Convergence Estimates for a Linear Backward Euler Scheme for the Saturation Equation

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Abstract: In a previous work, stability and consistency results were established for a linearized Euler scheme for the saturation equation. In this paper we continue the mathematical analysis of the scheme, in preparation for its numerical treatment in a future work. We use the regularity results, obtained previously, to establish error estimates in $L^2(\Omega)$ for the linear scheme. This work is done with the degenerate nature of the saturation equation in mind, but it is also valid for the non degenerate case like the concentration equation. We show that, if the regularization parameter β and the spatial discretization parameter h are carefully chosen in terms of the time stepping parameter Δt, the convergence is at least of order $O((\Delta t)^\alpha)$ for some determined $\alpha > 0$. Examples of choices of β and h are given. We also establish a new (at our knowledge) regularity result for the continuous Galerkin formulation of the Saturation Equation and a new regularity result for the linear scheme.

1. INTRODUCTION

In considering a numerical approximation of the saturation problem

$$\begin{cases} \varphi \dfrac{\partial S}{\partial t} + \nabla \cdot (f(S)u) - \nabla \cdot (k(S)\nabla S) = Q(S) & on\ \Omega \times (0, T_0] \\ (f(S)u - k(S)\nabla S) \cdot n = q & on\ \partial\Omega \times [0, T_0] \\ S(x,0) = S^0(x) & on\ \Omega \end{cases} \quad (1.1)$$

obtained from modeling a two-phase immiscible flow through a porous medium [1-4], where Ω is a bounded domain of $\mathbb{R}^n$, $n = 1, 2, 3$, one encounters two major problems: the nonlinearity of the equation and the degeneracy of the diffusion coefficient k. The degeneracy is often addressed by regularizing the problem in some way, though some works bypass this step (see [5]). One then produces a numerical scheme which, often, is still nonlinear. So a linearization in some sense follows in order to implement the scheme on a computer. Two earlier papers [6, 7] dealt with the latter part of the problem. This work is a sequel of these papers.

In (1.1), the unknown S is the saturation of the invading phase. The diffusion coefficient k (see Fig. **1**) is the conductivity of the medium and is assumed to satisfy the following conditions.

$$k(0) = k(1) = 0 \quad (1.2)$$

$$k(\xi) \geq \begin{cases} c_1 \xi^\mu & if\ 0 \leq \xi \leq \alpha_1 \\ c_2 & if\ \alpha_1 < \xi < \alpha_2 \\ c_3(1-\xi)^\mu & if\ \alpha_2 \leq \xi \leq 1, \end{cases} \quad (1.3)$$

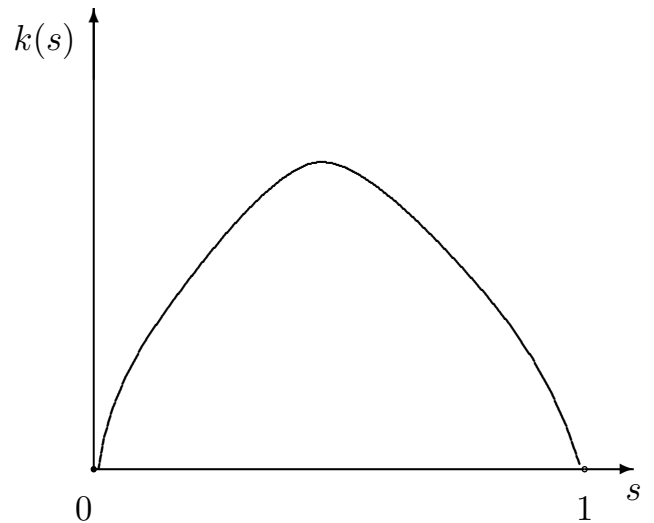where $0 < \alpha_1 < \dfrac{1}{2} < \alpha_2 < 1$, and $0 < \mu \leq 2$.

Fig. (1). Example of a graph of $k(s)$.

By [8] and [9], we have

$$| K(s_2) - K(s_1)|^2 \leq C\ | K(s_2) - K(s_1) || s_2 - s_1 |, \quad (1.4)$$

thanks to the fact that K is Lipschitz ($K'(s) = k(s)$), and

$$| s_2 - s_1 |^{1+\mu} \leq C\ | K(s_2) - K(s_1) |. \quad (1.5)$$

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For the derivation of the error estimates, we assume that k is continuously differentiable in the variable s.

The function f is twice continuously differentiable in the variable s, and

$$f'(0) = f'(1) = 0. \qquad (1.6)$$

We notice, by [8] and [9], that if (1.3) and (1.6) hold, then

$$|f(s_2) - f(s_1)|^2 \le C \,|K(s_2) - K(s_1)||\, s_2 - s_1| \qquad (1.7)$$

where

$$K(s) = \int_0^s k(\tau)d\tau. \qquad (1.8)$$

We also notice, through (1.7) and (1.8), that

$$|f'(s)| \le C\sqrt{k(s)}. \qquad (1.9)$$

We assume that the porosity of the medium, φ, satisfies the condtion

$$0 < \varphi_0 \le \varphi(x,t) = \varphi(x) \le \varphi_1 < \infty. \qquad (1.10)$$

As in [8-10] and [6], we assume that the Darcy velocity u is given and has the necessary regularity we need for this analysis.

The main purpose of this paper is to establish error estimates for a linearization of the Backward Euler scheme obtained by fully discretizing Problem 1.1 (see [6]) Problem 1.1 has been studied by many authors under various forms and conditions (see for instance [8-15], among others). Also similar studies have been done in [16-19] for the Richard's Equation which models water flow (single phase flow) through a porous medium, among other physical phenomena. We also refer to [20] which, in addition to our concern here, includes the study of problems with phase change (Stefan type problems). However, our approach in this work, is different from the approaches in the cited papers in the sense described below.

Because of the degeneracies ($k(0) = k(1) = 0$), Problem 1.1 has often been regularized into a family of nondegenerate problems whose solutions converge to the solution of (1.1) (see [8-10, 12, 13, 15]). Usually, the numerical approximation of the solution of (1.1) is done in three steps: regularization, continuous Galerkin method, and fully discretized Galerkin method. In the last step, some of the works cited above obtain a nonlinear implicit scheme (backward Euler). Therefore, one needs to linearize in some way for a computer implementation of the scheme. Often, a Picard iteration is used (see for instance [21] and [16]). For the third step (Discrete Galerkin Method), we choose, in this paper, to linearize the nonlinear functions intervening in (1.1) by Taylor approximations, of first order, of these quantities (see [6]). We then establish error estimates corresponding to this choice.

This paper is a continuation of [6] (where a method was proposed that linearizes the scheme), and of [7] (where some regularity results were established).

The remaining of the paper is structured as follow.

In section 2, we state some preliminary results established in previous works. We establish a new (at least at our knowledge) regularity result for the continuous Galerkin Method. The is helpful in the derivation of error estimates for the linear scheme.

In section 3, we state and prove our second main result, after establishing a regularity result for the linear scheme. Error estimates are obtained through a choice of β (the regularization parameter) and/or h (the spatial discretization parameter) in terms of the time-stepping parameter Δt.

Finally, we set additional notation which will be used throughout the remainder of this paper. We define $(f,g) := (f,g)_\Omega := \int_\Omega fg\,dx$ when this has a meaning. The notation $\|f\|_{L^p} := \|f\|_{L^p(\Omega)}$ is used for the standard Lebesgue norm of a measurable function, when this quantity is finite. Similarly, we denote by $\|f\|_{L^p(L^q)} := \|f\|_{L^p(0,T,L^q(\Omega))}$ the mixed Lebesgue norm for f. We will also denote $L^2(t^n, t^{n+1}, L^2(\Omega)) := L^2(\Omega \times [t^n, t^{n+1}])$. We use C, c, to denote positive constants which may change from line, but which are independent of the parameters β, h and Δt, unless otherwise explicitly specified. For $u, v \ge 0$, the notation $u \approx v$ means there exist positive constants c_1 and c_2 such that $c_1 v \le u \le c_2 v$.

2. PRELIMINARY RESULTS

In this section, we summarize previous results that are useful for the present analysis and establish a new regularity result.

