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Nonlinear Vibration Model for Initially Stressed Beam-Foundation System

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Abstract: An analytical solution for nonlinear vibration of an initially stressed beam with elastic end restraints resting on a nonlinear elastic foundation is obtained. As a first step in solving nonlinear vibration equation, the linear vibration mode functions for a beam with elastic end restraints resting on a linear elastic foundation are obtained. Then, the nonlinear vibration equation is solved by employing the linear mode functions to obtain frequency equation and nonlinear response using Jacobi elliptic integral. The nonlinearity due to lateral vibrations, the nonlinearity of foundations and lateral displacement due to lateral elastic restraints at beam ends not included in previous analytical work are considered in the present work. The effects of spring stiffness at the beam ends, foundation stiffness, axial load and vibration amplitude on the frequency parameter are studied. The present solution can be used to measure the accuracy of approximate methods.

Keywords: Nonlinear beam vibration, elliptic integrals, nonlinear foundations, mode functions and natural frequencies.

1. INTRODUCTION

Many practical engineering applications as railroad tracks, highway pavement, buried pipelines and foundation beams are modeled as beams resting on elastic foundations. To investigate the dynamics of the vibrational behavior of these applications, solutions need to be obtained. Few analytical solutions, limited to idealized cases for vibrations of such models are found in the literature. This is due to the intractable mathematical nature of the problem. Numerical methods such as finite element method [1-2], transfer matrix method [3], differential quadrature element method (DQM) [4-6], perturbation techniques [7-8] are used to obtain the vibration behavior of different types of linear or nonlinear beams resting on linear or nonlinear foundations.

Semi-analytical methods, such as series solutions, are suggested to obtain frequencies and mode functions of nonuniform beams resting on elastic foundation [9, 10]. Taha M.H. [11] studied the transient response of beams resting on viscoelastic foundation under stochastic dynamic loads. Taha M.H. and Abohadima S. [12] analyzed the vibrational behavior of a nonuniform flexural beam using Bessel's functions. Abohadima, S. and Taha M.H. [13] extended their works to the vibrational behavior of a nonuniform beam resting on a nonuniform foundation.

However, in most studies, the boundary conditions assumed to simulate the actual conditions at beam supports are idealized to obtain simple solutions. In fact, there is no absolutely clamped or pinned support, actually all supports always allow some movements. Also, few researchers studied the effects of axial load on the vibration behavior of such beams and nonlinearity due to stretching resulting from lateral vibrations.

In the present work, the effects of above mentioned parameters are taken into account. The nonlinear vibration of an initially stressed beam with elastic end restraints resting on a nonlinear elastic foundation is solved using elliptic integrals. The obtained solutions are verified against those obtained from numerical methods and found in close agreement. Parametric study to investigate the influences of foundation stiffness, elastic end restraints stiffnesses, initial axial load and vibration amplitude are conducted and results are depicted in graphs for a wide range of the different practical characteristics.

2. ANALYSIS

2.1. Vibration Equation

The equation of motion of a uniform beam with elastic end restrains, initially stressed by an axial load P_o, resting on a nonlinear elastic foundation, shown in Fig. (**1**) is given as:

$$EI\frac{\partial^4 Y}{\partial X^4}+P_o\frac{\partial^2 Y}{\partial X^2}+\mu\frac{\partial^2 Y}{\partial t^2}-\frac{EA}{2L}\{\int_0^L(\frac{\partial Y}{\partial X})^2\,dX\}\frac{\partial^2 Y}{\partial X^2}+ k_1Y(X,t)+k_2Y^3(X,t)=0 \quad (1)$$

where EI is the flexural stiffness of the beam, L is the beam length, μ is the beam mass per unit length, k_1 and k_2 are the foundation stiffnesses coefficients per unit length, E is the modulus of elasticity of the beam, A is the area of the beam cross section, $Y(X, t)$ is the lateral response of the beam, X is the coordinate along the beam and t is time.

Using the dimensionless parameters $x=X/L$ and $y=Y/L$, eqn. (1) may be rewritten as:

$$\frac{EI}{\mu L^4}\frac{\partial^4 y}{\partial x^4}+\frac{P_o}{\mu L^2}\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial t^2}-\frac{EA}{2\mu L^2}\{\int_0^1(\frac{\partial y}{\partial x})^2\,dx\}\frac{\partial^2 y}{\partial x^2}+ \frac{k_1}{\mu}y(x,t)+\frac{k_2L^2}{\mu}y^3(x,t)=0 \quad (2)$$

The solution of the nonlinear partial differential eqn. (2) is obtained by employing the linear mode functions and

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integrating over the domain of the dimensionless spatial variable x to separate the time variation. However, the solution of eqn. (2) depends on the boundary conditions at beam ends.

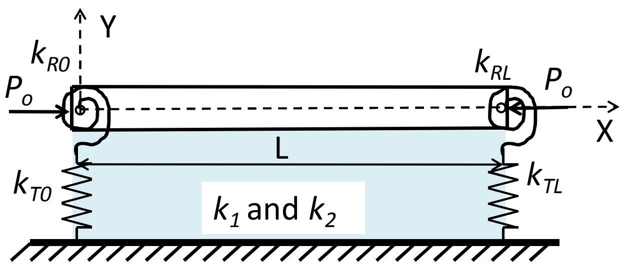


Fig. (1). Initially stressed beam-foundations system.

