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Generalized Synchronization of Fractional-Order Chaotic Systems with Unequal Orders

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Abstract: The synchronization problem between non-identical fractional orders chaotic systems has few investigated. This letter presents generalized synchronization of different fractional orders chaotic systems with non-identical orders. Based on the basic properties of fractional calculus and the stability theory of fractional order systems, a general method is obtained. Furthermore, the paper presents different dimensions chaotic systems to achieve the synchronization and the corresponding simulations show the feasibility of the method.

Keywords: Different dimensions, Fractional orders chaotic systems, Generalized synchronization, non-identical orders.

1. INTRODUCTION

Since Pecora and Carroll introduced a method [1] to synchronize two identical systems with different initial conditions in 1990, chaos synchronization has gained a lot of attention from various fields during the last two decades. Many types of chaos synchronization have been investigated, such as complete synchronization [2], lag synchronization [3], phase synchronization [4], projective synchronization [5], generalized synchronization [6], etc. Nowadays, generalized synchronization has been extensively studied because of generalized synchronization which states of two systems satisfy a functional relation or asymptotically satisfy a functional relation. However, most of them mainly concern the synchronization of integer order chaotic systems.

Recently, many fractional order nonlinear dynamics of chaotic systems have been investigated, for example, the fractional order unified system [7], the fractional order Liu system [8], the fractional order Chen system [9], the fractional order Rossler system [10], the fractional order Chua system [11], the fractional order Lu system, [12] etc. On the other hand, the synchronization phenomenon of fractional order chaotic systems have been intensively studied [13, 14].

In the aforementioned papers, the researchers are mainly studied the synchronization of the same orders chaotic systems, i.e., the drive and response systems are the same integer orders or the same fractional orders. For example, generalized synchronization between different fractional orders chaotic system with identical order in Ref. [15]. In the recent years, the synchronization problem between unequal orders chaotic systems, namely, between integer order and fractional order, or between fractional-order with unequal orders, has been attracted considerable interests of researchers [16, 17].

In this letter, generalized synchronization of different fractional orders chaotic systems with non-identical orders has been investigated. Based on several fundamental properties on fractional order calculus and the stability theory of fractional order systems, we introduce a general method to accomplish the generalized synchronization. Furthermore, this article extend the form of the work in Ref. [15].

This paper is organized as follows. In section 2, we present the some properties of fractional order calculus. In section 3, we give the general method and the relevant mathematical proof. In section 4, we present different dimensions chaotic systems to achieve the synchronization and the corresponding simulations show the feasibility of the method. A conclusion is given at the end.

2. PROPERTIES OF FRACTIONAL CALCULUS

In the paper, we consider the Caputo definition of fractional calculus, denote D^{α} as the simplified form of ${}_0^C D_t^{\alpha}(t \geq 0)$. Some essential properties of fractional derivatives and integrals are introduced [18, 19].

(A) For $\alpha = n$, where n is an integer, the operation $D^{\alpha} f(t)$ gives the same result as classical calculus of integer order n. Particularly, when $\alpha = 1$, the operation $D^{\alpha} f(t)$ is the ordinary derivation: $D^1 f(t) = \frac{df(t)}{dt}$

(B) Fractional differentiation and fractional integration are linear operations:

$$D^{\alpha}[af(t) + bg(t)] = aD^{\alpha} f(t) + bD^{\alpha} g(t)$$

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(C) For $\alpha \geq 0$, the following equation holds:
$D^{\alpha}D^{-\alpha}f(t) = D^0 f(t) = f(t)$

(D) The Laplace transform formula for the Caputo fractional derivative is as follows:

$$L[D^{\alpha}f(t)] = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k=1} f^k(0), (\alpha > 0, n-1 < \alpha \leq n).$$

In particular, when $\alpha \in (0,1]$, then

$L[D^{\alpha}f(t)] = s^{\alpha}F(s) - s^{\alpha-1}f(0)$.

3. PROBLEM FORMULATION AND SYNCHRONIZATION METHOD

Consider the following chaotic systems:

$$D^{\alpha}x = f(x) \quad (1)$$

and

$$D^{\beta}y = g(y) + U(x, y) \quad (2)$$

where $\alpha = [\alpha_1, \alpha_2, \mathrm{L}, \alpha_n]^T$ is an $n \times 1$ vector which denotes the fractional orders for each state of drive system, $\beta = [\beta_1, \beta_2, \mathrm{L}, \beta_m]^T$ is an $m \times 1$ vector which denotes the fractional orders for each state of response system, $x = [x_1, x_2, \mathrm{L}, x_n]^T \in R^n$ is the n dimensional state vector for drive system, $f: R^n \to R^n$ is a continuous vector function, $y = [y_1, y_2, \mathrm{L}, y_m]^T \in R^m$ is the m dimensional state vector for response system, $g: R^m \to R^m$ is a continuous vector function, and $U(x, y) \in R^m$ is a controller.

Definition

We say systems (1) and (2) are generalized synchronization with respect to vector map ϕ if there exit a controller $U(x, y) \in R^m$ and a given map $\phi: R^n \to R^m$ such that the solutions of systems (1) and (2) satisfy the following property: $\lim_{t\to\infty} |y(t) - \phi(x(t))| = 0$

Decompose the response system:

$D^{\beta}y = Ay + BG(y) + U(x, y)$

where A, B are matrix with same dimensions, $G(y)$ is the nonlinear.