2.1. The Regularized Problem

In order to get a family of nondegenerate problems approximating (1.1), we replace k by k_β, with $k_\beta \to k$ strongly as $\beta \to 0$ strongly, and with k_β satisfying the condition:

$$0 < k_\beta(s) \le \|k\|_{L^\infty} = \sup\{k(s), s \in [0,1]\}. \qquad (2.1)$$

For example, let $0 < \beta \le \dfrac{1}{2}$ and define k_β by

$$\delta = \min(k(\beta), k(1-\beta)). \qquad (2.2)$$

Define k_β by

$$\begin{cases} k_\beta(s) = k(s) & \text{if } k(s) \ge \delta \\ \dfrac{1}{2}\delta \le k_\beta(s) \le \delta & \text{otherwise.} \end{cases} \qquad (2.3)$$

(see Fig. **2**) Set

$$K_\beta(s) = \int_0^s k_\beta(\tau)d\tau. \qquad (2.4)$$

Then, for each β, we get the nondegenerate problem

$$\begin{cases} \varphi \dfrac{\partial S_\beta}{\partial t} + \nabla \cdot (f(S_\beta)u) - \nabla \cdot (k_\beta(S_\beta)\nabla S_\beta) = Q(S_\beta) & on\ \Omega \times (0, T_0] \\ (f(S_\beta)u - k(S_\beta)\nabla S_\beta) \cdot n = q & on\ \partial\Omega \times [0, T_0] \\ S_\beta(x, 0) = S^0(x) & on\ \Omega. \end{cases} \quad (2.5)$$
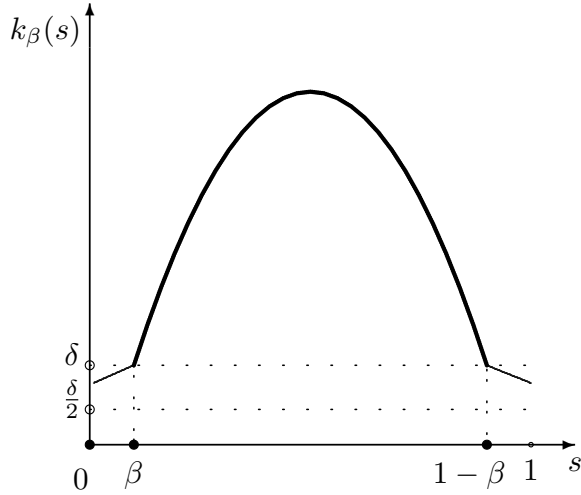

Fig. (2). Example of a perturbation of k.

For the remaining of this paper, we assume, to simplify, that $Q \equiv 0$ and $q \equiv 0$. Also, because of (1.10), we can assume, without lost of generality, that $\varphi \equiv 1$.

Let

$$C_0(\beta) = \| K_\beta(\cdot) - K(\cdot) \|_{L^\infty}^\gamma \quad (2.6)$$

where

$$\gamma = \frac{2 + \mu}{1 + \mu} \quad (2.7)$$

is the conjugate of $2 + \mu$. Let S and S_β be solutions of (1.1) and (2.5) respectively. Then, by [9],

$$\| K_\beta(S_\beta) - K(S) \|_{L^2(L^2)} \le C\beta^{2+\mu}. \quad (2.8)$$

2.2. Continuous Galerkin Method

Let $\{M_h\}_{h>0}$ be a family of finite dimensional spaces, with $M_h \subset H^1(\Omega)$, and assume that M_h has the approximation property:

$$\inf_{\chi \in M_h} \| f - \chi \|_{L^p(\Omega)} \le Ch^2 |f|_{W^{2,p}} \quad \text{for all } f \in W^{2,p}(\Omega). \,(2.9)$$

We will also need the *inverse estimate* assumption on M_h (see, for example, Section 4.5 of [22]):

$$\| \chi \|_{H^1} \le Ch^{-1} \| \chi \|_{L^2} \quad \text{for all } \chi \in M_h. \quad (2.10)$$

To account for possible numerical oscillations, extend k_β as follows (and call it again k_β):

$$k_\beta(\xi) = \begin{cases} k_\beta(1) & if\ \xi \ge 1 \\ k_\beta(-\xi) & if\ \xi \le 0. \end{cases} \quad (2.11)$$

For the same reason, extend the fractional function f as follows.

$$f(\xi) = \begin{cases} f(1), & if\ \xi \ge 1 \\ f(-\xi), & if\ \xi \le 0 \end{cases} \quad (2.12)$$

Then K_β is bijective from $\mathbb{R}$ to $\mathbb{R}$. We set

$$H_\beta = K_\beta^{-1}. \quad (2.13)$$

Consider the discretized problem: Find $V_h \in M_h$ such that

$$\left(\frac{\partial}{\partial t} H_\beta(V_h), \chi \right) - (f(H_\beta(V_h))u, \nabla\chi) + (\nabla V_h, \nabla\chi) = 0 \quad (2.14)$$

for all $\chi \in M_h$, and $t \in (0, T_0]$ with the initial condition:

$$P_h H_\beta(V_h(0)) = P_h S^0 \quad (2.15)$$

where S^0 is as in (1.1), and P_h the L^2 projection on M_h. V_h is hopefully the Galerkin approximation to $K(S)$ with S the solution to Problem 1.1. Indeed, by [10], we have

$$\| V_h - K(S) \|_{L^2(L^2)} \le C \left(h^{2\gamma} \beta^{\frac{-\mu}{1+\mu}} + \beta^{2+\mu} \right), \quad (2.16)$$

where

$$\gamma = \frac{2 + \mu}{1 + \mu}. \quad (2.17)$$

Remark 2.1 Differentiating (2.14) with respect to the time variable t, we obtain

$$\frac{d}{dt} \Big(((H_\beta(V_h))_t, \chi) - (f(H_\beta(V_h))u, \nabla\chi) + (\nabla V_h, \nabla\chi) \Big) = 0 \quad (2.18)$$

for all $\chi \in M_h$ and for all $t \in [0, T]$.

Because we will be using Taylor expansions of up to order 2 in this analysis, we assume that the solution, V_h, to (2.14) and (2.15) is three times differentiable in the time variable t.

2.3. Regularity Results for the Continuous Galerkin Method

The following regularity result was established in [7] and will be useful in this paper. We give a note on an improved proof of this lemma.

Lemma 2.2 Let V_h be the solution to problem (2.14)-(2.15). Then

$$\| V_{ht} \|_{L^\infty(L^2)} \le C(u)\delta^{-3}h^{-2}. \quad (2.19)$$

Remark 2.3 Note on the proof of the Lemma 2.2.

The proof of this lemma in [7] seemed to assume that

$$\| V_{ht} \|_{L^4(L^4)} \le C$$

independently of β and h. In fact we do not need this assumption. Using the last term on the righthand side of Equation (3.6) of [7], we can write

$$\left| \left(\sqrt{H_\beta'(V_h)} V_{ht}, \left(\sqrt{H_\beta'(V_h)} \right)_t V_{ht} \right) \right|$$

$$\leq \left\| \sqrt{H_\beta'(V_h)} V_{ht} \right\|_{L^2} \left\| \left(\sqrt{H_\beta'(V_h)} V_{ht} \right)_t \right\|_{L^2} \quad (2.20)$$

$$\leq C \left\| \sqrt{H_\beta'(V_h)} V_{ht} \right\|_{L^\infty(L^2)} \left\| H_\beta''(V_h) V_{ht}^2 \right\|_{L^2}$$

where we use the fact that

$$\left(\sqrt{H_\beta'(V_h)} \right)_t = \frac{1}{2} \frac{H_\beta''(V_h)}{\sqrt{H_\beta'(V_h)}} V_{ht}.$$

With this different manipulation of the terms, Equation (3.14) of [7] becomes

$$\frac{1}{2} \frac{d}{dt} \left\| \sqrt{H_\beta'(V_h)} V_{ht} \right\|_{L^2}^2 \leq C \left\| \sqrt{H_\beta'(V_h)} V_{ht} \right\|_{L^\infty(L^2)} \left\| H_\beta''(V_h) V_{ht}^2 \right\|_{L^2}$$

$$+ C(u) \left(\| \nabla V_{ht} \|_{L^2}^2 + \frac{1}{\delta} \| V_{ht} \|_{L^2}^2 + \| u_t \|_{L^2}^2 \right). \quad (2.21)$$

Now integrate (2.21) on the interval $(0, t)$, for $t \in (0, T_0)$, and notice that

$$\int_0^t \left\| \sqrt{H_\beta'(V_h)} V_{ht} \right\|_{L^\infty(L^2)} \left\| H_\beta''(V_h) V_{ht}^2 \right\|_{L^2} d\tau$$

$$= \left\| \sqrt{H_\beta'(V_h)} V_{ht} \right\|_{L^\infty(L^2)} \int_0^t \left\| H_\beta''(V_h) V_{ht}^2 \right\|_{L^2} d\tau$$

$$\leq \delta^{-3} \left\| \sqrt{H_\beta'(V_h)} V_{ht} \right\|_{L^\infty(L^2)} \int_0^t \| V_{ht} \|_{L^4}^2 d\tau \quad (2.22)$$

where we have used (3.51) and the fact that

$$\| v^2 \|_{L^2} = \| v \|_{L^4}^2, \quad (2.23)$$

for all $v \in L^4(\Omega)$. But then,

$$\int_0^t \| V_{ht} \|_{L^4}^2 d\tau \leq C \int_0^t \| V_{ht} \|_{H^1}^2 d\tau$$

$$\leq Ch^{-2} \int_0^t \| V_{ht} \|_{L^2}^2 d\tau$$

$$\leq Ch^{-2} \| V_{ht} \|_{L^2(L^2)}^2. \quad (2.24)$$

Going back to (2.22), we obtain

$$\int_0^t \left\| \sqrt{H_\beta'(V_h)} V_{ht} \right\|_{L^\infty(L^2)} \left\| H_\beta''(V_h) V_{ht}^2 \right\|_{L^2} d\tau$$

$$\leq \sigma_1 \left\| \sqrt{H_\beta'(V_h)} V_{ht} \right\|_{L^\infty(L^2)}^2 + \frac{1}{\sigma_1} \delta^{-6} h^{-4} \| V_{ht} \|_{L^2(L^2)}^4 \quad (2.25)$$

Finally take the sup over $[0, T]$ after integrating and then hide the first term of the righthand side of (2.25) in the corresponding term on the left hand side of the final result to get the lemma.