2.2. Boundary Conditions

The boundary conditions due to elastic end restraints at *x=0* are given as;

$$k_{T0}\, y(0,t) = -\frac{EI}{L^3}\frac{\partial^3 y(0,t)}{\partial x^3} \quad , \tag{3a}$$

$$k_{R0}\frac{\partial y(0,t)}{\partial x} = \frac{EI}{L}\frac{\partial^2 y(0,t)}{\partial x^2} \quad . \tag{3b}$$

and at *x=1* are:

$$k_{TL}\, y(1,t) = \frac{EI}{L^3}\frac{\partial^3 y(1,t)}{\partial x^3} \quad , \tag{3c}$$

$$k_{RL}\frac{\partial y(1,t)}{\partial x} = -\frac{EI}{L}\frac{\partial^2 y(1,t)}{\partial x^2} \quad . \tag{3d}$$

where k_{T0} and k_{TL} are the elastic stiffnesses of lateral restraints at *x=0, 1.0* respectively and k_{R0} and k_{RL} are the elastic rotation stiffnesses of beam support at *x=0, 1.0* respectively.

2.3. Solution of Linear Vibration Equation

A linear version of eqn. (2), neglecting the initial axial load, can be expressed as:

$$\frac{EI}{\mu L^4}\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} + \frac{k_1}{\mu}\, y(x,t) = 0 \tag{4}$$

Following the separation of variables analogy, the solution of eqn. (4) may be assumed as:

$$y(x,t) = y_o\, \phi(x)\psi(t) \tag{5}$$

where $\phi(x)$ is the linear mode function, $\psi(t)$ is a function representing the variation of the response with time and y_o is the dimensionless vibration amplitude (obtained from the initial conditions). Substituting eqn. (5) into eqn.(4), eqn.(4) is separated into:

$$\frac{d^4\phi}{dx^4}) - \lambda_f^4\, \phi(x) = o \tag{6}$$

$$\frac{d^2\psi}{dt^2} + \omega^2\psi(t) = 0 \tag{7}$$

where ω is the separation constant which represents the natural frequency and λ_f is the frequency parameter which is given as:

$$\lambda_f^4 = \frac{\mu L^4}{EI}(\omega^2 - \frac{k_1}{\mu}) \tag{8}$$

The general solution of eqn. (6) is given as:

$$\phi(x) = C_1\cos(\lambda_F x) + C_2\sin(\lambda_F x) + C_3\cosh(\lambda_F x) + C_4\sinh(\lambda_F x) \tag{9}$$

and the solution of eqn. (7), is:

$$\psi(t) = A_1\cos(\omega\, t) + A_2\sin(\omega\, t) \tag{10}$$

where constants A_1 and A_2 are obtained from initial conditions.

Substitution eqn. (9) into boundary conditions, eqns. (3), yields a system of homogeneous algebraic equations in unknown constants C_i, i=1, 2, 3, 4 with parameter λ_f. However, the condition of nontrivial solution for such system leads to the frequency equation as:

$$A_{11}A_{22} - A_{12}A_{21} = 0 \tag{11}$$

where:

$$A_{11} = -\xi_1 - \alpha_1\xi_2 + \alpha_2\xi_4 \tag{12a}$$

$$A_{12} = \xi_3 - \alpha_2\xi_2 + \alpha_1\xi_4 \tag{12b}$$

$$A_{21} = \eta_1 + \alpha_1\eta_2 + \alpha_2\eta_4 \tag{12c}$$

$$A_{22} = \eta_3 + \alpha_2\eta_2 + \alpha_1\eta_4 \tag{12d}$$

$$\alpha_1 = \frac{k_{R0}L}{2EI\lambda_f} - \frac{EI\lambda_f^3}{2k_{T0}L^3} \text{ and } \alpha_2 = \frac{k_{R0}L}{2EI\lambda_f} + \frac{EI\lambda_f^3}{2k_{T0}L^3} \tag{13}$$

$$\xi_1 = \cos(\lambda_f) + (K_{TL}/\lambda_f^3)\sin(\lambda_f) \tag{14a}$$

$$\xi_2 = \sin(\lambda_f) - (K_{TL}/\lambda_f^3)\cos(\lambda_f) \tag{14b}$$

$$\xi_3 = \cosh(\lambda_f) - (K_{TL}/\lambda_f^3)\sinh(\lambda_f) \tag{14c}$$

$$\xi_4 = \sinh(\lambda_f) - (K_{TL}/\lambda_f^3)\cosh(\lambda_f) \tag{14d}$$

$$\eta_1 = (K_{RL}/\lambda_f)\cos(\lambda_f) - \sin(\lambda_f) \tag{15a}$$

$$\eta_2 = (K_{RL}/\lambda_F)\sin(\lambda_F) + \cos(\lambda_F) \tag{15b}$$

$$\eta_3 = (K_{RL}/\lambda_f)\cosh(\lambda_f) + \sinh(\lambda_f) \tag{15c}$$

$$\eta_4 = (K_{RL}/\lambda_f)\sinh(\lambda_f) + \cosh(\lambda_f) \tag{15d}$$

and the restraints stiffness parameters are:

$$K_{T0} = \frac{k_{T0}L^3}{EI} \quad \text{and} \quad K_{TL} = \frac{k_{TL}L^3}{EI} \tag{16}$$

$$K_{R0} = \frac{k_{R0}L}{EI} \quad \text{and} \quad K_{RL} = \frac{k_{RL}L}{EI} \tag{17}$$

The frequency equation (11) can be solved using any proper iterative technique to obtain the frequency parameters λ_{fm}, where m=1,2, … is the mode number, hence the natural frequency ω_m can be calculated by means of eqn. (8).

