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Hermite Matrix and Its Eigenvalue-based Decomposition

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Abstract: In the MUSIC approach for multiple emitter location, the array covariance matrix is a Hermite matrix. In order to realize the MUSIC approach, we have to do the work of eigenvalue-based decomposition of the Hermite matrix. This paper proves that the problem of Hermite matrix decomposition can be transformed into the problem of real symmetric matrix decomposition, and the article gives the detailed transformation method. Using Jacobi diagonalization method, the eigenvalue-based decomposition of real symmetric matrix decomposition is realized on computer, so the eigenvalue-based decomposition of a Hermite matrix is realized on computer.

Keywords: MUSIC approach, Hermite matrix, real symmetric matrix, eigenvalue-based decomposition of a matrix, Jacobi diagonalization method.

1. INTRODUCTION

At first, we introduce MUSIC approach for multiple emitter location briefly [1-3].

Let

$S(t) = \left(s_1(t), s_2(t), \cdots, s_K(t)\right)^T$: K independent signals;

$N(t) = \left(n_1(t), n_2(t), \cdots, n_N(t)\right)^T$: noise vector received by uniform linear array with N antennas;

$X(t) = \left(x_1(t), x_2(t), \cdots, x_N(t)\right)^T$: signal vector received by uniform linear array with N antennas;

$A(\theta) = \left(a(\theta_1), a(\theta_2), \cdots, a(\theta_K)\right)$: direction matrix of signals;

$a(\theta_i) = \left(1, e^{j\frac{2\pi d}{\lambda}\sin\theta_i}, \cdots, e^{j(N-1)\frac{2\pi d}{\lambda}\sin\theta_i}\right)$ $(i = 1, 2, \cdots, K)$: direction vector of the *i*th signal,

then

$$X(t) = A(\theta)S(t) + N(t),$$

$$R_{xx} = E\left\{\left[X(t) - E(X(t))\right]\cdot\left[X(t) - E(X(t))\right]^H\right\}$$

$$= E\left\{X(t)\cdot X(t)^H\right\}$$

$$= A(\theta)E\left\{S(t)\cdot S^H(t)\right\}A^H(\theta) + E\left\{N(t)\cdot N^H(t)\right\}$$

$$= A(\theta)PA^H(\theta) + \sigma^2 I.$$

Since the signal vector

$$X(t) = (x_1(t), x_2(t), \cdots, x_N(t))^T$$

is complex, the array covariance matrix

$$R_{xx} = R_{xx}(t) = E\left\{\left[X(t) - E(X(t))\right]\cdot\left[X(t) - E(X(t))\right]^H\right\}$$

is a complex matrix.

Since $R_{xx} = R_{xx}(t) = A(\theta)PA^H(\theta) + \sigma^2 I$, it is easy to prove that $R_{xx}^H = R_{xx}$, so R_{xx} is a Hermite matrix.

For the complex matrix R_{xx}, the column of R_{xx} can be viewed as n-dimensional vector in linear space C^n. The linear space generated by column vectors of R_{xx} is called the matrix linear space of matrix R_{xx} [4].

Decompose the matrix linear space R_{xx} according to the eigenvalues, gain the linear space E_S which contains the signal direction vectors $a(\theta_i)(i = 1, 2, \cdots, K)$ and the signal zero space E_N, where $R_{xx} = E_S \oplus E_N$ [4, 5].

Define the spatial spectrum function

$$P(\theta) = \frac{1}{\left\|a^*(\theta)\cdot E_N\right\|^2}.$$

Scan the angle θ and observe the value of $P(\theta)$. Since $a(\theta_i) \perp E_N (i = 1, 2, \cdots, K)$, $a^*(\theta_i)\cdot E_N = 0$, $P(\theta)$ will have a peak when $\theta = \theta_i$. The DOA can be found according to the peak position θ_i of $P(\theta)$.

When we design the actual system for MUSIC approach, we always assume that the signals are short-time stable and the noise is stable, so $X(t)$ and $R_{xx}(t)$ are stable in short sampling time. In addition, the covariance function of general communication signals and Gauss white noise are er-

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godic, so the random process $R_{xx}(t)$ is ergodic. We can use the time-average of $R_{xx}(t)$ to replace the statistical-average of $R_{xx}(t)$, i.e. we can use

$$\hat{R}_{xx} = \frac{1}{M}\sum_{m=1}^{M} R_{xx}(t_m)$$

to replace $R_{xx}(t)$. According to the relation of $\hat{R}_{xx}$ and $R_{xx}(t)$, $\hat{R}_{xx}$ is a Hermite matrix. In actual computation, we just do the eigenvalue-based decomposition of $\hat{R}_{xx}$.