Next, we state and prove a new (at least at our knowledge) regularity result for the Continuous Galerkin Method which is useful in the analysis below. This is the first main result of this paper.

Lemma 2.4 Let V_h be the solution to problem (2.14)-(2.15). Then

$$\left\| \sqrt{H_\beta'(V_h)} V_{htt} \right\|_{L^2(L^2)}^2 + \eta \| \nabla V_{ht} \|_{L^\infty(L^2)}^2 \leq C(u) \delta^{-18} h^{-12}. \quad (2.26)$$

Proof of Lemma 2.4.

In (2.14), set $\chi = V_{httt}$. Then

$$\left((H_\beta(V_h))_t, V_{httt} \right) - (f(H_\beta(V_h))u, \nabla V_{httt}) + (\nabla V_h, \nabla V_{httt}) = 0. \quad (2.27)$$

Let us treat separately each term on the lefthand side of (2.27). Using the product rule, the first term can be written as

$$((H_\beta(V_h))_t, V_{httt}) = \frac{d}{dt}((H_\beta(V_h))_t, V_{htt}) - ((H_\beta(V_h))_{tt}, V_{htt})$$

$$= \frac{d}{dt}((H_\beta(V_h))_t, V_{htt})$$

$$- (H_\beta'(V_h) V_{htt}, V_{htt}) - (H_\beta''(V_h) V_{ht}^2, V_{htt})$$

$$= \frac{d}{dt}((H_\beta(V_h))_t, V_{htt})$$

$$- \left\| \sqrt{H_\beta'(V_h)} V_{htt} \right\|_{L^2}^2 - (H_\beta''(V_h) V_{ht}^2, V_{htt}). \quad (2.28)$$

We write the second term on the lefthand side of (2.27) as

$$(f(H_\beta(V_h))u, \nabla V_{httt}) = \frac{d}{dt}(f(H_\beta(V_h))u, \nabla V_{htt})$$

$$- ((f(H_\beta(V_h))u)_t, \nabla V_{htt}). \quad (2.29)$$

We rewrite the third term as

$$(\nabla V_h, \nabla V_{httt}) = \frac{d}{dt}(\nabla V_h, \nabla V_{htt}) - \frac{1}{2} \frac{d}{dt} \| \nabla V_{ht} \|_{L^2}^2 \quad (2.30)$$

Going back to (2.27) and putting (2.28), (2.29), and (2.30) together, we obtain

$$\left\| \sqrt{H_\beta'(V_h)} V_{htt} \right\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \| \nabla V_{ht} \|_{L^2}^2$$

$$= \frac{d}{dt}((H_\beta(V_h))_t, V_{htt}) - (H_\beta''(V_h) V_{ht}^2, V_{htt})$$

$$- \frac{d}{dt}(f(H_\beta(V_h))u, \nabla V_{htt})$$

$$+ ((f(H_\beta(V_h))u)_t, \nabla V_{htt}) + \frac{d}{dt}(\nabla V_h, \nabla V_{htt}). \quad (2.31)$$

Now, by (2.18), if we choose $\chi = V_{htt}$, we have

$$\frac{d}{dt}((H_\beta(V_h))_t, V_{htt}) - \frac{d}{dt}(f(H_\beta(V_h))u, \nabla V_{htt})$$

$$+ \frac{d}{dt}(\nabla V_h, \nabla V_{htt}) = 0. \quad (2.32)$$

So (2.31) becomes

$$\left\|\sqrt{H_\beta{}'(V_h)}V_{htt}\right\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\parallel \nabla V_{ht}\parallel_{L^2}^2$$

$$= -(H_\beta{}''(V_h)V_{ht}^2, V_{htt})$$

$$+((f(H_\beta(V_h))u)_t, \nabla V_{htt}), \tag{2.33}$$

which we treat as follows.

$$\left\|\sqrt{H_\beta{}'(V_h)}V_{htt}\right\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\parallel \nabla V_{ht}\parallel_{L^2}^2$$

$$\leq \left\|\frac{H_\beta{}''(V_h)V_{ht}^2}{\sqrt{H_\beta{}'(V_h)}}\right\|_{L^2}^2 + \frac{1}{4}\left\|\sqrt{H_\beta{}'(V_h)}V_{htt}\right\|_{L^2}^2$$

$$+h^{-2}\left\|\sqrt{\parallel k\parallel_{L^\infty}}(f(H_\beta(V_h))u)_t\right\|_{L^2}^2 + \frac{h^2}{4}\left\|\frac{1}{\sqrt{\parallel k\parallel_{L^\infty}}}\nabla V_{htt}\right\|_{L^2}^2$$

$$\leq \left\|\frac{H_\beta{}''(V_h)V_{ht}^2}{\sqrt{H_\beta{}'(V_h)}}\right\|_{L^2}^2 + \frac{1}{4}\left\|\sqrt{H_\beta{}'(V_h)}V_{htt}\right\|_{L^2}^2$$

$$+h^{-2}\left\|\sqrt{\parallel k\parallel_{L^\infty}}(f(H_\beta(V_h))u)_t\right\|_{L^2}^2 + \frac{h^2}{4}\left\|\frac{h^{-1}}{\sqrt{\parallel k\parallel_{L^\infty}}}V_{htt}\right\|_{L^2}^2$$

$$\leq \left\|\frac{H_\beta{}''(V_h)V_{ht}^2}{\sqrt{H_\beta{}'(V_h)}}\right\|_{L^2}^2 + \frac{1}{4}\left\|\sqrt{H_\beta{}'(V_h)}V_{htt}\right\|_{L^2}^2$$

$$+h^{-2}\left\|\sqrt{\parallel k\parallel_{L^\infty}}(f(H_\beta(V_h))u)_t\right\|_{L^2}^2 + \frac{1}{4}\left\|\sqrt{H_\beta{}'(V_h)}V_{htt}\right\|_{L^2}^2 \tag{2.34}$$

where we have used (2.1), the inverse estimate assumption (2.10), the fact that

$$H_\beta{}'(v) = \frac{1}{k_\beta(H_\beta(v))} \geq \frac{1}{\parallel k\parallel_{L^\infty}} \tag{2.35}$$

by (2.13) and (2.1), and the definition of k_β, (2.3). Also, notice that we have made use of the following fact:

$$\mid ab\mid = \left|\frac{ha}{\sqrt{2}}\sqrt{2}h^{-1}b\right| \leq \frac{1}{2}\left|\frac{ha}{\sqrt{2}}\right|^2 + \frac{1}{2}\left|\sqrt{2}h^{-1}b\right|^2$$

$$\leq \frac{1}{4}h^2a^2 + h^{-2}b^2 \tag{2.36}$$

Finally, hide the second and fourth terms on the left hand side of (2.34) in its righthand side and then integrate over $[0, T]$ to obtain

$$\left\|\sqrt{H_\beta{}'(V_h)}V_{htt}\right\|_{L^2(L^2)}^2 + \eta\parallel \nabla V_{ht}\parallel_{L^\infty(L^2)}^2$$

$$\leq C\left\{\left\|\frac{H_\beta{}''(V_h)V_{ht}^2}{\sqrt{H_\beta{}'(V_h)}}\right\|_{L^2(L^2)}^2\right.$$

$$+h^{-2}\left\|\sqrt{\parallel k\parallel_{L^\infty}}(f(H_\beta(V_h))u)_t\right\|_{L^2(L^2)}^2\right\}$$

$$+ \parallel \nabla V_{ht}\parallel_{L^2}^2|_{t=0} \tag{2.37}$$

The first term on the left hand side of (2.37) can be treated as follows.

$$\left\|\frac{H_\beta{}''(V_h)V_{ht}^2}{\sqrt{H_\beta{}'(V_h)}}\right\|_{L^2}^2 \leq C\delta^{-6}\parallel V_{ht}^2\parallel_{L^2}^2 \leq C\delta^{-6}\parallel V_{ht}\parallel_{L^4}^4$$

$$\leq C\delta^{-6}\parallel V_{ht}\parallel_{H^1}^4 \leq C\delta^{-6}h^{-4}\parallel V_{ht}\parallel_{L^2}^4 \tag{2.38}$$

where we have used (3.51), the inverse estimate assumption (2.10), and the fact that $H^1(\Omega)$ is continuously imbedded in $L^4(\Omega)$, for the spatial dimension $n = 2$ or 3 [23, 24].

