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On the Stability of Three-Dimensional Disturbances in Stratified Flow with Lateral and Vertical Shear

Iordanka N. Panayotova*,[1] and John P. McHugh[2]

[1]*Old Dominion University, Department of Mathematics and Statistics, 4700 Elkhorn Avenue, Norfolk, VA 23529, USA*

[2]*University of New Hampshire, Department of Mechanical Engineering, Durham, NH 03824, USA*

Abstract: Howard's semi-circle theorem is generalized for three-dimensional disturbances in a non-Boussinesq stratified fluid with a mean flow varying horizontally and vertically. In case of Boussinesq approximation Howard's and Rayleigh's theorem are extended for three-dimensional disturbances in a fluid with a mean flow varying only with respect to the latitude.

Keywords: Atmospheric waves, stability, Howard's theorem, shear flow.

1. INTRODUCTION

Howard's semi-circle theorem [1] states that the complex wavespeed for an unstable mode must fall within the range of values of the base flow, a conclusion that has proven to be of fundamental importance for a wide variety of flow phenomena. Howard's original proof applies to a two-dimensional basic state, $u(z)$, where z is the vertical distance and u is the horizontal velocity, with two dimensional disturbances. Yih [2] demonstrated that two-dimensional disturbances are the most unstable, making the three-dimensional problem unnecessary for $\bar{u}(z)$.

Howard's theorem was extended by Kochar and Jain [3], who proved that the complex wave velocity for any unstable mode lies within a semi-ellipse whose major axis coincides with the diameter of Howard's semi-circle, while its minor axis depends on the stratification. Banerjee *et al.* [4] found a more limiting version of Howard's theorem by restricting attention to homogeneous flow. Pedloski [5, 6] proved Howard's theorem for flow with a base rotation, which was later improved by Kanwar and Sinha [7]. Dahlburg *et al.* [8] treated the magnetohydrodynamics case.

Blumen [9] was the first who emphasized the importance of another base flow for geophysics, the basic state that varies with latitude, $\bar{u}(y)$. He has extended Howard's semi-circle theorem to infinitesimal two-dimensional non-divergent disturbances in a compressible fluid with a horizontal shear layer. Latter he has generalized further the semi-circle theorem for a class of three-dimensional long-wave perturbations (see [10]), in a Boussinesq stratified fluid. Scaling the variables for long length waves simplified greater governing equations and mathematical analysis eliminating the vertical velocity in the equations for mass conservation.

Ivanov and Morozov [11] used a numerical approach to study linear waves in a horizontally sheared fluid with a specific mean flow profile. Basovich and Tsimring [12] used the WKB method to also study linear waves in specific mean flow profiles. Staquet and Sommeria [13] review these works briefly.

Here we generalize Howard's theorem further for three-dimensional disturbances in the case of a flow with vertical stratification, $\bar{\rho}(z)$, and a mean flow varying in both lateral and vertical directions, $\bar{u}(y,z)$. Such a flow is inherently three-dimensional, and Yih's theory [2] no longer applies. It is also showed that in case when the basic flow varies only with latitude, $\bar{u}(y)$, the Howard's theorem is valid once the Boussinesq approximation is applied. This restriction is a result of mathematical obstacles that arise for non-Boussinesq case. Furthermore, Rayleigh's inflection point theorem is generalized for this case when the basic flow has its first derivative vanishing on the lateral boundaries.

2. HOWARD'S THEOREM WITH HORIZONTALLY AND VERTICALLY VARYING MEAN FLOW

Assume an incompressible, inviscid, stratified flow. The governing equations are the Euler equations, the continuity equation, and the equation of incompressibility (see [14]). The mean flow consists of a parallel flow, nonuniform in two directions $\bar{u}(y,z)$, and a stable density stratification, $\bar{\rho}(z)$. A non-divergent disturbance is added to the mean flow solution, and the governing equations are linearized by neglecting quadratic terms, resulting in

$$\bar{\rho}[u_t + \bar{u}u_x + \bar{u}_y v + \bar{u}_z w] = -\frac{\partial p}{\partial x}, \tag{1}$$

$$\bar{\rho}[v_t + \bar{u}v_x] = -\frac{\partial p}{\partial y}, \tag{2}$$

$$\bar{\rho}[w_t + \bar{u}w_x] = -\frac{\partial p}{\partial z} - \bar{\rho}r, \tag{3}$$