The normalized mode function is obtained assuming $C_1 = 1$ then, the values of the other three constants can be obtained in terms of α_1 and α_2. The m-mode function is obtained as:

$$\phi_m(x) = \sin\lambda_{fm}x + (\alpha_2\alpha_o - \alpha_1)\cos\lambda_{fm}x - \alpha_o\sinh\lambda_{fm}x + (\alpha_2 - \alpha_1\alpha_o)\cosh\lambda_{fm}x \quad (18)$$

where: $\alpha_o = \dfrac{A_{11}}{A_{12}}$

Substituting eqn. (10) and eqn. (18) into eq. (5), the lateral vibration for linear case of a beam resting on linear elastic foundations is obtained as:

$$y_m(x,t) = \sum \begin{pmatrix} \sin\lambda_{fm}x + \\ (\alpha_2\alpha_o - \alpha_1)\cos\lambda_{fm}x \\ -\alpha_o\sinh\lambda_{fm}x + \\ (\alpha_2 - \alpha_1\alpha_o)\cosh\lambda_{fm}x \end{pmatrix} (A\cos\omega_m t + B\cos\omega_m t) \quad (19)$$

Constants *A* and *B* are obtained from initial condition and orthogonality properties of mode functions.

2.4. Solution of the Nonlinear Vibration Equation

The linear m-mode function is employed in the nonlinear vibration equation to obtain the solution of nonlinear case. The nonlinear vibration is assumed as:

$$y(x,t) = y_o\,\phi_m(\lambda_{fm}x)\psi(t) \quad (20)$$

Substitution of eqn. (20) into eqn. (2) and integrating over the x-domain leads to:

$$\frac{\partial^2\psi}{\partial t^2} + \gamma_{1m}\,\psi(t) + \gamma_{2m}\,\psi^3(t) = 0 \quad (21)$$

where:

$$\gamma_{1m} = \omega_m^2 + \frac{P_o}{\mu L^2}\frac{\int_0^1 \phi'' dx}{\int_0^1 \phi_m\, dx} \quad (22a)$$

$$\gamma_{2m} = \frac{k_2 L^2 y_o^2}{\mu}\frac{\int_0^1 \phi_m^3\, dx}{\int_0^1 \phi_m\, dx} - \frac{EAy_o^2}{2\mu L^2}\{\int_0^1 (\phi_m^{'})^2\, dx\}\frac{\int_0^1 \phi_m^{''}\, dx}{\int_0^1 \phi_m\, dx} \quad (22b)$$

where ω_m is the linear m-mode natural frequency. Integrating eqn. (21) with respect to time and assuming at $t=0$, $\psi=1$ and $d\psi/dt=0$, eqn. (21) may be rewritten as:

$$\{\frac{d\psi}{d(\rho_m t)}\}^2 = (1-\psi^2)(k_m^2\psi^2 - k_m^2 + 1) \quad (23)$$

where:

$$\rho_m^2 = \gamma_{1m} + \gamma_{2m} \text{ and } k_m^2 = \frac{\gamma_{2m}}{2\rho_m^2} \quad (24a, b)$$

Substituting $\psi(t) = \cos(\varphi)$, where $\varphi = \varphi(t)$ into equation (23), one gets:

$$t\rho_m = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k_m^2\sin^2(\varphi)}}. \quad (25)$$

The integration of eqn. (25) is the elliptic integral of the first kind, its inversion yields the Jacobi elliptic function; $cn[\rho_m t, k_m]$.

Then, the variation in the lateral displacement of the beam at any location with time can be expressed as:

$$\psi_m(t) = cn[\rho_m t, k_m]. \quad (26)$$

The period of the Jacobi elliptic function is defined by the complete elliptic integral:

$$T_m = \frac{4}{\rho_m}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k_m^2\sin^2(\varphi)}}. \quad (27)$$

Then, the natural frequency Ω for m-mode in nonlinear vibration is:

$$\Omega_m = \frac{2\pi}{T_m} \quad (28)$$

2.5. Verification of Present Solution

To verify the obtained solutions, values of the frequency parameter calculated using the present solution and those obtained from numerical [1] and semi-analytical methods [10] are shown in Table **1** for conventional support condition. It is obvious that the obtained results are in close agreement with results obtained from FEM and semi-analytical method. Also, it is found that value of 1E5 for restraint stiffness parameter K is enough to represent absolutely rigid condition (no movement)

3. NUMERICAL RESULTS

The derived expressions are used to study the influence of different parameters on the natural frequency of the beam-foundation system. However, the natural frequency of the system increases as the overall stiffness of the system increases. The overall stiffness of the system depends on the flexural stiffness of the beam, the stiffness of the foundation, the stiffness of the elastic end restraints. In the case of compression axial load, the lateral component of the axial load in the deformed configuration is in the opposite direction of the resultant restoring force resulted from system overall stiffness and in the same direction in case of axial tension load. Therefore, as the compressive axial force increases, the resultant of the total restoring force decreases, and the natural frequency of the system decreases. As the magnitude of the axial compression load approaches a certain critical value, its lateral component compensates the effect of the system restoring force and the system transforms to aperiodic one approaching asymptotically the equilibrium deformed configuration. Indeed, this critical value is the buckling load of the beam-soil system.