The second term on the left hand side of (2.37) can be treated in the following way.

$$h^{-2}\left\|\sqrt{\parallel k\parallel_{L^\infty}}(f(H_\beta(V_h))u)_t\right\|_{L^2}^2 \leq Ch^{-2}\parallel f'(H_\beta(V_h))H_\beta{}'(V_h)V_{ht}u\parallel_{L^2}^2$$

$$+Ch^{-2}\parallel f(H_\beta(V_h))u_t\parallel_{L^2}^2$$

$$\leq Ch^{-2}\delta^{-1}\parallel u\parallel_{L^\infty}^2\parallel V_{ht}\parallel_{L^2}^2 +Ch^{-2}\parallel u_t\parallel_{L^2}^2 \tag{2.39}$$

where we have used (3.52), (1.9), and (2.3). Now by Remark 3.1 of [10],

$$\parallel V_{ht}\parallel_{L^2(L^2)}^2 \leq C(u). \tag{2.40}$$

We also have

$$\parallel V_{ht}\parallel_{L^4(L^2)}^4 \leq \parallel V_{ht}\parallel_{L^\infty(L^2)}^4 \leq C\delta^{-12}h^{-8}, \tag{2.41}$$

by Lemma 2.2.

Therefore, combining (2.37) through (2.39), we get the lemma.

3. LINEARIZATION AND ERROR ANALYSIS

In this section, we consider the perturbation given by (2.2) and (2.3) and establish error estimates for the linearized scheme proposed in [6].

3.1. The Linearized Scheme

As done in [6], we consider first order Taylor expansions of the functions H_β and $f \circ H_\beta$:

$$H_\beta(v_2) - H_\beta(v_1) = (v_2 - v_1)H_\beta'(v_1) + O((v_2 - v_1)^2), \tag{3.1}$$

$$(f \circ H_\beta)(v_2) - (f \circ H_\beta)(v_1) = (v_2 - v_1)(f \circ H_\beta)'(v_1) + O((v_2 - v_1)^2). \tag{3.2}$$

If we discard the second order terms in (3.1) and (3.2), and replace in (2.14), we obtain the approximate problem:

Find a sequence of functions $\{U_h^n\}_{n=0}^N$ of M_h verifying

$$\left(\frac{U_h^{n+1} - U_h^n}{\Delta t}H_\beta{}'(U_h^n), \chi\right) - \left(\left\{(f \circ H_\beta)(U_h^n) + (U_h^{n+1} - U_h^n)\times\right.\right.$$

$$(f \circ H_\beta)'(U_h^n)\} u^{n+1}, \nabla \chi \Big)$$

$$+ \Big(\nabla U_h^{n+1}, \nabla \chi \Big) = 0, \qquad \forall \ \chi \in M_h$$

$$0 \le n \le N - 1 \qquad\qquad (3.3)$$

$$P_h H_\beta U_h^0 = P_h S^0 \qquad\qquad (3.4)$$

where H_β is defined by (2.13). Here $u^n := u(\cdot, t^n) = u(\cdot, n\Delta t)$.

In this section we wish to show that the solution obtained through this scheme converges to the solution to Problem 1.1 (for $Q = 0$ and $q = 0$) in some functional space. For this, it is enough to show that the solution to Problem (3.3)-(3.4) converges to the solution to (2.14)-(2.15) in that same functional space, independently of β and h, by [9, 10].

Let A be the matrix of the system of linear algebraic equations given by (3.3)-(3.4) (see [6]). The following theorem, which shows the existence and uniqueness for the system above, was proved in [6].

Theorem 3.1 Let $v \in M_h$. Then under conditions (1.2)-(1.7), we have

$$({}^t v, Av) \ge c_2 \left(1 - \frac{\sqrt{\Delta t}\, c_3(u)}{2} \right) \| v \|_{L^2(\Omega)}^2 + \Delta t \| \nabla v \|_{L^2(\Omega)}^2 \qquad (3.5)$$

where c_2 and c_3 are independent of β, h, and Δt.

The following regularity result was established in [7] and will be used in the present work

Lemma 3.2 If $(U_h^n)_{n=0}^N \in M_h$ is the solution to the problem (3.3)-(3.4), then

$$\sum_{0 \le n \le N-1} \Delta t \left\| \sqrt{H'_\beta(U_h^n)} \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^2}^2 + \eta \max_{0 \le n \le N-1} \left\| \nabla(U_h^{n+1}) \right\|_{L^2}^2$$

$$\le C(u)\delta^{-1} \qquad\qquad (3.6)$$

for some $\eta > 0$ and δ as in (2.2), and $C = C(u, U_h^0, \nabla U_h^0)$

Next we state and prove a new discrete regularity result for the linearized full discretized scheme that will also be helpful in establishing our main result in this paper.

Lemma 3.3 If $(U_h^n)_{n=0}^N \in M_h$ is the solution to the problem (3.3)-(3.4), then

$$\max_{0 \le n \le N-1} \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^2}^2 + \eta \Delta t \max_{0 \le n \le N-1} \left\| \frac{\nabla(U_h^{n+1} - U_h^n)}{\Delta t} \right\|_{L^2}^2$$

$$\le C(\Delta t + 1)\delta^{-1} h^{-2} \qquad\qquad (3.7)$$

for some $\eta > 0$ and δ as in (2.2), and $C = C(u, U_h^0, \nabla U_h^0)$

Proof of Lemma.

Set $\chi = \dfrac{U_h^{n+1} - U_h^n}{\Delta t}$ in (3.3) to get

$$\left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \sqrt{H_\beta'(U_h^n)} \right\|_{L^2}^2 + \left(\nabla U_h^{n+1}, \frac{\nabla(U_h^{n+1} - U_h^n)}{\Delta t} \right)$$

$$= \left((f \circ H_\beta)(U_h^n) u^{n+1}, \frac{\nabla(U_h^{n+1} - U_h^n)}{\Delta t} \right)$$

$$+ \Big((U_h^{n+1} - U_h^n)(f \circ H_\beta)'(U_h^n) \times$$

$$u^{n+1}, \frac{\nabla(U_h^{n+1} - U_h^n)}{\Delta t} \Big). \qquad (3.8)$$

We split the second term on the lefthand side of (3.8) as follows.

$$\left(\nabla U_h^{n+1}, \frac{\nabla(U_h^{n+1} - U_h^n)}{\Delta t} \right) = \Delta t \left\| \frac{\nabla(U_h^{n+1} - U_h^n)}{\Delta t} \right\|_{L^2}^2$$

$$+ \left(\nabla U_h^n, \frac{\nabla(U_h^{n+1} - U_h^n)}{\Delta t} \right). \qquad (3.9)$$

Next, combining (3.8) and (3.9), and using the obvious inequality

$$| ab | \le \frac{1}{2} h^{-2} \sigma^{-1} a^2 + \frac{1}{2} h^2 \sigma b^2 \qquad (3.10)$$

by the Young Inequality, we get

$$\left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \sqrt{H_\beta'(U_h^n)} \right\|_{L^2}^2 + \Delta t \left\| \frac{\nabla(U_h^{n+1} - U_h^n)}{\Delta t} \right\|_{L^2}^2$$

$$\le C(u) \Big\{ \sigma^{-1} h^{-2} \| (f \circ H_\beta)(U_h^n) \|_{L^2}^2$$

$$+ \sigma^{-1} h^{-2} \| (f \circ H_\beta)'(U_h^n)(U^{n+1} - U_h^n) \|_{L^2}^2$$

$$+ \sigma^{-1} h^{-2} \| \nabla U_h^n \|_{L^2}^2 \Big\} + \sigma h^2 \left\| \frac{\nabla(U_h^{n+1} - U_h^n)}{\Delta t} \right\|_{L^2}^2. \qquad (3.11)$$

By the regularity assumption on f, we obviously have:

$$\| (f \circ H_\beta)(U_h^n) \|_{L^2} \le C. \qquad (3.12)$$

The second term on the righthand side of (3.11) can be handled in the following manner.