*Address correspondence to this author at the Old Dominion University, Department of Mathematics and Statistics, 4700 Elkhorn Avenue, Norfolk, VA 23529, USA; Tel: 757-683-6269; Fax: 757-683-3885;
E-mail: IPanayot@odu.edu

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \quad (4)$$

$$r_t+\bar{u}r_x-N^2w=0, \quad (5)$$

where u,v,w are the components of the disturbance velocity, p is the disturbance pressure, and r is the buoyancy, defined as

$$r=\frac{g(\rho-\bar{\rho})}{\bar{\rho}}, \quad (6)$$

and the Brunt-Vaisala frequency is

$$N^2=-\frac{g\bar{\rho}_z}{\bar{\rho}}. \quad (7)$$

note that $N=N(z)$.

Choose the normal mode form for the disturbances,

$$(u,v,w,p)=(\hat{u}(y,z),\hat{v}(y,z),\hat{w}(y,z),\hat{p}(y,z))e^{i(kx-\sigma t)}, \quad (8)$$

where k is a wavenumber, assumed real, and σ is the frequency, expected to be complex. Substitute (8) into equations (1)-(5) and drop the circumflex to obtain

$$ik(\bar{u}-c)(\bar{\rho}u)+\bar{u}_y(\bar{\rho}v)+\bar{u}_zw=ikp, \quad (9)$$

$$ik(\bar{u}-c)(\bar{\rho}v)=-p_y, \quad (10)$$

$$ik(\bar{u}-c)(\bar{\rho}w)=-p_z-\bar{\rho}r, \quad (11)$$

$$iku+v_y+w_z=0. \quad (12)$$

$$ik(\bar{u}-c)r-N^2w=0, \quad (13)$$

where c is the horizontal wave velocity, $c=\frac{\sigma}{k}$, also expected to be complex.

Reduce the equations to a more useful form by eliminating all variables but one; the pressure. To obtain the equation for pressure, eliminate r between (11) and (13) to get

$$[N^2-k^2(\bar{u}-c)^2]\bar{\rho}w=-ik(\bar{u}-c)p_z. \quad (14)$$

Eliminate v between (9) and (10) to obtain

$$u=-\frac{1}{\bar{\rho}}\left[\frac{p}{(\bar{u}-c)^2}+\frac{\bar{u}_y}{k^2(\bar{u}-c)^2}p_y-\frac{\bar{u}_zp_z}{N^2-k^2(\bar{u}-c)^2}\right]. \quad (15)$$

Finally, use (15), (10), and (14) to eliminate u, v, and w, respectively, in (12). After some manipulation, the result is

$$-\frac{d}{dy}\left[\frac{p_y}{\bar{\rho}(\bar{u}-c)^2}\right]+\frac{k^2p}{\bar{\rho}(\bar{u}-c)^2}-\frac{d}{dz}\left[\frac{p_z}{\bar{\rho}\left((\bar{u}-c)^2-M^2\right)}\right]=0, \quad (16)$$

where

$$M=\frac{N}{k}.$$

The boundary condition on the side walls is zero normal velocity: $v=0$ on $y=y_1,y_2$, and no vertical velocity on the horizontal boundaries: $w=0$ on $z=z_1,z_2$. Equation (10) may be used to conclude that the normal derivative of the pressure, in this case, p_y, will also vanish on the side walls. Equation (14) can be used to show that p_z will also vanish on the horizontal boundaries. Note that the sidewalls may extend to infinity.