In short, literatures [1-5] described the principle of MUSIC approach for multiple emitter location, and literatures [6-8] gave some specific simulation examples of MUSIC approach respectively. These literatures show that eigenvalue-based decomposition of the array covariance matrix is the basic task of MUSIC approach. Since the array covariance matrix is a Hermite matrix, so eigenvalue-based decomposition of Hermite matrix is very important for the MUSIC approach.

2. THE WEAKNESS OF USUAL EIGENVALUE-BASED DECOMPOSITION METHOD

The steps of usual eigenvalue-based decomposition of a matrix are as following [9-11]: at first, solve the eigen-equation $|\lambda I - \hat{R}_{xx}| = 0$ and get the eigenvalues of $\hat{R}_{xx}$; secondly, seek the eigenvectors belong to each eigenvalue.

In the array signal processing, the number of antennas, the order of array covariance matrix and the order of eigen-equation are same, and the number of antennas is usually large. With the usual eigenvalue-based decomposition method, we have to seek the solutions of high order algebraic equation. However, according to the algebraic theory, the algebraic equation with the order more than 5 has no formula solutions. In general, it is very difficult to seek the eigenvalues of the array covariance matrix, sometimes we have to seek the approximate eigenvalues. Moreover, it is neither easy to seek the eigenvectors belonging to each eigenvalue, because we have to solve the high order linear equations $(\lambda I - \hat{R}_{xx})q = 0$. If λ is the approximate eigenvalues, the linear equations have no solutions, then we must seek the least square solutions with more difficulty. It is visible that usual eigenvalue-based decomposition method is not suitable for MUSIC approach, we must look for other way of eigenvalue-based decomposition of Hermite matrix.

3. THE RELATION OF TWO EIGENVALUE-BASED DECOMPOSITIONS

There are many methods of real symmetric matrix decomposition. We hope to transform the problem of the Hermite matrix decomposition into the problem of real symmetric matrix decomposition.

Let A be a n order Hermite matrix, $A = A_r + iA_i$, real matrix A_r and A_i is the real part and the imaginary part of A respectively. Since $A^H = A_r^T - iA_i^T$ and $A^H = A$, $A_r^T = A_r$, $A_i^T = -A_i$, so A_r is a real symmetric matrix and A_i is a real antisymmetric matrix. Construct a $2n$ order real matrix

$$A' = \begin{pmatrix} A_r & -A_i \\ A_i & A_r \end{pmatrix}.$$

For $A'^T = A'$, A' is a real symmetric matrix. The relation of eigenvalue-based decomposition of Hermite matrix A and real symmetric matrix A' is as following.

Theorem 1 Hermite matrix A has the same eigenvalue set with the real symmetric matrix A'.

Proof Let λ be an eigenvalue of A, $q = q_r + iq_i$ be the eigenvector belonging to λ, then

$$Aq = \lambda q,$$

$$(A_r + iA_i)(q_r + iq_i) = \lambda(q_r + iq_i),$$

$$(A_r q_r - A_i q_i) + i(A_i q_r + A_r q_i) = \lambda q_r + i\lambda q_i,$$

$$\begin{cases} A_r q_r - A_i q_i = \lambda q_r, \\ A_i q_r + A_r q_i = \lambda q_i, \end{cases}$$

$$\begin{pmatrix} A_r & -A_i \\ A_i & A_r \end{pmatrix}\begin{pmatrix} q_r \\ q_i \end{pmatrix} = \lambda \begin{pmatrix} q_r \\ q_i \end{pmatrix}.$$

Let $q' = \begin{pmatrix} q_r \\ q_i \end{pmatrix}$, then $A'q' = \lambda q'$. So λ is an eigenvalue of A', and $q' = \begin{pmatrix} q_r \\ q_i \end{pmatrix}$ is the eigenvector of A' belonging to λ.

Conversely, let λ be an eigenvalue of A', $q' = \begin{pmatrix} q_r \\ q_i \end{pmatrix}$ be the eigenvector of A' belonging to λ, it is easy to prove that λ is an eigenvalue of A, $q = q_r + iq_i$ is the eigenvector of A belonging to λ.

In summary, Hermite matrix A has the same eigenvalue set with the real symmetric matrix A'.

