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Homotopy Analysis Method for a Class of Holling Model with the Functional Reaction

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Abstract: In this paper, the homotopy analysis method is proposed to solve a class of holling model with the functional reaction. The model is characterized with the nonlinear equations in the denominator and it is difficult to obtain the closed approximation solutions. The series solutions of the model are obtained and the results show that the presented method is efficiency and simplicity. The convergence of this algorithm is also proved.

Keywords: Holling Model, Homotopy Analysis Method, Approximation Solutions.

1. INTRODUCTION

The prey-predator system is a very important model that has been studied by many ecologists and mathematicians. Recently, the holling model with functional reaction becomes a very attractive topic. Many works [1-5] focused on discussing the stability and the limit cycles, and so on. The direct and simple numerical simulations are lack. So finding explicit analytic solutions of system is extremely important in biology.

It is difficult to get ecological model's solution because of nonlinearity in the denominator. In recent years, many powerful methods have been developed to construct explicit analytic solution of nonliear system. In 1992, Liao [6] employed a general analytic method for nonlinear problems, namely homotopy analysis method (HAM). This technique has successfully been applied to solve many nonlinear problems, such as the KdV-type equations [7, 8], nonlinear heat transfer [9], nonlinear water equations [10], differential difference equations [11], and so on. The validity of HAM is independent of whether or not there exists small parameter in the considered equation. Besides, different from all previous numerical and analytical methods, it contains a certain auxiliary parameter *h*, which provides us with a simple way to adjust and control the convergence of solution series.

In this work, the HAM is used to solve the predator-prey models. The rest of this paper is organized as follows. In section 2, a brief introduction about the method is provided, and we provide the convergent theorem for nonlinear equation. In section 3, Holling III's model, is documented, the obtained results suggest that newly improvement technique is promising tool and powerful improvement for nonlinear equations. The last section includes our conclusions.

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2. BASIC IDEAS OF HAM

In this section, we give some basic concepts of the homotopy analysis method. To do this, consider the following nonlinear equation

$$N[u(t)] = 0 \tag{1}$$

where N is a nonlinear operator, t denote independent variables, and $u(t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way. With the generalized traditional homotopy method, Liao [6] constructs the so-called zero-order deformation equations

$$(1-p)L[\varphi(t;p)-u_0(t)] = p\hbar H(t)N[\varphi(t;p)] \tag{2}$$

where $p\in[0,1]$ is an embedding parameter,, $\hbar \neq 0$ is an auxiliary parameter, $H(t)\neq 0$ an auxiliary function, L is an auxiliary linear operator. $u_0(t)$ is an initial guess of $u(t)$, $\varphi(t;p)$ is an unknown function, respectively. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $p=0$ and $p=1$,

$$\varphi(t,0) = u_0(t); \qquad \varphi(t,1) = u(t) \tag{3}$$

Therefore, as p increases from 0 to 1, $\varphi(t;p)$ vary or deform from the initial $u_0(t)$ to the exact solution $u(t)$ governed by Eq. (1). This is the basic idea of the homotopy and this kind of variation is called deformations in topology. Thus, by Taylor's theorem and Eq. (3), we can express

$$\varphi(t;p) = u_0(t) + \sum_{k=1}^{\infty} u_k(t)p^k \tag{4}$$

where

$$u_k(t)=\frac{1}{k!}\frac{\partial^k\varphi(t;p)}{\partial p^k}\Big|_{p=0} \tag{5}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter h are so properly chosen, Eq. (4) is convergent at $p=1$, then one has

$$u(t)=\varphi(t,1)=u_0(t)+\sum_{k=1}^{\infty}u_k(t) \tag{6}$$

which must be one of the solutions of the original nonlinear Eq. (1), as proved by Liao [6]. Now, define the vectors

$$\vec{u}_n=\{u_0(t),u_1(t),\cdots,u_n(t)\} \tag{7}$$

Differentiating the zero-order deformation Eq. (2) m times with respect to t and then dividing them by $m!$ and finally setting $p=0$, we have the mth-order deformation equations