$$\| (f \circ H_\beta)'(U_h^n)(U^{n+1} - U_h^n) \|_{L^2}^2 =$$

$$\left\| f'(H_\beta(U_h^n)) H_\beta'(U_h^n) \Delta t \left(\frac{U_h^{n+1} - U_h^n}{\Delta t} \right) \right\|_{L^2}^2$$

$$= \left\| f'(H_\beta(U_h^n)) \sqrt{H_\beta'(U_h^n)} \Delta t \left(\sqrt{H_\beta'(U_h^n)} \frac{U_h^{n+1} - U_h^n}{\Delta t} \right) \right\|_{L^2}^2$$

$$\le C(\Delta t)^2 \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \sqrt{H_\beta'(U_h^n)} \right\|_{L^2}^2, \qquad (3.13)$$

where we have used (1.9) and the fact that

$$H_\beta{}'(s) = \frac{1}{k_\beta(H_\beta(s))}, \tag{3.14}$$

by (2.4) and (2.13). Thus, by Lemma 3.2, we have

$$\| (f \circ H_\beta)'(U_h^n)(U^{n+1} - U_h^n) \|_{L^2}^2 \le C \Delta t \, \delta^{-1}. \tag{3.15}$$

Also by the same lemma,

$$\| \nabla U_h^n \|_{L^2}^2 \le C \delta^{-1}. \tag{3.16}$$

Thanks to the inverse estimate assumption (2.10), and because $U_h^{n+1} - U_h^n \in M_h$, the last term of the righthand side of (3.11) can be treated as follows.

$$\left\| \frac{\nabla(U_h^{n+1} - U_h^n)}{\Delta t} \right\|_{L^2}^2 \le C h^{-2} \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^2}^2. \tag{3.17}$$

We also notice that

$$H_\beta{}'(s) \ge \frac{1}{\| k(\cdot) \|_{L^\infty}}, \tag{3.18}$$

by (2.1) and (3.14). Hence the first term on the lefthand side of (3.11) is bounded below as follows.

$$\left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \sqrt{H_\beta{}'(U_h^n)} \right\|_{L^2}^2 \ge \| k(\cdot) \|_{L^\infty}^{-1} \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^2}^2. \tag{3.19}$$

Substituting (3.19) in (3.11), multiplying the new inequality by $\| k(\cdot) \|_{L^\infty}$, and using (3.12), (3.13), (3.15), and (3.16), we obtain

$$\left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^2}^2 + \| k(\cdot) \|_{L^\infty} \Delta t \left\| \frac{\nabla(U_h^{n+1} - U_h^n)}{\Delta t} \right\|_{L^2}^2$$

$$\le C(u)\sigma^{-1}h^{-2}(1 + \Delta t \delta^{-1} + \delta^{-1})$$

$$+ \sigma \| k(\cdot) \|_{L^\infty} \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^2}^2. \tag{3.20}$$

Now, choose σ in the arithmetic-geometric mean inequality (3.10) so that

$$\sigma \| k(\cdot) \|_{L^\infty} \le \frac{1}{2}, \tag{3.21}$$

in order to hide the last term on the righthand side of (3.20) in the corresponding term of the lefthand side. Finally, take the maximum of both sides over $0 \le n \le N$ to get the Lemma.

3.2. Error Estimates

We now establish the second main result of this paper.

3.2.1. Preparation for the Main Theorem

To prepare for the statement and the proof of our main results, we do some preliminary work.

Let $V_h \in M_h$ be the solution to Problem (2.14)-(2.15). Let $V_h^n := V_h(\cdot, t^n)$ with $t^n = n\Delta t$.

Using a first order Taylor expansion of the function $H_\beta(v)$ in the variable v, we obtain

$$H_\beta(v_2) - H_\beta(v_1) = (v_2 - v_1)H_\beta'(v_1) + \frac{(v_2 - v_1)^2}{2} H_\beta''(v_0) \tag{3.22}$$

with v_0 between v_1 and v_2. Hence, we have

$$\frac{H_\beta(U_h^{n+1}) - H_\beta(U_h^n)}{\Delta t} = \frac{U_h^{n+1} - U_h^n}{\Delta t} H_\beta'(U_h^n)$$

$$+ \frac{(U_h^{n+1} - U_h^n)^2}{2\Delta t} H_\beta''(\psi_h^n) \tag{3.23}$$

for some ψ_h^n between U_h^n and U_h^{n+1}.

Similarly, using a first order Taylor expansion of the function $(f \circ H_\beta)(v)$, we get

$$(f \circ H_\beta)(U_h^{n+1}) = (f \circ H_\beta)(U_h^n) + \left(U_h^{n+1} - U_h^n\right)\left(f \circ H_\beta\right)'(U_h^n)$$

$$+ \frac{(U_h^{n+1} - U_h^n)^2}{2} \left(f \circ H_\beta\right)''(\psi_h^n) \tag{3.24}$$

for some φ_h^n between U_h^n and U_h^{n+1}.

We can rewrite (2.14) as

$$\left(\frac{H_\beta(V_h^{n+1}) - H_\beta(V_h^n)}{\Delta t}, \chi \right) - \left((f \circ H_\beta)(V_h^{n+1})u^{n+1}, \nabla\chi \right)$$

$$+ \left(\frac{\partial(H_\beta(V^n))}{\partial t} - \frac{H_\beta(V_h^{n+1}) - H_\beta(V_h^n)}{\Delta t}, \chi \right)$$

$$+ \left(\nabla V_h^{n+1}, \nabla\chi \right) = 0, \ \forall \chi \in M_h. \tag{3.25}$$

Next using (3.24) and (3.23) in (3.3), we get

$$\left(\frac{H_\beta(U_h^{n+1}) - H_\beta(U_h^n)}{\Delta t}, \chi \right) - \left(\frac{(U_h^{n+1} - U_h^n)^2}{2\Delta t} H_\beta''(\psi_h^n), \chi \right)$$

$$- \left(f \circ H_\beta(U_h^{n+1})u^{n+1}, \chi \right)$$

$$- \left(\frac{(U_h^{n+1} - U_h^n)^2}{2} (f \circ H_\beta)''(\varphi_h^n)u^{n+1}, \nabla\chi \right)$$

$$+ \left(\nabla U_h^{n+1}, \nabla\chi \right) = 0, \qquad \forall \ \chi \in M_h$$

$$0 \le n \le N - 1 \tag{3.26}$$

Now, subtract (3.25) from (3.26) and rewrite to get

$$\left(\frac{H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})}{\Delta t} - \frac{H_\beta(U_h^n) - H_\beta(V_h^n)}{\Delta t}, \chi \right)$$

$$+ \left(\nabla\left(U_h^{n+1} - V_h^{n+1}\right), \nabla\chi \right)$$

$$- \left(((f \circ H_\beta)(U_h^{n+1}) - (f \circ H_\beta)(V_h^{n+1}))u^{n+1}, \nabla\chi \right)$$

$$-\left(\frac{(U_h^{n+1}-U_h^n)^2}{2}(f\circ H_\beta)''(\varphi_h^n)u^{n+1},\nabla\chi\right)$$

$$-\left(\frac{\partial(H_\beta(V^n))}{\partial t}-\frac{H_\beta(V_h^{n+1})-H_\beta(V_h^n)}{\Delta t},\chi\right)$$

$$-\left(\frac{\left(U_h^{n+1}-U_h^n\right)^2}{2\Delta t}H_\beta''(\psi_h^n),\chi\right)=0,\ \forall\chi\in M_h. \tag{3.27}$$

We need the following for the proof of Theorem 3.4 below.

3.2.2. The Discrete Analogue of the Poisson Solution Operator

We give some properties of the discrete analogue of the Poisson operator, which we need in the proof of our second main result below. For more details on this operator, see the appendix of [7], for instance.

We define the operator T_h from $(H^1(\Omega))^*$ into M_h (defined in section 2) by $T_h f=\omega_h$, where ω_h is the unique solution of the discretized elliptic problem:

$$\begin{cases}(\nabla\omega_h,\nabla\chi)=(f-f_\Omega,\chi) & \text{for all }\chi\in M_h,\\(\omega_h)_\Omega=f_\Omega\end{cases} \tag{3.28}$$

where $g_\Omega:=\dfrac{1}{|\Omega|}\int_\Omega g(x)dx$, is the mean value of g over Ω. In the present analysis, we assume that $|\Omega|=1$, to simplify.

Then we have the following properties.

$$(\nabla T_h f,\nabla\chi)=(f,\chi)-f_\Omega\chi_\Omega,\ \forall\chi\in M_h, \tag{3.29}$$

$$(T_h\chi,\chi)\approx\|\chi\|_{(H^1(\Omega))^*}^2=\|\chi\|_{H_h^{-1}}^2\ \ \forall\chi\in M_h, \tag{3.30}$$

$$\|\nabla T_h\chi\|_{L^2}\le\|\chi\|_{H_h^{-1}}\ \ \forall\chi\in M_h, \tag{3.31}$$

and

$$\|T_h\chi\|_{L^2}\le\|T_h\chi\|_{H^1}\approx\|\chi\|_{H_h^{-1}},\ \forall\chi\in M_h. \tag{3.32}$$

Finally, by definition of T_h, and by the fact that

$$f_\Omega=\int_\Omega fdx=(f,1)=(P_h f,1)=(P_h f)_\Omega, \tag{3.33}$$

(since $1\in M_h$), we have

$$T_h f=T_h P_h f \tag{3.34}$$

for all $f\in(H^1(\Omega))^*$, where P_h is the L^2-projection onto M_h.