On the other hand, in case of initial axial tension, it causes an increase in the vibration frequency of the system. Moreover, as the vibration amplitude increases, the

Table 1. Frequency Parameter λ_{fm} for Linear Vibration

Elastic End Restraints Stiffness				Mode (m)				Analysis	Supports
K_{T0}	K_{TL}	K_{R0}	K_{RL}	1	2	3	4		
1E5	1E5	1E5	0	3.924	7.061	10.191	13.312	Present	Clamped – Pinned
1E5	1E5	1E5	0	3.927	7.07	10.21	13.352	[1]	
1E5	1E5	1E5	0	3.93	7.075	10.216	13.361	[10]	
1E5	1E5	1E5	1E5	4.725	7.839	10.963	14.072	Present	Clamped – Clamped
1E5	1E5	1E5	1E5	4.73	7.782	11.013	14.155	[1]	
1E5	1E5	1E5	1E5	4.694	7.794	10.917	14.04	[10]	
1E5	1E5	0	0	3.141	6.282	9.416	12.546	Present	Pinned -Pinned
1E5	1E5	0	0	3.141	6.283	9.425	12.566	[1]	

[1] Naidu using FEM.
[10] Wang Using Fourier transforms.

stretching due to deformed configuration produces an axial tensile force which increases the frequency parameter of the system. Bearing in mind these facts, the effects of different parameters on the vibration behavior of the beam-foundation system can be predicted qualitatively.

For the case of linear vibration of a beam resting on linear elastic foundations (P_o=0, k_2=0 and stretching due to transverse vibration is neglected) values for frequency parameter λ_{fm} against different values of end restraints are shown in Table **2**. It is clear that the frequency parameter increases as the stiffnesses of the end restraints increase. The frequency parameter for subsequent mode is shifted by π for conventional end supports and by value less than π for elastic end restrains. However, the effect of stiffness of elastic end restrains is more noticeable for frequency parameter of lower modes.

The m-frequency parameter λ_{fm} for the nonlinear case is defined as:

$$\lambda_{fm}^4 = \frac{\mu L^4 \Omega_m^2}{EI} \tag{29}$$

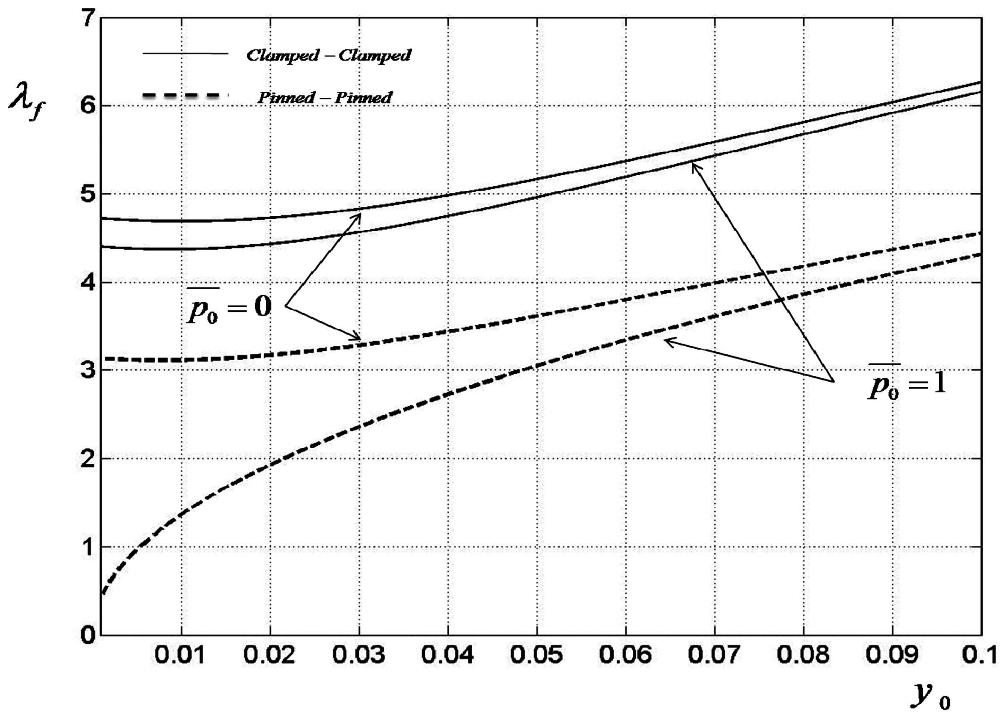
Using dimensionless parameters with respect to geometric properties of the beam eliminates the effect of the geometric properties of the beam on the frequency parameter.

The effects of dimensionless vibration amplitude on the fundamental frequency parameter λ_f (m=1) for different values of load parameter $\overline{P}_o$, end conditions and foundation stiffnesses ($\overline{k}_1$, $\overline{k}_2$) are shown in Fig. (**2**). Fig. (**2A**) represents the case of a beam without foundation and Fig. (**2B**) represents the beam-foundation system. The dimensionless load parameter and foundation parameters are defined as:

Table 2. Frequency Parameter λ_{fm} for Linear Vibration

Elastic End Restraints Stiffness				Mode (m)				Designation
K_{T0}	K_{TL}	K_{R0}	K_{RL}	1	2	3	4	
1E5	0	1E5	0	1.874	4.691	7.847	10.979	Clamped - Free
1E5	10	1E5	0	2.638	4.791	7.868	10.986	
1E5	100	1E5	0	3.639	5.613	8.077	11.058	
1E5	1E5	1E5	0	3.924	7.061	10.191	13.312	Clamped - Pinned
1E5	0	1E5	100	2.206	5.101	8.149	11.221	
1E5	100	1E5	100	3.739	5.703	8.324	11.289	
1E5	1E5	1E5	100	4.231	7.26	10.339	13.428	
1E5	0	1E5	1000	2.341	5.425	8.528	11.627	
1E5	100	1E5	1000	3.827	5.796	8.628	11.667	
1E5	1E5	1E5	1000	4.616	7.672	10.746	13.816	
1E5	1E5	1E5	1E5	4.725	7.839	10.963	14.072	Clamped - Clamped
0	0	0	0	4.73	7.853	10.995	14.144	Free - Free
1E5	1E5	0	0	3.141	6.282	9.416	12.546	Pinned - Pinned

(**A**) Beam without foundation ($\bar{k}_1 = \bar{k}_2 = 0$)



(**B**) Beam – foundation system ($\bar{k}_1 = 1E2$ and $\bar{k}_2 = 1E5$)

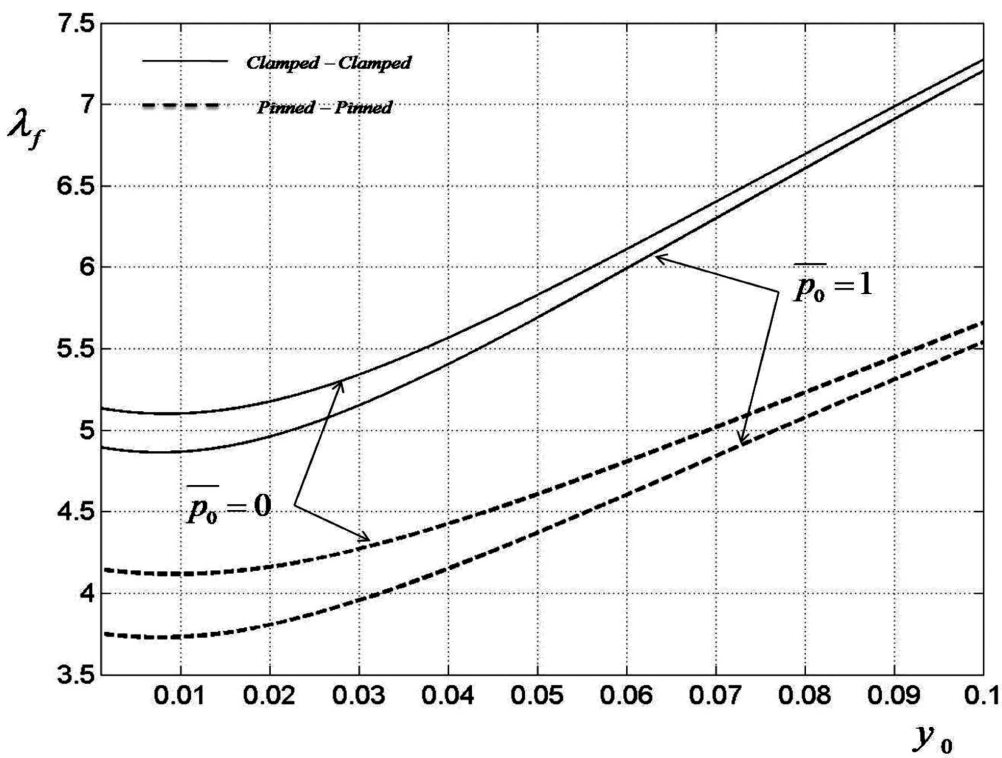


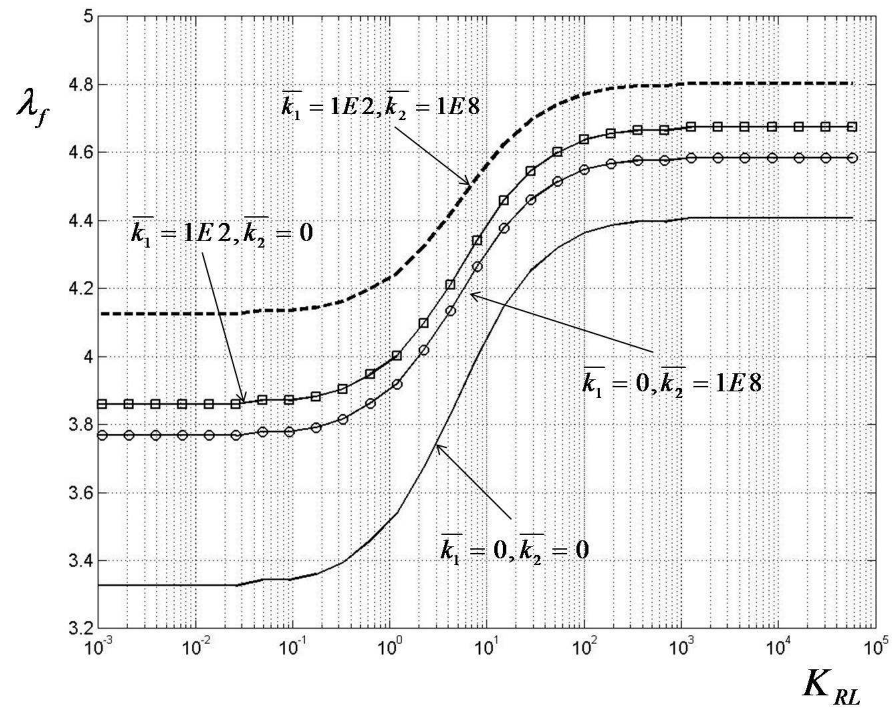
Fig. (2). Influence of dimensionless vibration amplitude y_o on frequency Parameter λ_f.