$$L[u_m(t)-\chi_m u_{m-1}(t)]=hH(t)R(\vec{u}_{m-1}) \tag{8}$$

where

$$R(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(x,t;p)]}{\partial p^{m-1}}\Bigg|_{p=0} \tag{9}$$

and

$$\chi_m=\begin{cases}0 & m\le 1\\ 1 & m>1\end{cases} \tag{10}$$

The solution of the mth-order deformation Eq. (8) is readily found to be

$$u_m(t)=\chi_m u_{m-1}(t)+hH(t)L^{-1}[R(\vec{u}_{m-1})] \tag{11}$$

It should be emphasized that $u_i(t)$ is governed by Eq. (7) with the initial conditions that come from the original problems, which can be easily solved by symbolic computation software such as Maple and Mathematics.

3. CONVERGENCE THEOREM

Theorem If the series (6), i.e. $\sum_{k=0}^{\infty}u_k(t)$ converge to function $u(t)$, then $u(t)$ must be the exact solution of Eq. (1).

Proof Since the series $\sum_{k=0}^{\infty}u_k(t)$ is convergent, we must have

$$\lim_{n\to\infty}u_n(t)=0 \tag{12}$$

Using the definition (10), we have

$$\sum_{m=1}^{n}[u_m(t)-\chi_m u_{m-1}(t)]=u_1(t)+[u_2(t)-u_1(t)]$$
$$+\cdots+[u_n(t)-u_{n-1}(t)]=u_n(t) \tag{13}$$

Therefore from (12), we lead to

$$\sum_{m=1}^{\infty}[u_m(t)-\chi_m u_{m-1}(t)]=\lim_{n\to\infty}u_n(t)=0 \tag{14}$$

Now, from the above expression and (11) we have

$$hH(t)\sum_{m=1}^{\infty}[R_{1m}(u_{m-1})]=$$

$$\sum_{m=1}^{\infty}[u_m(t)-\chi_m u_{m-1}(t)]=0 \tag{15}$$

Since $h\neq 0$ and $H(t)\neq 0$ then the above equation gives

$$\sum_{m=1}^{\infty}[R_{1m}]=0$$

Now, from (1) and definitions (5) and (9), it holds that

$$\sum_{m=1}^{\infty}[R_m]=\sum_{m=1}^{\infty}\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(x,t;p)]}{\partial p^{m-1}}\Bigg|_{p=0}$$

$$=N[\sum_{m=1}^{\infty}\frac{1}{(m-1)!}\frac{\partial^{m-1}\varphi(x,t;p)}{\partial p^{m-1}}\Bigg|_{p=0}]$$

$$=N[u(t)]=0 \tag{16}$$

In the above equations we used from this fact that L is a linear operator and the series $\sum_{k=0}^{\infty}u_k(t)$ converges to $u(t)$, respectively. Hence, $u(t)$ are the exact solutions of Eq. (1) and the proof is completed.

4. APPLICATION TO THE MODEL

Now we consider the Holling III' model [12]:

$$\begin{cases}x'=rx-\dfrac{\alpha x^2 y}{1+\omega x^2}\\ y'=y(-d+e\dfrac{\alpha x^2}{1+\omega x^2}-by)\end{cases} \tag{17}$$

Here $x(t)$, $y(t)$ represent the population density of two species respectively at time t, a susceptible pest and infected pest which live on the crop. All parameters are positive constants which meaning omitted here (see [12]). Type on both sides of the Eq. (17) with $1+\omega x^2$, note $\omega=a_1, r\omega=a_2, \alpha=a_3$, $e\alpha-d\omega=b_1, b\omega=b_2$, this leads to the dimensionless equation

$$\begin{cases}x'-rx+a_1x^2x'-a_2x^3+a_3x^2y=0\\ y'+dy+by^2+a_1x^2y'-b_1x^2y+b_2x^2y^2=0\end{cases} \tag{18}$$

with the initial conditions $x(0)=A$, $y(0)=B$. Due to the governing Eq. (18), the auxiliary linear operators are choosed:

$$L_1[X(t;p)]=\frac{\partial X(t;p)}{\partial t}+X(t;p)$$

$$L_2[Y(t;p)]=\frac{\partial Y(t;p)}{\partial t}+Y(t;p)$$

which satisfy $L_1[C_1e^{-t}]=0, L_2[C_2e^{-t}]=0$, where C_1, C_2 are integral constants, and the $X(t;p)$, $Y(t;p)$ are unknown real functions. Furthermore, due to (18), the non-linear operators are defined

$$N_1[X(t;p),Y(t;p)]=\frac{dX}{dt}-rX+a_1X^2X'$$

$$-a_2X^3+a_3X^2Y. \quad N_2[X(t;p),Y(t;p)]=\frac{dY}{dt}+dY+bY^2+$$

$$a_1X^2Y'-b_1X^2Y+b_2X^2Y^2$$

Then, introducing a non-zero auxiliary h, the zero-order deformation equations

$$(1-p)L_1[X(t;p)-x_0(t)]=$$

$$h_1H_1(t)pN_1[X(t;p),Y(t;p)] \tag{19.1}$$

$$(1-p)L_2[Y(t;p)-y_0(t)]=$$

$$h_2H_2(t)pN_2[X(t;p),Y(t;p)] \tag{19.2}$$

Obviously, when $p=0$ and $p=1$,

$$X(t;0)=x_0(t), Y(t;0)=y_0(t) \tag{20.1}$$

$$X(t;1)=x(t), Y(t;1)=y(t) \tag{20.2}$$

Therefore, as the embedding parameter p increases from 0 to 1, $X(t;p), Y(t;p)$ vary from the initials $x_0(t), y_0(t)$ to the exact solution $x(t), y(t)$ governed by (13). This is the basic idea of the homotopy and this kind of variation is called deformations in topology. Expanding $X(t;p)$ and $Y(t;p)$ in Taylor series with respect to p admits

$$X(t,p)=x_0(t)+\sum_{k=1}^{\infty}x_k(t)p^k \tag{21.1}$$

$$Y(t,p)=y_0(t)+\sum_{k=1}^{\infty}y_k(t)p^k \tag{21.2}$$

where

$$\begin{cases} x_k(t)=\dfrac{1}{k!}\dfrac{\partial^k X(t;p)}{\partial p^k}\Big|_{p=0} \\ y_k(t)=\dfrac{1}{k!}\dfrac{\partial^k Y(t;p)}{\partial p^k}\Big|_{p=0} \end{cases} \tag{22}$$

If the auxiliary linear parameter, the initial conditions, and the auxiliary parameters $h_1=h_2=h$, $H_1(t)=H_2(t)$ $=1$ are chosen, the above series converge at $p=1$, and one has

$$x(t)=x_0(t)+\sum_{k=1}^{\infty}x_k(t),\ y(t)=y_0(t)+\sum_{k=1}^{\infty}y_k(t)$$

Differentiating the zero-order deformation equations (19) m times with respect to p and then dividing them by $m!$ and finally setting $p=0$, the mth-order deformation equations read

$$L_1[x_m(t)-\chi_m x_{m-1}(t)]=hR_{1m}[x_{m-1},y_{m-1}] \tag{23.1}$$

$$L_2[y_m(t)-\chi_m y_{m-1}(t)]=hR_{2m}[x_{m-1},y_{m-1}] \tag{23.2}$$

Where

$$R_{1m}(t)=x'_{m-1}-rx_{m-1}-\sum_{i=0}^{m-1}[a_1\sum_{j=0}^{i}\frac{dx_i}{dt}x_{i-j}$$

$$-a_2\sum_{j=0}^{i}x_jx_{i-j}+a_3\sum_{j=0}^{i}x_jy_{i-j}]x_{m-1-i} \tag{24.1}$$

$$R_{2m}(t)=y'_{m-1}+dy_{m-1}+b\sum_{i=0}^{m-1}y_iy_{m-1-i}+\sum_{i=0}^{m-1}[\sum_{j=0}^{i}\frac{dy_j}{dt}x_{i-j}$$

