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Two Methods for Solving Constrained Bi-Matrix Games

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Abstract: In real life there are game problems in which players face with certain restrictions in the choice of strategy. These decision problems lead to constrained games. The quadratic programming problem equivalent to a constrained bi-matrix game is shown which provides a general method solving constrained bi-matrix games and shows a perfect correspondence between games and programming problems. Besides, a two-step method for constrained games is proposed whose theme is transforming the constrained game into an equivalent ordinary game. In the end, an example is shown to illustrate consistency of the two methods.

Keywords: Bi-matrix game, constrained bi-matrix game, quadratic programming.

1. INTRODUCTION

There are game problems in real life where the strategies of players are constrained to satisfy certain linear equations or inequalities rather than being in whole strategy space. These decision problems lead to constrained games first introduced by Charnes [1, 2]. He proved that constrained games can be solved by solving a pair of mutually dual linear programming problems and extended the concept to n-person constrained games. Later it was studied by Kawaguchi and Maruyama [3] in somewhat more generality which widens the scope of game-theoretic approaches. It is shown that the proposed approaches can handle uncertainty in the inequality constraints. More recently, constraints have been introduced in NTU coalitional games and Markov games [4, 5]. Also, a certain constrained dynamic game is shown to be equivalent to a pair of symmetric dual variational problems by Husain and Ahmad [6]. Various duality results are proved under convexity and generalized convexity assumptions on the appropriate functionals. Slightly different from the above-mentioned study, following Charnes's chart, we are more concerned about whether constrained bi-matrix games exist similar results as in constrained matrix games.

2. PRELIMINARIES ON MATRIX GAMES AND BI-MATRIX GAMES

In this section, we present certain basic definitions and preliminaries with regard to matrix games and bi-matrix games.

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative orthant. Let $A \in R^{m\times n}$ be m×n real matrix and $e^T = (1, 1, \ldots, 1)$ be a vector of 'ones' whose dimension is specified context. By a two person zero-sum matrix game we mean the triplet MG = (S^m, S^n, A) where $S^m = \{x \in \mathbb{R}^m_+,\ e^T x = 1\}$ and $S^n = \{x \in \mathbb{R}^n_+,\ e^T y = 1\}$. In the terminology of game theory, S^m (respectively S^n) is called the *strategy space* for Player I (respectively Player II) and A are called the *pay-off matrix*. Usually two person zero-sum matrix game is abbreviated as matrix game. If Player I chooses i^{th} pure strategy and Player II chooses j^{th} pure strategy then a_{ij} is the amount paid by Player II to Player I. The quantity $E(x, y) = x^T Ay$ is called the *expected pay-off* of Player I by Player II since elements of S^m (respectively S^n) can be thought of as a set of all probability distribution over $I = \{1, 2, \ldots, m\}$ (respectively $J = \{1, 2, \ldots, n\}$).

Definition 2.1 (Solution of game). Let $MG = (S^m, S^n, A)$ be the given matrix game. A triplet $(x^*, y^*, v^*) \in S^m \times S^n \times \mathbb{R}$ is called a solution of the game MG if

$E(x^*, y) \geq v^*, \forall y \in S^n$,

and

$E(x, y^*) \leq v^*, \forall x \in S^m$.

Here x^* is called an optimal strategy for Player I, y^* is called an optimal strategy for PlayerII, v^* is called the value of the game MG.

Theorem 2.1 (Existence theorem). Let $MG = (S^m, S^n, A)$. Then $\max_{x\in S^m}\min_{y\in S^n} x^T Ay$ and $\min_{y\in S^n}\max_{x\in S^m} x^T Ay$ both exists and are equal.

Theorem 2.1 guarantees that every matrix game has a solution. If there is no solution in the pure form then there is certainly a solution in the mixed form. Not long after the invention of simplex method, Kuhn and Tucker *et al.* [7] pointed out that solving a matrix game is equivalent to solving a pair of primal-dual linear programming.

In matrix game, one player's gain is just the other player's loss. Obviously there are situations in which the interests of two players are not exactly opposite. Such situations give rise to two person non-zero sum matrix games, also called bi-matrix games. Some well known bi-matrix game examples are "The Prisoner's Dilemma", "The Battle of Sexes" and "The Bargaining Problem".

A bi-matrix game is expressed as $BG = (S^m, S^n, A, B)$, where A and B are $m\times n$ real matrices representing the pay-offs to Player I and PlayerII respectively.