$$\bar{P}_o = \frac{P_o L^2}{\pi^2 EI} \quad (30)$$

$$\bar{k}_1 = \frac{k_1 L^4}{EI} \text{ and } \bar{k}_2 = \frac{k_2 L^6}{EI} \quad (31)$$

In case of beam without foundation, as the value of $\bar{P}_o$ approaches the buckling value, the frequency parameter approaches zero (Aperiodic system). Also, it should be noted that, the value for critical axial load (buckling load) in case of beam-foundation system is greater than its value in case of beam without foundation.

(**A**) Effects of foundation parameters ($\bar{P}_o = 1$)



(**B**) Effects of load parameter ($\bar{k}_1 = 1E2$ and $\bar{k}_2 = 1E5$)

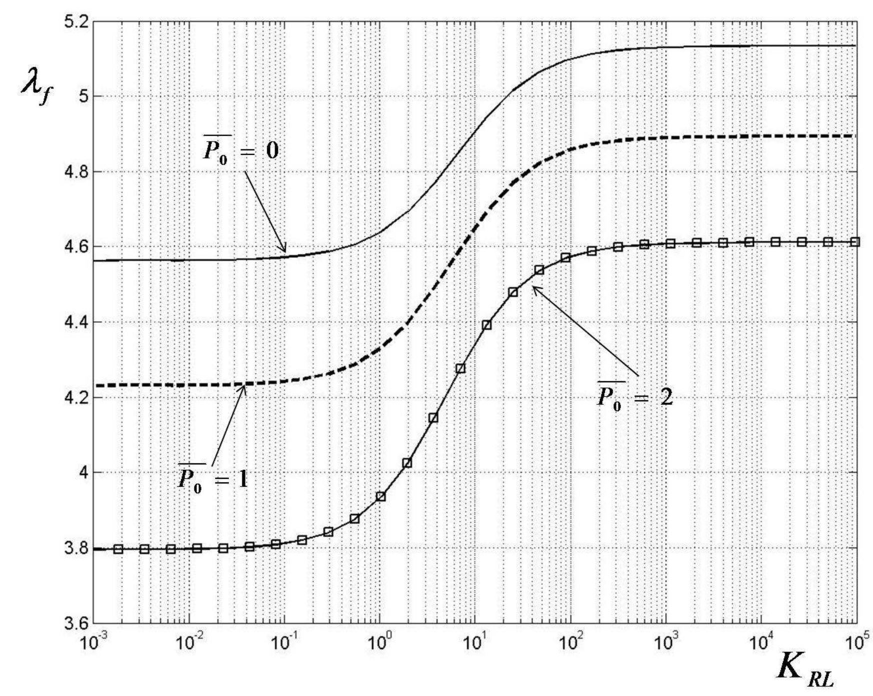
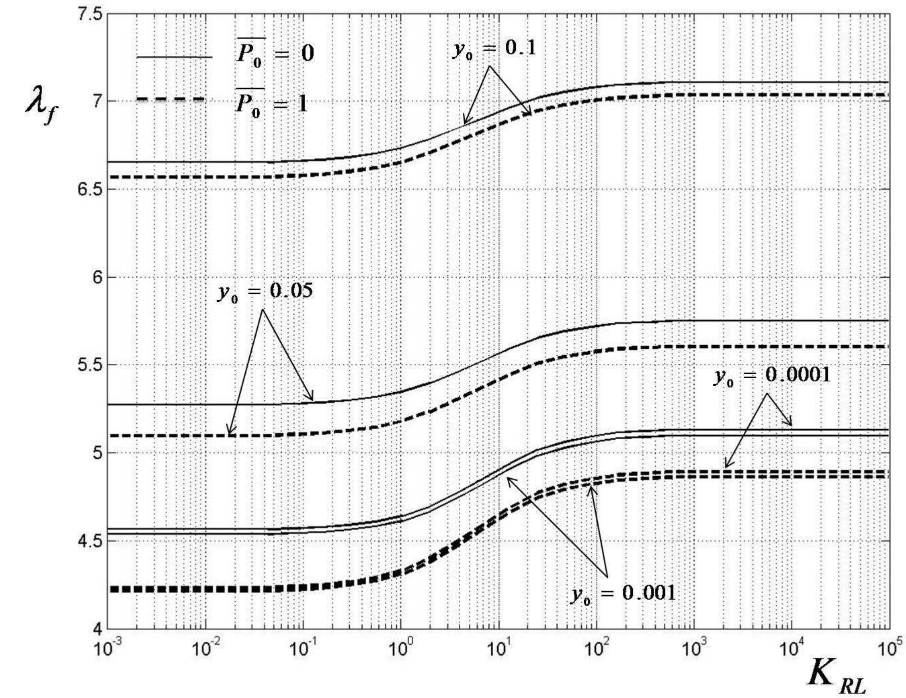


Fig. (3). Influence of rotational stiffness parameter on frequency parameter $\lambda_f(K_{T0} = K_{TL} = K_{R0} = 1E5)$.