Theorem 1 shows that eigenvalue-based decomposition of n order Hermite matrix A is equivalent to eigenvalue-based decomposition of $2n$ order real symmetric matrix A'. However, it is not difficult to find a problem: n order Hermite matrix A has n eigenvalues, and $2n$ order real symmetric matrix A' has $2n$ eigenvalues, the eigenvalue numbers of two matrixes are not equal why do the two matrixes have the same eigenvalue set?

In order to have an intuitive understanding of the problem, we give an example to illustrate.

Hermite matrix

$$A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has 2 eigenvalues, $\lambda_1 = 0, \lambda_2 = 2$.

The real symmetric matrix which is constructed by A

$$A' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

has 4 eigenvalues, $\mu_1 = \mu_2 = 0, \mu_3 = \mu_4 = 2$.

The eigenvectors of A' belonging to eigenvalue $\mu_1 = \mu_2 = 0$ are

$$q_1' = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \ q_2' = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

The eigenvectors of A constructed by q_1', q_2' are

$$q_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \ q_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Since $q_2 = iq_1$, q_2 is the same eigenvector with q_1.

The example illustrates that $\lambda_1 = 0$, the 1 order eigenvalue of A, corresponts to $\mu_1 = \mu_2 = 0$, the 2 order eigenvalue of A', and q_1 (or q_2), the eigenvector of A belonging to eigenvalue $\lambda_1 = 0$, splits into 2 eigenvectors of A' belonging to eigenvalue $\mu_1 = \mu_2 = 0$, q_1' and q_2'.

Similarly, the eigenvalue $\lambda_2 = 2$ and the eigenvector of A belonging to eigenvalue $\lambda_2 = 2$ has the same property.

Next, for general Hermite matrix A, A' is the real symmetric matrix constructed by A, we discuss the relation of the multiplicity of the eigenvalue λ of A and the multiplicity of the eigenvalue λ of A'.

Theorem 2 Let λ be an eigenvalue of Hermite matrix A and real symmetric matrix A', then the multiplicity of eigenvalue λ in A' is double as in A.

Proof Let $q = q_r + iq_i$ be the eigenvector of A belonging to λ, then $iq = -q_i + iq_r$ is the eigenvector of A belonging to λ.

According to the proof of theorem 1,

$$q_1' = \begin{pmatrix} q_r \\ q_i \end{pmatrix}, \quad q_2' = \begin{pmatrix} -q_i \\ q_r \end{pmatrix}$$

are all the eigenvectors of A' belonging to λ. Now we prove q_1' and q_2' are linear unrelative. Assume there are two real number k_1 and k_2, such that

$$k_1 q_1' + k_2 q_2' = 0,$$

or

$$k_1 \begin{pmatrix} q_r \\ q_i \end{pmatrix} + k_2 \begin{pmatrix} -q_i \\ q_r \end{pmatrix} = 0,$$

i.e.

$$\begin{cases} k_1 q_r = k_2 q_i, \\ k_1 q_i = -k_2 q_r. \end{cases} \tag{1}$$

Compute the inner product of q_r and the first formula of Eq. (1), the inner product of q_i and the second formula of Eq. (1), and get

$$\begin{cases} k_1 \|q_r\|^2 = k_2 (q_i, q_r), \\ k_1 \|q_i\|^2 = -k_2 (q_i, q_r). \end{cases} \tag{2}$$

Add two eqations of Eq. (2), and get

$$k_1 (\|q_r\|^2 + \|q_i\|^2) = 0.$$

Because $q_1' = \begin{pmatrix} q_r \\ q_i \end{pmatrix}$ is the eigenvector of A',

$$\|q_r\|^2 + \|q_i\|^2 \neq 0,$$

so $k_1 = 0$.

For the same reason, $k_2 = 0$.

Thus q_1' and q_2' are linear unrelative.

The argument shows that an eigenvector $q = q_r + iq_i$ of A splits into two linear unrelative eigenvectors of A':

$$q_1' = \begin{pmatrix} q_r \\ q_i \end{pmatrix}, \ q_2' = \begin{pmatrix} -q_i \\ q_r \end{pmatrix},$$

so the multiplicity of eigenvalue λ in A' is double as in A.

According to theorem 1 and theorem 2, we can transform the problem of an n order Hermite matrix decomposition into the problem of a $2n$ order real symmetric matrix decomposition. Therefore, in order to do the work of eigenvalue-based decomposition of the Hermite matrix, we just need do the work of eigenvalue-based decomposition of real symmetric matrix. Next we discuss the eigenvalue-based decomposition of real symmetric matrix [12, 13].