Definition 2.2 (Equilibrium point of *BG*). A pair $(x^*, y^*) \in S^m \times S^n$ is said to be an equilibrium point of the bi-matrix game BG if

$x^TAy^* \le x^{*T}Ay^*$,

and

$x^{*T}Ay \le x^{*T}Ay^*$,

for all $x\in S^m$ and $y\in S^n$.

It's straight to find that a matrix game $MG = (S^m, S^n, A)$ is a special case of the bi-matrix game BG with $B = -A$.

In the context of bi-matrix game, the following theorem due to Nash is very basic as it guarantees the existence of an equilibrium point of the bi-matrix game BG.

Theorem 2.2 (Nash existence theorem [8]). Every bi-matrix game $BG = (S^m, S^n, A, B)$ has at least one equilibrium point.

As already mentioned that every matrix game can be solved by solving a suitable primal-dual linear programming problems, Mangasarian and Stone [9] established a similar result to show that an equilibrium point of a bi-matrix game can be obtained by solving an appropriate quadratic programming problem.

Theorem 2.3 (Equivalence theorem). Let $BG = (S^m, S^n, A, B)$ be the given bi-matrix game. A necessary and sufficient condition that (x^*, y^*) be an equilibrium point of BG is that it is a solution of the following quadratic programming problem.

$$\max\ x^T(A+B)y-\alpha-\beta$$

$$s.t.\begin{cases} Ay-\alpha e\le 0 \\ B^Tx-\beta e\le 0 \\ e^Tx-1=0 \\ e^Ty-1=0 \\ x,\ y\ge 0 \\ \alpha,\ \beta\in \mathbb{R} \end{cases}$$

Further, if $(x^*, y^*, \alpha^*, \beta^*)$ is a solution of the above problem then $\alpha^* = x^{*T}Ay^*$, $\beta^* = x^{*T}By^*$.

If let $B = -A$, theorem 1.3 is reduced to a dual pair of linear programming that is equivalent to a matrix game.

3. CONSTRAİNED Bİ-MATRİX GAMES AND EQUİVALENT QUADRATİC PROGRAMMİNG

There are certain game problems in real life where the strategies of players are constrained to satisfy several linear inequalities rather than being in S^m or S^n only. These decision problems lead to constrained games.

Let $S_1 = \{x\in\mathbb{R}^m, Cx\le c, x\ge 0\}$, $S_2 = \{y\in\mathbb{R}^n, D^Ty\ge d, y\ge 0\}$, where $c\in\mathbb{R}^s$, $d\in\mathbb{R}^t$, $A, B\in R^{m\times n}$, $C\in R^{s\times m}$, $D\in R^{n\times t}$. Then the constrained bi-matrix games CBG is denoted as $CBG = (S_1, S_2, A, B)$. Note that $e^Tx = 1$ or $e^Ty= 1$ can be rewritten in two inequalities. If $B = -A$, then a constrained bi-matrix game degenerates into a constrained matrix game.

Definition 3.1 (Equilibrium point of *CBG*). A pair $(x^*, y^*) \in S_1\times S_2$ is said to be an equilibrium point of the constrained bi-matrix game CBG if

$x^TAy^* \le x^{*T}Ay^*$,

and

$x^{*T}By \le x^{*T}By^*$,

for all $x\in S_1$ and $y\in S_2$.

A main result due to of Charnes [1] in the constrained matrix game theory, as in usual matrix games, assert that every CMG is equivalent to two primal-dual linear programming problems.

Then a natural question is coming: Is there a quadratic programming problem equivalent to a given CBG? The answer is yes.

Since the constraints are linear, if not empty, the strategy set of Player I, namely S_1 (respectively S_2) is a convex set on S^m (S^n). Let $\{x_1, x_2, \ldots, x_s\}$ ($\{y_1, y_2, \ldots, y_t\}$)be the vertices of S_1 (S_2), then we have the following theorem.

Theorem 3.1 (Equivalence theorem of *CBG*). Let $CBG = (S^m, S^n, A, B)$ be the given constrained bi-matrix game. A necessary and sufficient condition that (x^*, y^*) be an equilibrium point of CBG is that it is a solution of the following quadratic programming problem (QPP).

$$\max\ x^T(A+B)y-\alpha-\beta$$

$$s.t.\begin{cases} x_i^TAy-\alpha\le 0 & i=1,\ 2,\ldots,s \\ x^TBy_j-\beta\le 0 & j=1,\ 2,\ldots,t \\ Cx\le c \\ D^Ty\ge d \\ x,\ y\ge 0 \\ \alpha,\ \beta\in \mathbb{R} \end{cases}$$

Further, if $(x^*, y^*, \alpha^*, \beta^*)$ is a solution of the above problem then $\alpha^* = x^{*T}Ay^*$, $\beta^* = x^{*T}By^*$.