$$+b_1\sum_{j=0}^{i}x_iy_{j-i}+b_2\sum_{j+n+l=0}x_jx_ny_l]x_{m-1-i} \tag{24.2}$$

Now, the solutions of the mth-order deformation Eqs. (23) for $m\geq 1$ become

$$x_m=\chi_m x_{m-1}+h\int_0^t \exp[r(t-s)]R_{1m}(s)ds+c_1e^{-t} \tag{25.1}$$

$$y_m=\chi_m y_{m-1}+h\int_0^t \exp(d(s-t))R_{2m}(s)ds+c_2e^{-t} \tag{25.2}$$

Where the constants c_1 and c_2 are determined by the initial conditions $x_m(0)=0, y_m(0)=0$. Mathematica software is used to solve the linear Eqs. (25) under the initial conditions up to first few order of approximations. We have

$$x(t)=x_0(t)+x_1(t)+\cdots=Ae^{-t}-A_{11}e^{-t}h$$

$$+A_{12}e^{-t}ht+A_{11}e^{-3t}h+\cdots \tag{26.1}$$

$$y(t)=y_0(t)+y_1(t)+\cdots=Be^{-t}+B_{11}e^{-3t}h$$

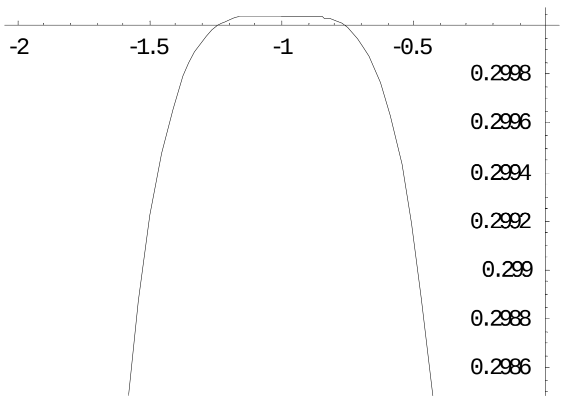
$$-B_{12}e^{-t}h-A_{12}e^{-t}ht+\cdots \tag{26.2}$$

where

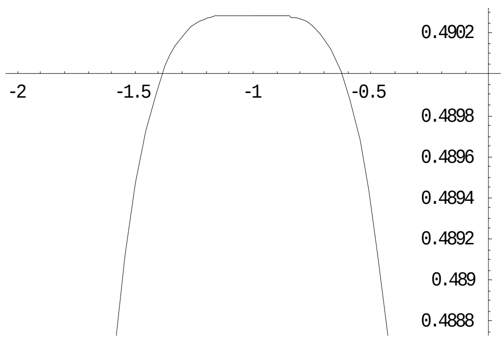
$A_{11}=[A(a_1+a_2)-a_3B]/2, A_{12}=A(1+r)$;

$B_{11}=A_{11}A^2, B_{12}=A^2[a_3-A(a_1+a_2)]/2$.

The solutions obtained by (25) contain the parameter h. We plot the so-called h-curves to ensure solution series converge, as suggested by Liao [6]. The valid region of h is a horizontal line segment. Thus the valid region of h in this case is $-1.2<h<-0.8$, as shown in Fig. (**1**) when $r=0.1, b_1=0.02, A=0.3, B=0.5, a_1=0.3, t=0.01$ $b_2=0.003, b=0.011, a_2=0.002, a_3=0.01,$



(a) h -curves of the 4-th order approximation for x



(b) h-curves of the 4-th order approximation for y

Fig. (1). the h -curves of the 4-th order approximation for $x(t)$ and $y(t)$.

$d = 0.015$. So and we can discuss the qualitative of Eq. (18).

4. CONCLUSIONS

The ordinary equations which arise from ecologists problems are usually nonlinear and such equations with Functional Reaction are difficult to estimate numerically or analytically. In this paper the HAM was applied to solve the prey-predator system and the approximate analytic solutions are obtained. We can study the model's property by the approximate analytic solutions. This approach was very efficient and powerful technique in finding the solutions of the proposed equations.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflicts of interest.

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