4. JACOBI DIAGONALIZATION METHOD

There are many eigenvalue-based decomposition methods of real symmetric matrix, among them Jacobi diagonalization method is the most commonly used one.

In the program of Jacobi diagonalization transformation, each transformation of Jacobi diagonalization method eliminates a pair of off-diagonal elements of real symmetric matrix A. To eliminate a pair of off-diagonal elements a_{pq} and a_{qp}, implement the orthogonal trans-formation to A with orthogonal matrix

$$N=\begin{array}{cc}\left(\begin{array}{ccccccccccc}
1 &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& \cos\alpha &&&& -\sin\alpha &&& \\
&&&& \ddots &&&&&& \\
&&&&& 1 &&&&& \\
&&&&&& \ddots &&&& \\
&&& \sin\alpha &&&& \cos\alpha &&& \\
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{array}\right) & \begin{array}{l} \\ \\ \\ row\ p \\ \\ \\ \\ row\ q \\ \\ \\ \\ \end{array} \\
\begin{array}{ccccccccccc} &&& col\ p &&&& col\ q &&& \end{array} &
\end{array}$$

Select an α in the orthogonal transformation N such that $\overline{a}_{pq}=\overline{a}_{qp}=0$ in $\overline{A}=N^T AN$. i.e. α meets the condition

$$\overline{a}_{pq}=(-a_{pq}+a_{qq})\cos\alpha\sin\alpha+a_{pq}(\cos^2\alpha-\sin^2\alpha)=0$$

that is

$$\tan 2\alpha=\frac{2a_{pq}}{a_{pp}-a_{qq}} \tag{3}$$

In the orthogonal transformation N, it is not necessary to seek α. However, we just need know sinα and cosα, thus Eq.(3) can be rewritten as

$$r^2=(a_{pp}-a_{qq})^2+4a_{pq}^2 \tag{4a}$$

$$\sin^2\alpha=\frac{1}{2}-\frac{a_{pp}-a_{qq}}{2r} \tag{4b}$$

$$\cos^2\alpha=\frac{1}{2}+\frac{a_{pp}-a_{qq}}{2r} \tag{4c}$$

$$\sin\alpha\cos\alpha=\frac{a_{pq}}{r} \tag{4d}$$

Calculate r by Eq.(4a), calculate sinα by Eq.(4b) and calculate cosα by Eq.(4c) or Eq.(4d).

The orthogonal transformation $\overline{A}=N^T AN$ changes column p, column q, row p and row q of A only, other columns and rows of A are not changed. After the transformation, two diagonal elements are

$$\overline{a}_{pp}=a_{pp}\cos^2\alpha+a_{qq}\sin^2\alpha+2a_{pq}\sin\alpha\cos\alpha$$

$$=\frac{a_{pp}+a_{qq}+r}{2} \tag{5a}$$

$$\overline{a}_{qq}=a_{pp}\sin^2\alpha+a_{qq}\cos^2\alpha-2a_{pq}\sin\alpha\cos\alpha$$

$$=\frac{a_{pp}+a_{qq}-r}{2} \tag{5b}$$

Other elements of column p, column q, row p and row q are

$$\overline{a}_{ip}=\overline{a}_{pi}=a_{ip}\cos\alpha+a_{iq}\sin\alpha$$

$$\overline{a}_{iq}=\overline{a}_{qi}=-a_{ip}\sin\alpha+a_{iq}\cos\alpha$$

$$(i=1,2,\cdots,n,\ i\neq p,\ i\neq q) \tag{6}$$

In Eq. (5a) and Eq. (5b), $\overline{a}_{pp}>\overline{a}_{qq}$. For each orthogonal transformation, let $p<q$, then the eigenvalues in the last diagonal matrix will be in descending order.

A series of transform is needed to eliminate all off-diagonal elements generally. Because the eliminated elements do not always keep zeros, several iterations are required. Sign the k transformation matrix as N_k , the gotten matrix of the k transformation as $A^{(k)}$, then the $k+1$ transformation can be written as $A^{(k+1)}=N_{k+1}^T A^{(k)} N_{k+1}$. The eigenvectors matrix $Q^{(k+1)}$ of $A^{(k+1)}$ and the eigenvectors matrix $Q^{(k)}$ of $A^{(k)}$ satisfy $Q^{(k+1)}=N_{k+1}^T Q^{(k)}$.