Proof. Let S be the set of all feasible solutions of the above problem. Suppose $S\neq\varnothing$. For any $(x, y, \alpha, \beta)\in S$,

$$x^TAy-\alpha = (\sum a_ix_i)^TAy-\sum a_i\alpha = \sum a_i(x_i^TAy-\alpha)\le 0.$$

Similarly, $x^TBy-\beta\le 0$ and therefore we have $x^T(A+B)y-\alpha-\beta\le 0$. Now suppose that (x^*, y^*) is an equilibrium point of the CBG. Let $\alpha^* = x^{*T}Ay^*$, $\beta^* = x^{*T}By^*$, then $(x^*, y^*, \alpha^*, \beta^*)\in S$ and $x^{*T}(A+B)y^*-\alpha^*-\beta^* = 0$. Therefore, $(x^*, y^*, \alpha^*, \beta^*)$ is a solution of the above quadratic programming problem and the optimal value of the QPP is 0.

Conversely, let $(x^*, y^*, \alpha^*, \beta^*)$ be a solution of the above QPP, then $(x^*, y^*, \alpha^*, \beta^*)\in S$ and $x^{*T}(A+B)y^*-\alpha^*-\beta^* = 0$. Since $x_i^TAy^*-\alpha^*\le 0$ and $x^{*T}By_j-\beta^*\le 0$, then for arbitrary $(x, y)\in S_1\times S_2$, we have

$x^TAy^*\le\alpha^*$ and $x^{*T}By\le\beta^*$

and

$x^{*T}Ay^{*} \le \alpha^{*}$ and $x^{*T}By^{*} \le \beta^{*}$.

Notice that $x^{*T}(A+B)y^{*} - \alpha^{*} - \beta^{*} = 0$, it's straightforward to find that $x^{T}Ay^{*} \le x^{*T}Ay^{*} = \alpha^{*}$ and $x^{*T}By \le x^{*T}By^{*} = \beta^{*}$. That is, (x^{*}, y^{*}) is an equilibrium point of the *CBG*. Here ends the proof.

For a given *CBG*, if there is an equilibrium point of its corresponding *BG* satisfying the constraints then it certainly is an equilibrium point of the given *CBG*. But even if no equilibrium point of the corresponding *BG* satisfies the constraints, we can not ensure that the *CBG* must not have an equilibrium point. Here's a simple example. Consider a bi-matrix game with the following payoff matrices

(1, 1)	(2, 0)
(0, 2)	(1, 1)

The game has only one equilibrium point in mixed form, *i.e.*, (1/2, 1/2; 1/2, 1/2). For Player I, assume that the probability of the first strategy being selected must be greater than 0.6. It's not difficult to find the equilibrium point of this constrained bi-matrix game. That is, (0.6, 0.4; 1, 0).

Example 3.1 Consider a bi-matrix game with the following payoff matrices:

	(1, 1)	(2, 0)	(0, 2)
(*A, B*)=	(0, 2)	(1, 1)	(2, 0)
	(2, 0)	(0, 2)	(1,1)

Noting that this bi-matrix game is a modified version of the famous rock paper scissors game by adding 'one' to all players' payoff in each situation, then it's straight to find the unique equilibrium point, *i.e.*, (1/3, 1/3, 1/3; 1/3, 1/3, 1/3). Now suppose that Player I faces with the following restriction:

$x_1 \ge 0.5$.

Apparently the previous equilibrium point is no longer feasible. Now Player I's strategy space S_1 $\{x\in\mathbb{R}^m, x_1 \ge 0.5, e^T x = 1, x \ge 0\}$ is a triangle with three vertices, namely, {(1, 0, 0), (1/2, 1/2, 0), (1/2, 0, 1/2)}. Solving the following quadratic programming problem,

$$\max\ x^T(A+B)y - \alpha - \beta$$

$$s.t.\begin{cases} y_1 + 2y_2 - \alpha \le 0 \\ y_1/2 + 3y_2/2 + y_3 - \alpha \le 0 \\ 3y_1/2 + y_2 + y_3/2 - \alpha \le 0 \\ x_1 + 2x_2 - \beta \le 0 \\ x_2 + 2x_3 - \beta \le 0 \\ 2x_1 + x_3 - \beta \le 0 \\ x_1 + x_2 + x_3 = 1 \\ x_1 \ge 0.5 \\ y_1 + y_2 + y_3 = 1 \\ x,\ y \ge 0 \\ \alpha,\ \beta \in \mathrm{R} \end{cases}$$

we get $x^* = (1/2, 1/3, 1/6)^T$, $y^* = (1/3, 0, 2/3)^T$, $\alpha^* = 5/6$, $\beta^* = 7/5$.