3.2.3 The Main Theorem

Thanks to the above preparatory work, we can now state and prove our second main result.

Theorem 3.4 Suppose conditions (1.2)-(1.9) hold. Let V_h be the solution to (2.14)-(2.15). Denote

$V_h^n:=V_h(\cdot,t^n)=V_h(\cdot,n\Delta t)$, $0\le n\le N$. Let $\{U_h^n\}_{0\le n\le N}$ be the solution to (3.3)-(3.4). Then

$$\max_{0\le n\le N-1}\left\|P_h H_\beta(U_h^{n+1})-P_h H_\beta(V_h^{n+1})\right\|_{H_h^{-1}}^2$$

$$+\eta\sum_{0\le n\le N-1}\Delta t((U_h^{n+1}-V_h^{n+1}),H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1}))$$

$$\le C(u)((\Delta t)^2(h^{-12}\delta^{-19})), \tag{3.35}$$

for some $\eta>0$.

Proof.

We first notice that, for $\chi\in M_h$, we have

$$(H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1}),\chi)=(P_h(H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1})),\chi) \tag{3.36}$$

Also, by [7], page 376,

$$\frac{1}{2}\|P_h(H_\beta(U_h^{n^*})-H_\beta(V_h^{n+1}))\|_{H_h^{-1}}^2-\frac{1}{2}\|P_h(H_\beta(U_h^n)-H_\beta(V_h^n))\|_{H_h^{-1}}^2$$

$$\le\left(P_h(H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1})),T_h(H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1}))\right). \tag{3.37}$$

In (3.27), set $\chi=T_h\left(H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1})\right)$, and use (3.32), (3.36), (3.37), and the arithmetic-geometric inequality to obtain

$$\frac{1}{2\Delta t}\left\{\left\|P_h\left(H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1})\right)\right\|_{H_h^{-1}}^2\right.$$

$$\left.-\|P_h(H_\beta(U_h^n)-H_\beta(V_h^n))\|_{H_h^{-1}}^2\right\}$$

$$+(\nabla(U_h^{n+1}-V_h^{n+1}),\nabla T_h(H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1})))$$

$$\le\sigma_1\|(f(H_\beta(U_h^n))-f(H_\beta(V_h^n)))u^{n+1}\|_{L^2}^2+E$$

$$+F+G+C\|\nabla T_h(H_\beta(U_h^{n+1})-H_\beta(V^{n+1}))\|_{L^2}^2 \tag{3.38}$$

where σ_1 can be made arbitrary small thanks to the arithmetic-geometric inequality, and where the terms E, F, and G are to be specified and treated below.

Using (3.29), the third term on the left side of (3.38), can be treated as follow.

$$(\nabla(U_h^{n+1}-V_h^{n+1}),\nabla T_h(H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1})))$$

$$=(U_h^{n+1}-V_h^{n+1},H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1}))$$

$$-\left(U_h^{n+1}-V_h^{n+1}\right)_\Omega\left(H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1})\right)_\Omega. \tag{3.39}$$

We rewrite (3.38) using (3.31), (3.34), and (3.39):

$$\frac{1}{2\Delta t}\left\{\left\|P_h\left(H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1})\right)\right\|_{H_h^{-1}}^2\right.$$

$$\left.-\|P_h(H_\beta(U_h^n)-H_\beta(V_h^n))\|_{H_h^{-1}}^2\right\}$$

$$+((U_h^{n+1}-V_h^{n+1}),(H_\beta(U_h^{n+1})-H_\beta(V_h^{n+1})))$$

$$\leq \left(U_h^{n+1} - V_h^{n+1}\right)_\Omega \left(H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})\right)_\Omega$$

$$+ \sigma_1 \| (f(H_\beta(U_h^{n+1})) - f(H_\beta(V_h^{n+1})))u^{n+1} \|_{L^2}^2 + E$$

$$+ F + G + C \| P_h(H_\beta(U_h^{n+1}) - H_\beta(V^{n+1})) \|_{H_h^{-1}}^2 \tag{3.40}$$

Next, we treat each of the terms on the righthand side of (3.40) separately. The first term on the righthand side of (3.40) is treated as follows.

$$\left| \left(U_h^{n+1} - V_h^{n+1}\right)_\Omega \left(H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})\right)_\Omega \right| \leq \sigma_2 \left(U_h^{n+1} - V_h^{n+1}\right)_\Omega^2$$

$$+ \frac{1}{\sigma_2} \left(H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})\right)_\Omega^2. \tag{3.41}$$

We have

$$\left| (U_h^{n+1} - V_h^{n+1})_\Omega \right| = \left| \int_\Omega (U_h^{n+1} - V_h^{n+1}) dx \right|$$

$$\leq \| U_h^{n+1} - V_h^{n+1} \|_{L^2}, \tag{3.42}$$

by Holder Inequality (using the assumption that $|\Omega| = 1$). In the same way, we have

$$\left| \left(H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})\right)_\Omega \right| = \left| \int_\Omega (H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})) dx \right|$$

$$= \left| \int_\Omega P_h(H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})) dx \right|$$

$$\leq \| P_h(H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})) \|_{H_h^{-1}} \| 1 \|_{H^1}$$

$$\leq \| P_h(H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})) \|_{H_h^{-1}}, \tag{3.43}$$

where we have used (3.33).

We can now rewrite (3.41) using (3.42) and (3.43):

$$\left| \left(U_h^{n+1} - V_h^{n+1}\right)_\Omega \left(H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})\right)_\Omega \right|$$

$$\leq \sigma_2 \| U_h^{n+1} - V_h^{n+1} \|_{L^2}^2$$

$$+ C \| P_h(H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})) \|_{H_h^{-1}}^2. \tag{3.44}$$

The second term on the righthand side of (3.40) is bounded as follows.

$$\| (f(H_\beta(U_h^{n+1})) - f(H_\beta(V_h^{n+1})))u^{n+1} \|_{L^2}^2$$

$$\leq C \| u \|_{L^\infty(L^\infty)}^2 \left(U_h^{n+1} - V_h^{n+1}, H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1})\right), \tag{3.45}$$

by (1.7).

To treat the term

$$F = \left\| \frac{\partial(H_\beta(V^n))}{\partial t} - \frac{H_\beta(V_h^{n+1}) - H_\beta(V_h^n)}{\Delta t} \right\|_{L^2}^2,$$

we notice, using Taylor expansion, that

$$\frac{\partial(H_\beta(V^n))}{\partial t} - \frac{H_\beta(V_h^{n+1}) - H_\beta(V_h^n)}{\Delta t}$$

$$= \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} (H_\beta(V_h(\cdot, \tau)))_{tt} (\tau - t^n) d\tau. \tag{3.46}$$

Hence, the term F is bounded as follows.

$$F = \left\| \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} (H_\beta(V_h(\cdot, \tau)))_{tt} (\tau - t^n) d\tau \right\|_{L^2}^2$$

$$\leq \left(\frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \| (H_\beta(V_h))_{tt} \| (\tau - t^n) d\tau \right)^2$$

$$\leq \Delta t \| (H_\beta(V_h))_{tt} \|_{L^2(t^n, t^{n+1}; L^2(\Omega))}^2. \tag{3.47}$$

To bound the terms E, F, and G, one can check easily that, for $v \in \mathbb{R}$,

$$H_\beta''(v) = \frac{-k_\beta'(H_\beta(v))}{\left(k_\beta(H_\beta(v))\right)^3}. \tag{3.48}$$

and

$$(f \circ H_\beta)''(v) = \frac{f''(H_\beta(v))k_\beta(H_\beta(v)) - f'(H_\beta(v))k_\beta'(H_\beta(v))}{\left(k_\beta(H_\beta(v))\right)^3} \tag{3.49}$$

where we have used (2.4) and (2.13).

Thus, with the assumption that k is continuously differentiable and f twice continuously differentiable, we get

$$| (f \circ H_\beta)''(v) | \leq \frac{C}{\delta^3} \tag{3.50}$$

and

$$| (H_\beta)''(v) | \leq \frac{C}{\delta^3}, \tag{3.51}$$

for all $v \in \mathbb{R}$. Also

$$| H_{\beta'}(v) | = \left| \frac{1}{k_\beta(H_\beta(v))} \right| \leq \frac{C}{\delta}, \tag{3.52}$$

by (2.3) and (2.13).