Multiply (16) by p^*, the complex conjugate of p, and integrate across the domain. Simplify with integration by parts and use the sidewall boundary conditions to obtain

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\frac{|p_y|^2}{\bar{\rho}(\bar{u}-c)^2}dydz+\int_{z_1}^{z_2}\int_{y_1}^{y_2}\frac{k^2|p|^2}{\bar{\rho}(\bar{u}-c)^2}dydz+\int_{z_1}^{z_2}\int_{y_1}^{y_2}\frac{|p_z|^2}{\bar{\rho}\left((\bar{u}-c)^2-M^2\right)}dydz=0. \quad (17)$$

The real and imaginary parts are

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}[(\bar{u}-c_r)^2-c_i^2]Qdydz=\int_{z_1}^{z_2}\int_{y_1}^{y_2}\frac{M^2|p_z|^2}{\bar{\rho}q^2}dydz, \quad (18)$$

$$c_i\int_{z_1}^{z_2}\int_{y_1}^{y_2}(\bar{u}-c_r)Qdydz=0, \quad (19)$$

where

$$q=|(\bar{u}-c)^2-M^2|,$$

$$s=|\bar{u}-c|,$$

and

$$Q=\frac{k^2|p|^2}{\bar{\rho}s^4}+\frac{|p_z|^2}{\bar{\rho}q^2}+\frac{|p_y|^2}{\bar{\rho}s^4},$$

which is positively definite for the statically stable fluid under consideration. Equation (18) and (19) are similar as Howard's [1961] equations (3.2) and (3.3), except for the definition of the positive definite quantities. The remainder of the theory follows that of Howard, and will be reiterated here for completeness.

Assume the flow is unstable, which implies that the imaginary part of the wave velocity is not zero, $c_i\neq 0$, for which (19) shows that $(\bar{u}-c)$ must change sign somewhere in the flow, that is

$$a<c_r<b, \qquad a=\min_{y,z}\bar{u}, \quad b=\max_{y,z}\bar{u}, \quad (20)$$

i.e. c_r lies in the range of $\bar{u}$. Limits of the maximum growth rate can be also predicted. Equation (18) gives

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\left[\bar{u}^2-2\bar{u}c_r+c_r^2-c_i^2\right]dydz>0,$$

which after applying (19) in the form

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\bar{u}Qdydz=c_r\int_{z_1}^{z_2}\int_{y_1}^{y_2}Qdydz, \quad c_i\neq 0, \quad (21)$$

results in the equation

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\left(\bar{u}^2 - c_r^2 - c_i^2\right)Qdydz > 0. \qquad (22)$$

Now since $(a-\bar{u}) \le 0$ and $(b-\bar{u}) \ge 0$, it is always true that

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}(a-\bar{u})(b-\bar{u})Qdydz \le 0,$$

or equivalently

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\left[ab+\bar{u}^2-\bar{u}(a+b)\right]Qdydz \le 0.$$

Using (22) gives

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\left[ab+c_r^2+c_i^2-\bar{u}(a+b)\right]Qdydz \le 0,$$

and after applying identity (21) it becomes

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\left[ab+c_r^2+c_i^2-c_r(a+b)\right]Qdydz \le 0.$$

Since the quantity within the brackets is independent of y and z, and the $Q>0$, then we have that the expression within the brackets must be negative. With some rearrangement this condition can be written as

$$\left[c_r - \frac{1}{2}(a+b)\right]^2 + c_i^2 \le \left[\frac{1}{2}(b-a)\right]^2. \qquad (23)$$

In addition (23) says that the maximum growth rate is limited by

$$kc_i \le \frac{k}{2}(b-a). \qquad (24)$$

3. HOWARD'S THEOREM FOR BOUSSINESQ FLOW WITH $\bar{u}(y)$

Consider a base flow that varies only with the latitude, $\bar{u}(y)$. Howard's theorem may be proven for such a flow, but only once Boussinesq flow has been assumed.

