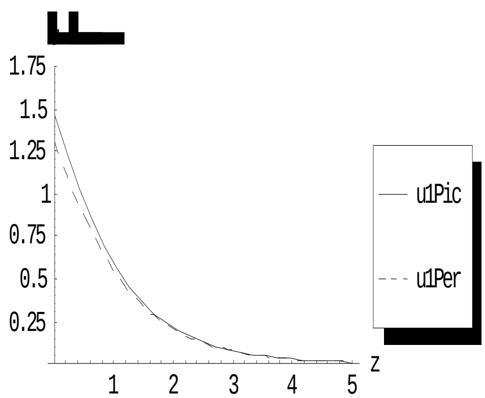

Fig. (43). Comparison between Picard approximation and Perturbation method for first order at $\varepsilon = 1, \gamma = 1$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, t = 3$.

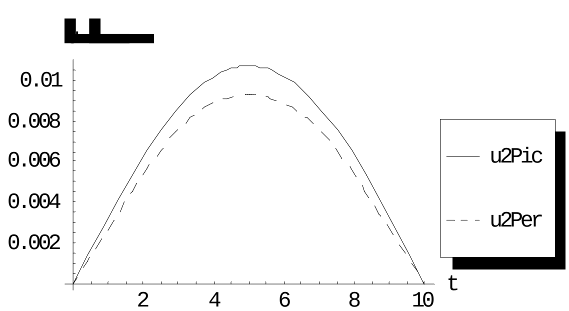

Fig. (44). Comparison between Picard approximation and Perturbation method for second order at $\varepsilon = 1, \gamma = 1$ and $\alpha, \rho_1, \rho_2 = 1$, $T = 10, z = 5$.

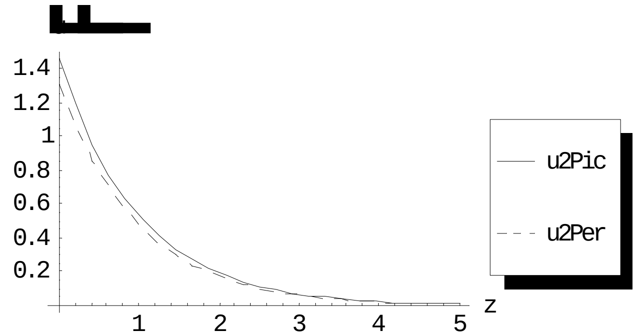

Fig. (45). Comparison between Picard approximation and Perturbation method for second order at $\varepsilon = 1, \gamma = 1$ and $\alpha, \rho_1, \rho_2 = 1$, $T = 10, t = 3$.

We can notice that the two approximations are very near in the presence of gamma. In fact we noticed this fact for all case studies.

Taking the case $f_1(t) = \rho_1,\ f_2(t) = \ \rho_2 \sin\left(\frac{\pi}{T}t\right), \gamma = 0$, the following selective results, Figs. (**42-45**) are obtained:

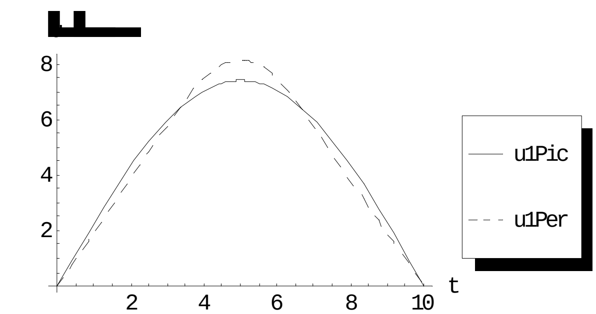

Fig. (46). Comparison between Picard approximation and Perturbation method for first order at $\varepsilon = 1, \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, z = 5$.

Taking the case $f_1(t) = \rho_1,\ f_2(t) = \ \rho_2 \sin\left(\frac{\pi}{T}t\right), \gamma = 1$, we obtained, Fig. (**46**):

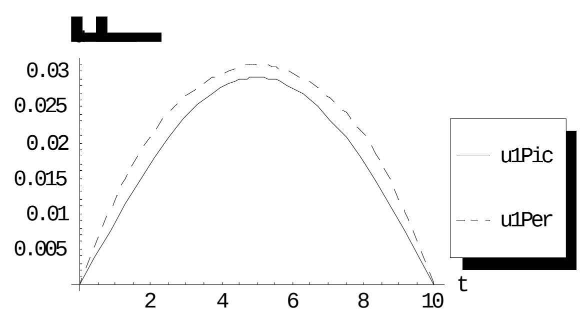

Fig. (47). Comparison between Picard approximation and Perturbation method for first order at $\varepsilon = 1, \gamma = 1$ and $\alpha, \rho_1, \rho_2 = 1, T = 10,\ z = 4$.

Taking the case $f_1(t) = \rho_1 e^{-t}, f_2(t) = \rho_2 e^{-t}, \gamma = 0$, the following selective results, Figs. (**47**, **48**) are obtained:

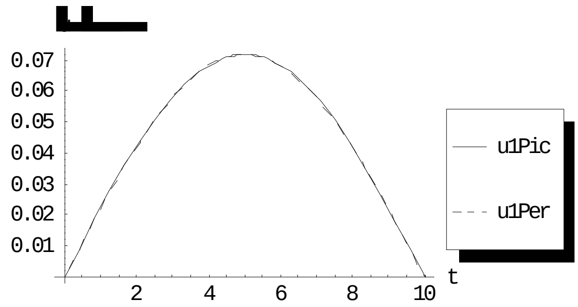

Fig. (48). Comparison between Picard approximation and Perturbation method for first order at $\varepsilon = 1, \gamma = 0$ and $\alpha, \rho_1, \rho_2 = 1, T = 10, z = 5$.

CONCLUSIONS

The stability of the solution of the cubic nonlinear homogeneous Schrodinger equation is highly affected in the absence of gamma. The perturbation as well as the Picard methods introduce approximate solutions for such problems where second or third order of approximations can be obtained from which some parametric studies can be achieved to illustrate the solution behavior under the change of the problem physical parameters. The use of Mathematica, or any other symbolic code, makes the use of the solution algorithm possible and can develop a solution procedure which can help in getting some knowledge about the solution.

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