Now

$$\left(H_\beta(V_h)\right)_{tt} = H_\beta''(V_h)V_{ht}^2 + H_\beta'(V_h)V_{htt}, \tag{3.53}$$

and

$$\left\| H_\beta'(V_h)V_{htt} \right\|_{L^2} = \left\| \sqrt{H_\beta'(V_h)} \sqrt{H_\beta'(V_h)} V_{htt} \right\|_{L^2}$$

$$\leq \frac{1}{\sqrt{\delta}} \left\| \sqrt{H_\beta'(V_h)} V_{htt} \right\|_{L^2}. \tag{3.54}$$

Therefore

$$\| (H_\beta(V_h))_{tt} \|_{L^2_{(t^n,t^{n+1},L^2(\Omega))}} \leq \| H_\beta^{''}(V_h)V_{ht}^2 \|_{L^2_{(t^n,t^{n+1},L^2(\Omega))}}$$

$$+ \frac{1}{\sqrt{\delta}} \| \sqrt{H_\beta^{'}(V_h)}V_{htt} \|_{L^2_{(t^n,t^{n+1},L^2(\Omega))}} \quad (3.55)$$

where we have used the fact that $H^1(\Omega)$ is continuously imbedded in $L^4(\Omega)$, for $n = 2$ or 3 (see for instance [23, 24]), and (3.58) below.

Hence, (3.47) becomes

$$F \leq C\Delta t \left(\| H_\beta^{''}(V_h)V_{ht}^2 \|^2_{L^2_{(t^n,t^{n+1},L^2(\Omega))}} \right)$$

$$+ \frac{1}{\delta} \| \sqrt{H_\beta^{'}(V_h)} V_{htt} \|^2_{L^2_{(t^n,t^{n+1},L^2(\Omega))}} \quad (3.56)$$

The term E is bounded as follows,

$$E = \left\| \frac{(U_h^{n+1} - U_h^n)^2}{2} (f \circ H_\beta)''(\varphi_h^n) \right\|^2_{L^2}$$

$$\leq \frac{C(\Delta t)^2}{\delta^6} \left((\Delta t)^2 \left\| \left(\frac{U_h^{n+1} - U_h^n}{\Delta t} \right)^2 \right\|^2_{L^2} \right). \quad (3.57)$$

Given that

$$\| v^2 \|_{L^2} = \| v \|^2_{L^4}, \text{ for all } v \in L^4, \quad (3.58)$$

estimate (3.57) becomes

$$E \leq \frac{C(\Delta t)^2}{\delta^6} \left(\Delta t \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|^2_{L^4} \right)^2. \quad (3.59)$$

The term G is treated as the term E and we get:

$$G = \left\| \frac{(U_h^{n+1} - U_h^n)^2}{2\Delta t} H_\beta''(\psi_h^n) \right\|^2_{L^2}$$

$$\leq \frac{C}{\delta^6} \left(\Delta t \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|^2_{L^4} \right)^2. \quad (3.60)$$

Combining (3.44), (3.45), (3.56), (3.59), and (3.60) with (3.40), we get

$$\frac{1}{2\Delta t} \left\{ \left\| P_h \left(H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1}) \right) \right\|^2_{H_h^{-1}} \right.$$

$$\left. - \| P_h(H_\beta(U_h^n) - H_\beta(V_h^n)) \|^2_{H_h^{-1}} \right\}$$

$$+ ((U_h^{n+1} - V_h^{n+1}), (H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1}))$$

$$\leq \sigma_2 \| U_h^{n+1} - V_h^{n+1} \|^2_{L^2}$$

$$+ \sigma_1 \| u \|^2_{L^\infty(L^\infty)} \left(U_h^{n+1} - V_h^{n+1}, H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1}) \right)$$

$$+ \frac{C(\Delta t)^2}{\delta^6} \left(\Delta t \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|^2_{L^4} \right)^2$$

$$+ C\Delta t \left(\| H_\beta^{''}(V_h)V_{ht}^2 \|^2_{L^2_{(t^n,t^{n+1},L^2(\Omega))}} \right)$$

$$+ \delta^{-1} \| \sqrt{H_\beta^{'}(V_h)}V_{htt} \|^2_{L^2_{(t^n,t^{n+1},L^2(\Omega))}} \right)$$

$$+ \frac{C}{\delta^6} \left(\Delta t \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|^2_{L^4} \right)^2$$

$$+ C \| P_h(H_\beta(U_h^{n+1}) - H_\beta(V^{n+1})) \|^2_{H_h^{-1}}. \quad (3.61)$$

Next, in view of (1.7), choose, in the arithmetic-geometric inequality, σ_2 such that

$$\sigma_2 \| U_h^{n+1} - V_h^{n+1} \|^2_{L^2} \leq \frac{1}{4} \left(U_h^{n+1} - V_h^{n+1}, H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1}) \right). \quad (3.62)$$

In the same manner, choose σ_1 such that

$$\sigma_1 \| u \|^2_{L^\infty(L^\infty)} \leq \frac{1}{4}. \quad (3.63)$$

Then the first two terms on the righthand side of (3.61) can be hidden in its lefthand side. After hiding these terms, multiply (3.61) by Δt, and then sum for $0 \leq n \leq m$, with $0 \leq m \leq N - 1$, to get

$$\frac{1}{2} \left\{ \left\| P_h \left(H_\beta(U_h^{m+1}) - H_\beta(V_h^{m+1}) \right) \right\|^2_{H_h^{-1}} \right.$$

$$\left. - \| P_h(H_\beta(U_h^0) - H_\beta(V_h^0)) \|^2_{H_h^{-1}} \right\}$$

$$+ \frac{1}{2} \sum_{0 \leq n \leq m} \Delta t((U_h^{n+1} - V_h^{n+1}), (H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1}))$$

$$\leq C(\| u \|_{L^\infty(L^\infty)}) \left\{ \frac{\Delta t^3 + \Delta t}{\delta^6} \sum_{0 \leq n \leq m-1} \left(\Delta t \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|^2_{L^4} \right)^2 \right.$$

$$\left. + \Delta t^2 \left(\| H_\beta^{''}(V_h)V_{ht}^2 \|^2_{L^2(L^2)} + \delta^{-1} \left\| \sqrt{H_\beta^{'}(V_h)}V_{htt} \right\|^2_{L^2(L^2)} \right) \right\}$$

$$+ C \sum_{0 \leq n \leq m} \Delta t \| P_h(H_\beta(U_h^{n+1}) - H_\beta(V^{n+1})) \|^2_{H_h^{-1}}. \quad (3.64)$$

The second term on the left hand side of (3.64) vanishes thanks to (2.15) and (3.4).

We notice the following obvious fact.

$$\sum_{0 \leq i \leq n} \| v \|^p_{L^p(t^i,t^{i+1},L^p(\Omega))} = \| v \|^p_{L^p(0,T_0,L^p(\Omega))}, \ 1 \leq p < \infty, \quad (3.65)$$

for all $v \in L^p(0, T_0, L^p(\Omega))$. This justifies the presence of the second and third terms in the righthand side of (3.64).

Next, to deal with the first term in the righthand side of (3.64), we notice the obvious fact: If (a_i) are nonnegative real quantities for $1 < i < n$, then

$$\sum_{1 \le i \le n} a_i^2 \le \left(\sum_{1 \le i \le n} a_i \right)^2 . \tag{3.66}$$

Using this fact, we obtain

$$\sum_{0 \le n \le m-1} \left(\Delta t \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^4}^2 \right)^2 \le \left(\sum_{0 \le n \le m-1} \Delta t \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^4}^2 \right)^2$$

$$\le \left(\sum_{0 \le n \le m-1} \Delta t \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{H^1}^2 \right)^2$$

$$\le C h^{-4} \left(\sum_{0 \le n \le m-1} \Delta t \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^2}^2 \right)^2 , \tag{3.67}$$

where we have used the fact that $H^1(\Omega)$ is continuously imbedded in $L^4(\Omega)$ and the inverse estimate assumption (2.10). Now, use (2.35) and Lemma 3.3 to see that

$$\sum_{0 \le n \le m-1} \left(\Delta t \left\| \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^4}^2 \right)^2 \le C(u) T_0 \Delta t \, h^{-8} \delta^{-2} . \tag{3.68}$$

Use lemma 2.4, (2.38), (2.41), and (3.68) in (3.64) to obtain

$$\frac{1}{2} \left\{ \left\| P_h \left(H_\beta(U_h^{m+1}) - H_\beta(V_h^{m+1}) \right) \right\|_{H_h^{-1}}^2 \right.$$

$$+ \frac{1}{2} \sum_{0 \le n \le m} \Delta t ((U_h^{n+1} - V_h^{n+1}), (H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1}))$$

$$\le C(u)((\Delta t)^2 (h^{-12} \delta^{-19} + h^{-8} \delta^{-8}))$$

$$+ C \Delta t \sum_{0 \le n \le m} \| P_h (H_\beta(U_h^{n+1}) - H_\beta(V^{n+1})) \|_{H_h^{-1}}^2 . \tag{3.69}$$

Finally, apply the discrete Gronwall Lemma to (3.69) and then take the maximum for $0 \le m \le N - 1$ to get the theorem.