The Boussinesq equations are

$$u_t + \bar{u}u_x + \bar{u}_y v = -p_x,$$

$$v_t + \bar{u}v_x = -p_y,$$

$$w_t + \bar{u}w_x = -p_z - r, \qquad (25)$$

$$u_x + v_y + w_z = 0,$$

$$r_t + \bar{u}r_x - N^2 w = 0,$$

where p is now the pressure divided by a reference density and N is a constant. Assume the normal mode form,

$$(u,v,w,p) = (\hat{u}(y),\hat{v}(y),\hat{w}(y),\hat{p}(y))e^{i(kx+mz-kct)},$$

drop the circumflex and proceed as before to obtain

$$-\left[\frac{p_y}{(\bar{u}-c)^2}\right]_y + \frac{k^2 p}{(\bar{u}-c)^2} + \frac{m^2 p}{(\bar{u}-c)^2 - M^2} = 0,. \qquad (26)$$

where $M = \dfrac{N}{k}$, as before. Multiply by p^* and integrate (by parts if necessary) to get

$$\int_{y_1}^{y_2}\frac{|p_y|^2}{(\bar{u}-c)^2}dy + \int_{y_1}^{y_2}\frac{k^2|p|^2}{(\bar{u}-c)^2}dy + \int_{y_1}^{y_2}\frac{m^2|p|^2}{(\bar{u}-c)^2-M^2}dy = 0, \quad (27)$$

where the velocity boundary conditions have been used to set the normal derivative of pressure at the boundary to zero.

Real and imaginary parts of (27) are

$$\int_{y_1}^{y_2}[(\bar{u}-c_r)^2 - c_i^2]Qdy = \int_{y_1}^{y_2}\frac{M^2|p|^2}{q^2}dy, \qquad (28)$$

$$c_i\int_{y_1}^{y_2}(\bar{u}-c_r)Qdy = 0, \qquad (29)$$

where

$$q = |(\bar{u}-c)^2 - M^2|,$$

$$s = |\bar{u}-c|,$$

$$Q = \frac{k^2|p|^2}{s^4} + \frac{m^2|p|^2}{q^2} + \frac{|p_y|^2}{s^4},$$

all positive definite. Note that q and s are defined as before, but Q is different. Howard's theorem follows in the same manner as before.

4. RAYLEIGH'S THEOREM WITH MEAN FLOW VARYING WITH LATITUDE

Return to equation (26) and change variables using

$$\psi = \frac{p}{(\bar{u}-c)}.$$

After some simplification, the result is

$$-\psi_{yy} + \psi\left(\frac{-\bar{u}_{yy}}{(\bar{u}-c)} + \frac{2\bar{u}_y^2}{(\bar{u}-c)^2}\right) + k^2\left[1 - \frac{m^2(\bar{u}-c)^2 - N^2}{k^2(\bar{u}-c)^2 - N^2}\right]\psi = 0. \qquad (30)$$

Multiply (30) by ψ^*, integrate across the domain, and simplify to obtain

$$\int_{y_1}^{y_2}|\psi|^2\left(\frac{-\bar{u}_{yy}}{(\bar{u}-c)} + \frac{2\bar{u}_y^2}{(\bar{u}-c)^2} + \frac{k^2m^2(\bar{u}-c)^2}{k^2(\bar{u}-c)^2 - N^2}\right)dy = \qquad (31)$$

$$= -\int_{y_1}^{y_2}(|\psi_y|^2 + k^2|\psi|^2)dy - |\psi|^2\frac{\bar{u}_y}{(\bar{u}-c)}\Big|_{y_1}^{y_2}.$$

Note that to obtain the above equation we used that on the lateral boundaries

$$\psi_y = \frac{p_y}{(\bar{u}-c)} - \frac{p\bar{u}_y}{(\bar{u}-c)^2} = -\frac{p\bar{u}_y}{(\bar{u}-c)^2} = -\frac{\psi\bar{u}_y}{(\bar{u}-c)},$$

as $p_y = 0$ for $y = y_1$ and $y = y_2$. To proceed further we assume that the first derivative of the background flow vanishes on the boundaries, i.e. $\bar{u}_y(y_1) = \bar{u}_y(y_2) = 0$, then the boundary term disappears, and (31) takes the form

$$\int_{y_1}^{y_2} |\psi|^2 \left(\frac{-\bar{u}_{yy}}{(\bar{u}-c)} + \frac{2\bar{u}_y^2}{(\bar{u}-c)^2} + \frac{k^2 m^2 (\bar{u}-c)^2}{k^2(\bar{u}-c)^2 - N^2} \right) dy = -\int_{y_1}^{y_2} (|\psi_y|^2 + k^2 |\psi|^2) dy \tag{32}$$