Fig. (**3**) shows the influences of elastic rotational stiffness at one end on frequency parameter λ_f for different values of foundation parameters in Fig. (**3A**) and for different values of load parameter in Fig. (**3B**). It is obvious that the frequency parameter increases as the system stiffness increases and as the load parameter decreases. The limiting values of the frequency parameter represent the case of conventional end conditions.

(**A**) Effects of rotation stiffness (K_{TL} = *1E5*)



(**B**) Effects of lateral translation stiffness (K_{RL} = *1E5*)

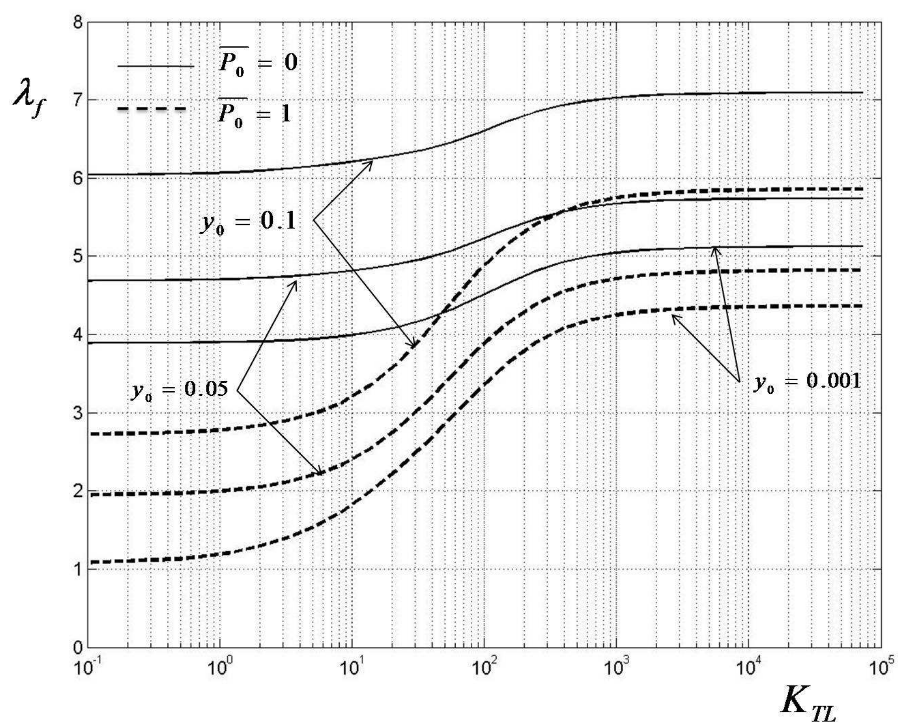
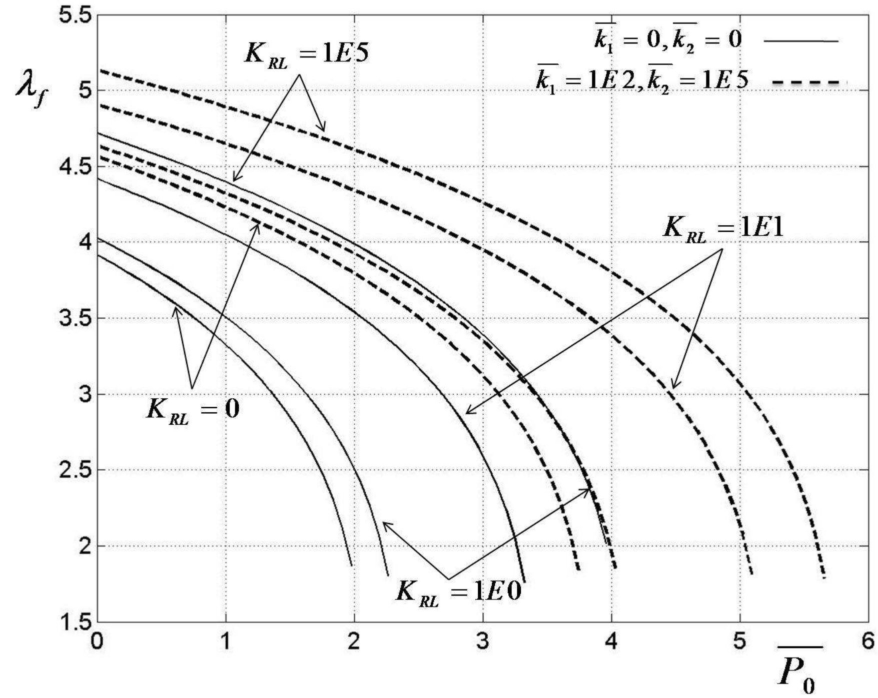


Fig. (4). Influences of vibration amplitude and load parameter on frequency parameter λ_f($K_{T0} = K_{R0} = 1E5$ and $\overline{k}_1 = 1E2$, $\overline{k}_2 = 1E5$).

Fig. (**4**) depicts the influences of dimensionless vibration amplitude y_o and elastic stiffness at one end on the frequency parameter. Fig. (**4A**) represents the effects of elastic rotation stiffness at one end and Fig. (**4B**) represents the effects of elastic lateral translation stiffness.