Remark 3.5 The application of the discrete Gronwall Lemma to (3.69) needs some justification. Split the last term of (3.69) as follows.

$$C \Delta t \sum_{0 \le n \le m} \| P_h (H_\beta(U_h^{n+1}) - H_\beta(V^{n+1})) \|_{H_h^{-1}}^2$$

$$= C \Delta t \, \| P_h (H_\beta(U_h^{m+1}) - H_\beta(V^{m+1})) \|_{H_h^{-1}}^2$$

$$+ C \Delta t \sum_{0 \le n \le m-1} \| P_h (H_\beta(U_h^{n+1}) - H_\beta(V^{n+1})) \|_{H_h^{-1}}^2 . \tag{3.70}$$

Now, for Δt sufficient small so that

$$\frac{1}{2} - C \Delta t \ge c_0 > 0, \tag{3.71}$$

for some $c_0 > 0$, the first term on the righthand side of (3.70) can be hidden in the first term of the lefthand side of (3.69).

Theorem 3.4 can be made specific thanks to (1.5) and (1.7). We do this through the following.

Corollary 3.6 Under the conditions of Theorem 3.4, we have

$$\sum_{0 \le n \le N} \Delta t \, \| U_h^{n+1} - V_h^{n+1} \|_{L^2}^2 \le C(u)(\Delta t)^2 \, h^{-12} \delta^{-19}, \tag{3.72}$$

$$\sum_{0 \le n \le N} \Delta t \, \left\| H_\beta(U_h^{n+1}) - H_\beta(V_h^{n+1}) \right\|_{L^{2+\mu}}^{2+\mu} \le C(u)(\Delta t)^2 \, h^{-12} \delta^{-19}, \tag{3.73}$$

where μ is as in (1.3).

Clearly, Theorem 3.4 (through Corollary 3.6) states that, for fixed β (thus δ), the regularization parameter, and h, the spatial discretization parameter, the solution to problem (3.3) and (3.4) converges to the solution to problem (2.14) and (2.15). More interestingly, it also states that for a fixed β, the solution to problem (3.3) and (3.4) converges to the solution to problem 2.5 uniformly in h, if h is appropriately chosen in terms of Δt. This mean that scheme (3.3)-(3.4), as we expected, will converge faster when applied to the non degenerate case i.e. the case where $k(s) \ge k_0 > 0$, for all $s \in [0, 1]$.

4. CONCRETE EXAMPLES AND DISCUSSION

We illustrate these through examples of appropriate choices of β and h in terms of Δt.

By (1.3) and (2.2), we have

$$\delta \ge c \beta^\mu. \tag{4.1}$$

As in [13, 12], for the corollary below, we make the assumption

$$k(s) \le c_3 s^\mu. \tag{4.2}$$

We choose β and h in terms of Δt as follows.

$$\beta \approx h^{\lambda_1}, \tag{4.3}$$

and

$$h \approx (\Delta t)^{\lambda_2}, \tag{4.4}$$

with $\lambda_1 > 0$ and $\lambda_2 > 0$.

Define

$$\alpha = 2 - 19 \mu \lambda_1 \lambda_2 - 12 \lambda_2, \tag{4.5}$$

4.1. Choice 1

In view of (4.3), if we choose λ_1 such that (2.16) becomes

$$\| V_h - K(S) \|_{L^2(L^2)} = O\left(\beta^{2+\mu} \right) = O\left(h^{\lambda_1(2+\mu)} \right), \tag{4.6}$$

then it suffices that,

$$\lambda_1 = \frac{4 + 2\mu}{2 + 4\mu + \mu^2}. \tag{4.7}$$

Set

$$K^n(\cdot) := K(S(\cdot, t^n)) = K(S(\cdot, n\Delta t)). \qquad (4.8)$$

Then, thanks to the triangle inequality, we have the following

Corollary 4.1 Under the conditions of Theorem 3.4, and conditions (4.3) through (4.8), we have

$$\sum_{0 \le n \le N-1} \Delta t \parallel K^n - U_h^{n+1} \parallel_{L^2(L^2)}^2 \le C(u)(\Delta t)^\alpha. \qquad (4.9)$$

Next we choose λ_1 as in (4.7), with μ as in (1.3). If we then set

$$\lambda_2 = \frac{3}{4} \frac{2 + 4\mu + \mu^2}{12 + 62\mu + 25\mu^2}, \qquad (4.10)$$

then

$$\alpha = \frac{1}{2}.$$

For λ_1 given by (4.7), if we set

$$c_\mu = \frac{2 + 4\mu + \mu^2}{12 + 62\mu + 25\mu^2},$$

we get Table **1**. As expected, we see that the rate of convergence α_2 decreases as λ_2 increases.

Table 1. Variation of the Rate of Convergence as a Function of λ_2

$\lambda_1 = \dfrac{4 + 2\mu}{2 + 4\mu + \mu^2}$					
λ_2	$\dfrac{7c_\mu}{8}$	$\dfrac{5c_\mu}{6}$	$\dfrac{3c_\mu}{4}$	$\dfrac{2c_\mu}{3}$	$\dfrac{3c_\mu}{5}$	$\dfrac{c_\mu}{2}$
α	$\dfrac{1}{4}$	$\dfrac{1}{3}$	$\dfrac{1}{2}$	$\dfrac{2}{3}$	$\dfrac{4}{5}$	1

4.2. Choice 2

Putting (2.16) and (3.72) together and using the triangle inequality, we get

$$\sum_{0 \le n \le N-1} \Delta t \parallel K^n - U_h^{n+1} \parallel_{L^2(L^2)}^2 \le C(u)(\beta^{2+\mu}$$

$$+ h^{2\gamma}\beta^{-\frac{\mu}{\mu+1}} + (\Delta t)^\alpha). \qquad (4.11)$$

Now, choose λ_1 and λ_2 in such a way that

$$\sum_{0 \le n \le N-1} \Delta t \parallel K^n - U_h^{n+1} \parallel_{L^2(L^2)}^2 = O\left(\beta^{2+\mu}\right) = O\left((\Delta t)^{\lambda_1 \lambda_2 (2+\mu)}\right).$$

$$(4.12)$$

For this purpose, from (4.11), it suffices that

$$\begin{cases} 2\gamma\lambda_2 - \left(\dfrac{\mu}{1+\mu}\right)\lambda_1\lambda_2 = \lambda_1\lambda_2(2+\mu) \\ 2 - 19\mu\lambda_1\lambda_2 - 12\lambda_2 = \lambda_1\lambda_2(2+\mu). \end{cases} \qquad (4.13)$$

Using (2.7) and solving system (4.13) for λ_1 and λ_2, we obtain

$$\begin{cases} \lambda_1 = \dfrac{4 + 2\mu}{2 + 4\mu + \mu^2} \\ \lambda_2 = \dfrac{1}{2}\dfrac{2 + 4\mu + \mu^2}{8 + 33\mu + 13\mu^2}, \end{cases} \qquad (4.14)$$

which yields

$$\alpha = \frac{(2 + \mu)^2}{8 + 33\mu + 13\mu^2}. \qquad (4.15)$$

5. CONCLUSION

This paper has established three main results: a regularity result for the continuous Galerkin formulation for the saturation equation, a regularity result for a linear scheme for the saturation equation, and error estimates for a fully discretized and linearized scheme for the same equation, under reasonable conditions on the data, and without any pretention to optimal estimates. The choices made here are not necessarily optimal, but they do show there is convergence when the perturbation, spatial discretization, and time-stepping parameters, β, h, and Δt are chosen in such away that all converge to 0, with the first two as some powers of the latter. How good is the proposed method will hopefully come from a continuation of this work which will concern itself with improving the present results by improving, for instance, Lemma 2.4 above, and, especially, doing some numerical experiments. The results obtained in this paper show, as expected, that we have a higher rate of convergence for the non degenerate case, i.e. for the case $k(s) \ge k_0 > 0$, for all $s \in [0, 1]$. Another sequel of this paper will look more in depth at the special case of nondegenerate problems like the concentration problem.

The author intends to pursue investigations in the H^1-norm i.e. investigations for estimates for

$$\sum_{0 \le n \le N} \Delta t \parallel \nabla(U_h^{n+1} - V_h^{n+1}) \parallel_{L^2}^2.$$

As stated above, our perspective for a next paper (in association with other authors) is to implement effectively the scheme (3.3)-(3.4) on a computer and compare our results with the present estimates, and with existing methods.

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CONFLICT OF INTERESTS

Authors confirm that this article content has no conflicts of interest.

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