The imaginary part of this equation is

$$2ic_i \int_{y_1}^{y_2} |\psi|^2 \left[\frac{-\bar{u}_{yy}}{r^2} + \left(\frac{2k^2 m^2 N^2}{q^2} + \frac{4\bar{u}_y^2}{r^4} \right)(\bar{u} - c_r) \right] dy = 0. \tag{33}$$

For instability, $c_i \neq 0$, resulting in

$$\int_{y_1}^{y_2} P\bar{u}_{yy} dy = \int_{y_1}^{y_2} R(\bar{u} - c_r) dy,$$

where $P = |\psi|^2 / r^2$ and $R = \frac{2k^2 m^2 N^2}{q^2} + \frac{4\bar{u}_y^2}{r^4}$ are both positive definite.

By Howard's theorem, c_r must be in the range of $\bar{u}$ for instability, leading to

$$\int_{y_1}^{y_2} (\bar{u}_{min} - c_r) R dy \le \int_{y_1}^{y_2} P\bar{u}_{yy} dy \le \int_{y_1}^{y_2} (\bar{u}_{max} - c_r) R dy.$$

Since ($(\bar{u}_{min} - c_r) \le 0$) and ($(\bar{u}_{max} - c_r) \ge 0$), then

$$\int_{y_1}^{y_2} P\bar{u}_{yy} dy = 0.$$

Therefore, $\bar{u}_{yy}$ must change sign, and Rayleigh's theorem is proven for this case.

5. CONCLUDING REMARKS

With stratification, two types of disturbance modes are expected to be present; 1) modes associated with the parallel flow that would exist in a similar form without stratification, and 2) internal waves that would exist without the mean flow, but are distorted by presence of the mean flow. The above theorems of course apply to both types of modes. One may thus conclude that internal waves moving faster than the mean flow cannot be linearly unstable. Internal waves that do travel with the range of the mean flow may be unstable, and could grow into large amplitude waves.

REFERENCES

[1] Howard LN. Note on a paper by John Miles. J Fluid Mech 1961; 10: 509-12.
[2] Yih CS. Stability of two-dimensional parallel flow to three-dimensional disturbances. Quart Appl Math 1955; 12: 434-5.
[3] Kochar G T, Jain RK. Note on Howard's semicircle theorem. J Fluid Mech 1979; 91: 489-91.
[4] Barnerjee MB, Gupta JR, Subbiah M. On reducing Howard's semicircle for homogeneous shear flows. J Math Anal Appl 1988; 130: 398-402.
[5] Pedlosky J. The stability of currents in the atmosphere and the oceans. Part 1. J Atmos Sci 1964a; 21: 201-19.
[6] Pedlosky J. The stability of currents in the atmosphere and the oceans. Part 2. J Atmos Sci 1964b; 21: 342-52.
[7] Kanwar V, Sinha A. A new upper bound on the growth rate of linear instability of baroclinic zonal flows in a two-layer model on a beta-plane. Phys Fluids 1999; 11: 2925-7.
[8] Dahlburg RB, Boncinelli P, Einaudi G. The evolution of a plane jet in a neutral sheet. Phys Plasmas 1998; 5: 79-93.
[9] Blumen W. Shear layer instability of an inviscid compressible fluid. J Fluid Mech 1970; 40(4): 769-81.
[10] Blumen W. Hydrostatic neutral waves in a parallel shear flow of a stratified fluid. J Atmosph Sci 1971; 28: 340-4.
[11] Ivanov Y A, Morozov Y G. Deformation of internal gravity waves by a stream with horizontal velocity shear. Okeanoligie 1974; 14: 376-80.
[12] Basovich AY, Tsimring LS. Internal waves in horizontally inhomogeneous flow. J Fluid Mech 1984; 142: 233-49.
[13] Staquet C, Sommeria J. Internal gravity waves: from instabilities to turbulence. Rev Fluid Mech 2002; 34: 559-593.
[14] Turner J S. Buoyancy effects in fluids. Camb Univ. Press 1973.

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