4. A TWO-STEP METHOD FOR CONSTRAİNED GAMES

Consider a constrained bi-matrix game *CBG*, let $\{x_1, x_2, \ldots, x_s\}$ $(\{y_1, y_2, \ldots, y_t\})$be the vertices of S_1 (S_2), then any $x \in S_1$ $(y \in S_2)$ can be expressed by a convex combination of $\{x_1, x_2, \ldots, x_s\}$ $(\{y_1, y_2, \ldots, y_t\})$. Given $(x, y) \in S_1 \times S_2$, Player I's payoff is

$$\begin{aligned} x^T Ay &= \left(\sum_i p_i x_i\right)^T A\left(\sum_j q_j y_j\right) \\ &= (p_1,\ldots,p_s)\begin{pmatrix}x_1 & \ldots & x_s\end{pmatrix}^T_{s\times m} A_{m\times n}\begin{pmatrix}y_1 & \ldots & y_t\end{pmatrix}_{n\times t}\begin{pmatrix}q_1\\ \vdots \\ q_t\end{pmatrix} \\ &= p^T\left[\begin{pmatrix}x_1 & \ldots & x_s\end{pmatrix}^T_{s\times m} A_{m\times n}\begin{pmatrix}y_1 & \ldots & y_t\end{pmatrix}_{n\times t}\right]q \\ &= p^T A' q, \end{aligned}$$

where $p, q \ge 0$ and $e^T p = e^T q = 1$.

By same operation, Player II's payoff can be rewritten as $p^T B' q$. So far, the constrained bi-matrix game has been transformed into an ordinary bi-matrix game.

Example 3.1 (continued). In this example, only Player I faces restrictions, hence

$$\begin{aligned} A' &= \begin{pmatrix}x_1 & \ldots & x_s\end{pmatrix}^T_{s\times m} A_{m\times n} \\ &= \begin{pmatrix}1 & 1/2 & 1/2\\ 0 & 1/2 & 0\\ 0 & 0 & 1/2\end{pmatrix}^T \begin{pmatrix}1 & 2 & 0\\ 0 & 1 & 2\\ 2 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 0\\ 1/2 & 3/2 & 1\\ 3/2 & 1 & 1/2\end{pmatrix} \end{aligned}$$

$$B' = \begin{pmatrix}x_1 & \ldots & x_s\end{pmatrix}^T_{s\times m} B_{m\times n} = \begin{pmatrix}1 & 0 & 2\\ 3/2 & 1/2 & 1\\ 1/2 & 1 & 3/2\end{pmatrix}.$$

According theorem 1.3, solving the quadratic programming problem that is equivalent to the bi-matrix game with payoff matrices (A', B'),

$$\max\ p^T(A'+B')q - \alpha - \beta$$

$$s.t.\begin{cases} A'q - \alpha e \le 0 \\ B'^T p - \beta e \le 0 \\ e^T p - 1 = 0 \\ e^T q - 1 = 0 \\ p,\ q \ge 0 \\ \alpha,\ \beta \in \mathrm{R} \end{cases}$$

we get $p^* = (0, 2/3, 1/3)^T$, $q^* = (1/3, 0, 2/3)^T$, $\alpha^* = 5/6$, $\beta^* = 7/5$. Not surprisingly, the two methods get the same result since $x^* = \sum_i p_i x_i = (1/2,\ 1/3,\ 1/6)^T$ and $y^* = q^*$.

Just as already seen, 'unconstrained' or 'constrained' is a relative distinction, a constrained game is an ordinary game itself. As long as the strategy space and payoff function of a constrained game satisfy appropriate conditions, see Glicksberg [10], then this constrained game exists equilibrium. And therefore we do not discuss this issue further here.

CONCLUSION AND DISCUSSION

In this paper we proposed two methods to deal with constrained bi-matrix games. One is solving equivalent quadratic programming problems and the other is transforming constrained bi-matrix games into ordinary unconstrained bi-matrix games. Unlike constrained matrix games, if the constraints change continuously, we still can not trace equilibrium of constrained bi-matrix games dynamically unless we are able to trace the change of strategy set dynamically. Moreover, constraints in actual game problems are not necessarily linear. All these need further studies in the future.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

ACKNOWLEDGEMENTS

This work is supported by the National Natural Science Foundation of China (Nos. 71201089), and the Natural Science Foundation Youth Project of Shandong Province, China (ZR2012GQ005).

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Received: September 16, 2014 Revised: December 23, 2014 Accepted: December 31, 2014