(**A**) Beam clamped at one end ($K_{R0} = 1E5$)



(**B**) Beam pinned at one end ($K_{R0} = 0$)

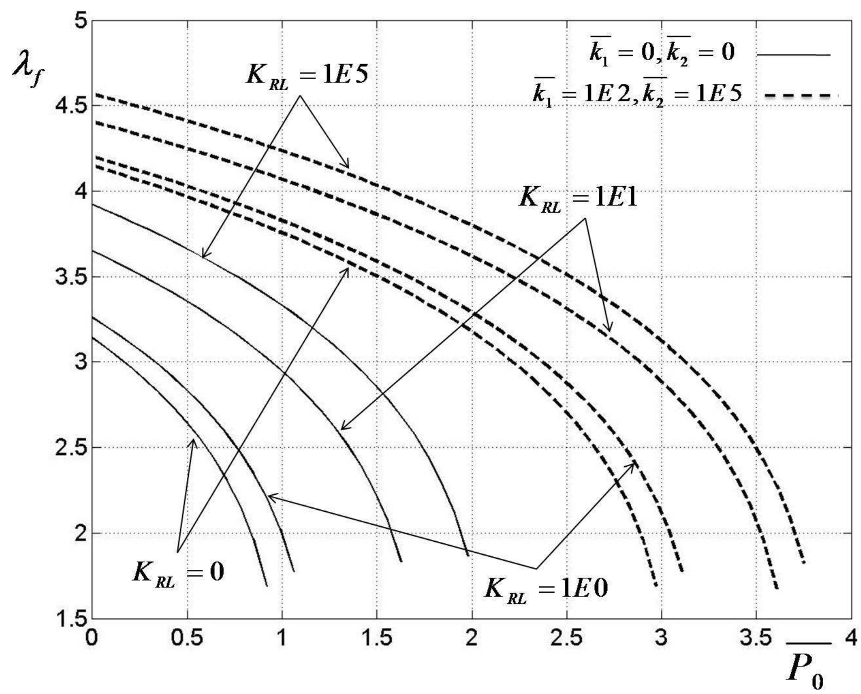


Fig. (5). Influences of load parameter and rotational stiffness on frequency parameter $\lambda_f(K_{T0} = K_{TL} = 1E5)$.

In Fig. (**5**), the effects of the load parameter on the frequency parameter for different end restrains are shown. The figure predicts both the frequency parameter for beam-foundation system of different characteristics and the critical (Buckling) load for such cases.

Fig. (**6**) shows the influence of end lateral translation stiffness parameter on the frequency parameter λ_f for different values of foundation parameters, while the load parameter takes the value of Euler critical load for simply supported beam without foundation. The effect of variation

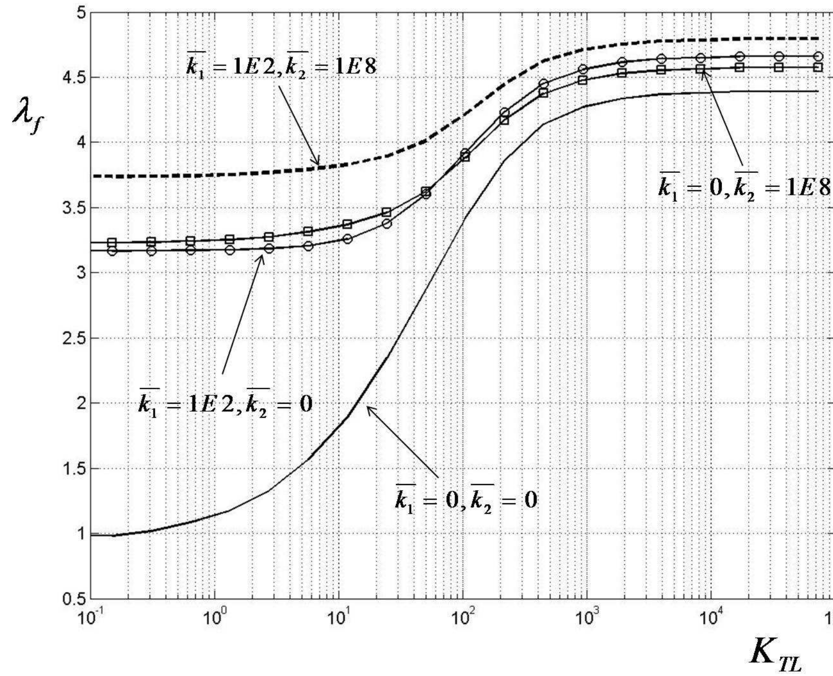


Fig. (6). Influence of lateral translation stiffness on frequency parameter λ_f ($\overline{P}_o = 1$ and $K_{T0} = K_{R0} = K_{RL} = 1E5$).

of elastic end restrains stiffness is less noticeable for beams resting on foundations.

4. CONCLUSIONS

Analytic expressions for the natural frequencies of nonlinear vibration of an initially stressed beam with elastic end restraints resting on a nonlinear foundation are obtained. It is found that the natural frequency of the beam-foundation system increases as the overall stiffness of system increases. The overall stiffness of the system is composed of the flexural stiffness of the beam, the stiffness of elastic restraints at ends and the foundation stiffness. The natural frequency of the system decreases as the axial compression load increases. However, as the axial compression load approaches certain value, the system transforms to aperiodic system and approaches asymptotically the deformed equilibrium configuration. Indeed, this critical value is called the buckling load. The stretching due to lateral vibration amplitude releases the effect of axial compression load, leading to an increase in the natural frequency of the system. Furthermore, the shifting of higher modes in case of elastic end restraints is smaller than π while it is approximately equal π for conventional end conditions.

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CONFLICT OF INTEREST

The authors confirm that this article content has no conflicts of interest.

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