Suppose there are s transformations to diagonalize A, then $Q^{(s)}=N_s^T N_{s-1}^T\cdots N_2^T N_1^T Q$, where Q is the eigenvectors matrix of A. For $Q^{(s)}$ is the eigenvectors matrix of diagonal matrix, $Q^{(s)}$ can be unit matrix I, i.e. $Q^{(s)}=I$. So

$$Q=N_1N_2\cdots N_s \tag{7}$$

5.THE REALIZATION OF JACOBI DIAGONALIZATION METHOD ON COMPUTER

There are three realization means of Jacobi diagonalization method on computer.

(1) Tracking Search Method

Each transformation makes the square-sum of the diagonal elements increase $2a_{pq}^2$ while the square-sum of the off-diagonal elements decreases $2a_{pq}^2$. If we select the largest module element of the off-diagonal elements for each transformation, then the efficiency to eliminate the off-diagonal elements is the highest.

Search for each row of the matrix, find out the largest module element of the off-diagonal elements, and record the column number of the element. The above information of the matrix is recorded in a $1\times n$ real array and a $1\times n$ integer array. So the largest module element of off-diagonal elements of the matrix can be found by searching for the real array, and the column number can be gotten from the corresponding position in the integer array. In the transformation process, these two arrays have to be revised continually to shorten the search process. If a transformation eliminate the element a_{pq} and a_{qp} , the pth element and the qth element in two arrays must be modified.

It is easy to see that the search process will spend a lot of time to find out the largest module element of the off-diagonal elements after each transformation.

(2) Jacobi Method in Order

Each transformation doesn't eliminate the largest module element of off-diagonal elements, but eliminate the off-diagonal elements in certain order, such as (p, q)=(1, 2), (1, 3), … ,(1, n), (2, 3), (2, 4), … , (2, n), … ,(n-1, n). When all off-diagonal elements are eliminated once, repeat the process time and time again until each off-diagonal element is zero.

This method omits the searching process, it can overcome long-searching-time shortcomings of tracking search method, but it needs a lot of transformations and a large amount of calculation.

(3) Jacobi Method with Threshold

This method is the compromise between tracking search method and Jacobi method in order. By this method, a threshold is given, although the off-diagonal element is eliminated in order, only those off-diagonal elements whose module is larger than the threshold are eliminated, the off-diagonal elements whose module is lower than the threshold remain unchanged. When the modules of all off-diagonal elements are less than the threshold, reduce the threshold and repeat the process again. End the iterative process when the threshold is small enough.

We can actualize Jacobi method with threshold by the following steps.

At first, calculate the square root of square sum of the off-diagonal elements of A, and get

$$v_0 = \left(2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}.$$

Secondly, set threshold $v_1 = v_0 / n$. Scan the off-diagonal elements of the matrix A in accordance with the line and seek the first element a_{pq} which module is larger than the threshold v_1, implement the orthogonal transfor-mation to A with orthogonal matrix N to eliminate a_{pq}. Repeat the process again and again until the modules of all off-diagonal elements are less than the threshold v_1.

Thirdly, set threshold $v_2 = v_1 / n$ and repeat the process.

At last, end the iterative process when the threshold is small enough, e.g. $v_k < \varepsilon$.

We choose Jacobi method with threshold evaluating the effectiveness of above three methods comprehensively.

6. THE STEPS OF EIGENVALUE-BASED DECOMPOSITION OF HERMITE MATRIX

According to above discussion, we obtain the steps of eigenvalue-based decomposition of Hermite matrix A.

(1) Separate A into real part and imaginary part

$$A = A_r + iA_i.$$

(2) Construct a real symmetric matrix

$$A' = \begin{pmatrix} A_r & -A_i \\ A_i & A_r \end{pmatrix}.$$

(3) Eliminate the off-diagonal elements of A' by Jacobi method with threshold, get a diagonal matrix. The diagonal elements of the matrix are all eigenvalues of A'.

(4) All transformation matrixes N_i sequentially multiply and get the eigenvector matrix of A according to Eq. (7).

7. CONCLUSIONS

In the MUSIC approach, eigenvalue-based decomposition of the Hermite matrix is the basic work. By transforming the problem of Hermite matrix decomposition into the problem of real symmetric matrix decomposition, the basic work has been done.

Literatures [7, 8] gave some specific simulation examples of MUSIC approach, and the simulation results showed that the methods discussed in this article was correct, feasible and effective.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflicts of interest.

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