Define the generalized synchronization error:

$e(t) = y(t) - \phi(x(t))$

In order to achieve the synchronization, we separate the controlle $U(x, y) \in R^m$ function:

$U(x, y) = U_1(x, y) + U_2(x, y)$

and propose the

$U_1(x, y) = (D^{-(\alpha-\beta)} - I)[g(y)]$

where the I is the identity operator. By inserting $U_1(x, y)$ into the response system, we obtain:

$D^{\beta}y = D^{-(\alpha-\beta)}g(y) + U_2(x, y)$

By applying the Laplace transform to the system and letting $Y(s) = L\{y(t)\}$, we get:

$s^{\beta}Y(s) - s^{-(\alpha-\beta)}y(0) = s^{-(\alpha-\beta)}L\{g(y)\} + L\{U_2(x, y)\}$

Multiplying the $s^{\alpha-\beta}$ to both the left and right sides of the above equation and applying the inverse Laplace transform to the result, we obtain:

$D^{\alpha}y = g(y) + D^{(\alpha-\beta)}U_2(x, y)$

By introducing the $U_1(x, y)$, we then reduce the problem to the synchronization of fractional order systems with identical orders.

In this paper, we propose the nonlinear $U_2(x, y)$ of the following form

$U_2(x, y) = D^{-(\alpha-\beta)}[D\phi f(x) - K(y - \phi(x))$
$- BG(y) - A\phi(x)]$

then the error system is:

$D^{\alpha}e = D^{\alpha}(y - \phi(x))$
$= D^{\alpha}y - D\phi D^{\alpha}x$
$= Ay + BG(y) + D^{\alpha-\beta}U_2(x, y) - D\phi \cdot D^{\alpha}x$
$= Ay + BG(y) - D\phi \cdot D^{\alpha}x + D\phi f(x) - K(y - \phi(x))$
$- BG(y) - A\phi(x)$
$= (A - K)e$

Where $K = [k_1, k_2, \mathrm{L}, k_n]^T \in R^{m\times m}$ is a feedback control matrix, $D\phi$ is the Jacobian matrix of ϕ.

If the eigenvalues $\lambda_1, \lambda_2, \mathrm{L}, \lambda_n$ of $A - K$ satisfy $|\arg(\lambda_i)| > \frac{q\pi}{2} (i = 1,2, \mathrm{L}, n)$. According to the stability theorems of fractional order system, the error system will be asymptotically stable. Hence the generalized synchronization between different orders chaotic systems is achieved.

4. NUMERICAL SIMULATION

The fractional order Liu system [15]

$$\begin{cases} D^{\alpha}x_1 = m(x_2 - x_1) \\ D^{\alpha}x_2 = nx_1 - kx_1x_3 \\ D^{\alpha}x_3 = -lx_3 + hx_1^2 \end{cases}$$

Where m, n, l, h, k are the constants. when $\alpha = 0.9, m = 10, n = 40, l = 2.5, h = 4, k = 1$, the system is chaotic. Fig. (**1**) displays the chaotic attractors of the Liu fractional order chaotic system.

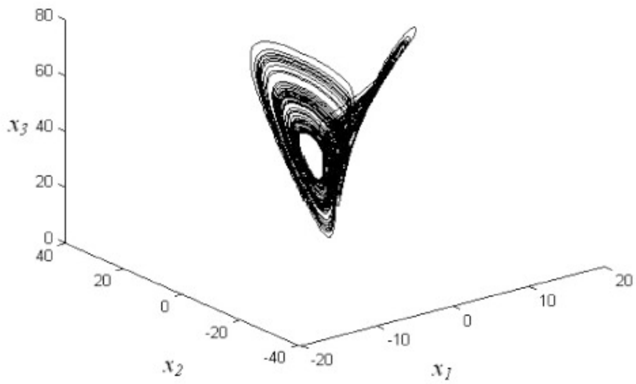

Fig. (1). Phase plot of fractional order Liu chaotic system in x1 − x2 − x3 space.

The novel four dimensional fractional order system [15]:

$$\begin{cases} D^{\beta} y_1 = a(y_2 - y_1) + y_4 \\ D^{\beta} y_2 = dy_1 - y_1 y_3 + cy_2 \\ D^{\beta} y_3 = y_1 y_2 - by_3 \\ D^{\beta} y_4 = y_2 y_3 + ry_4 \end{cases}$$

where $a = 35, b = 3, c = 12, d = 7, r = 0.2$ (Figs. **2**-**4**) displays the hyperchaotic attractors of the new fractional order hyperchaotic system when $\beta = 0.95$.

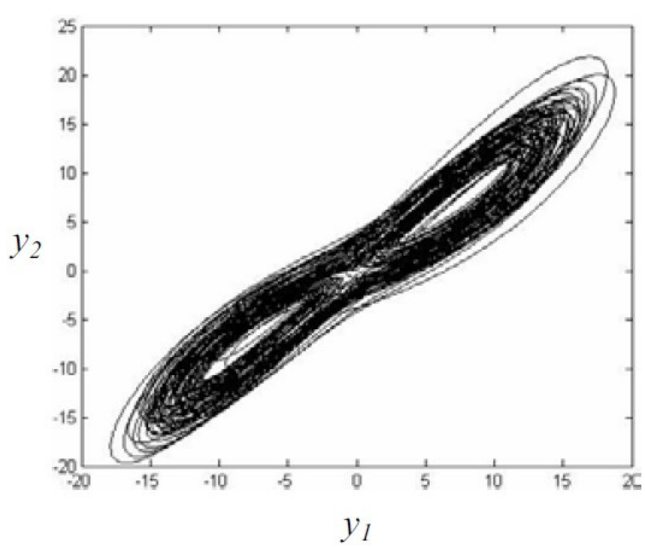

Fig. (2). Phase plot of the new hyperchaotic system in y1 − y2 plane.

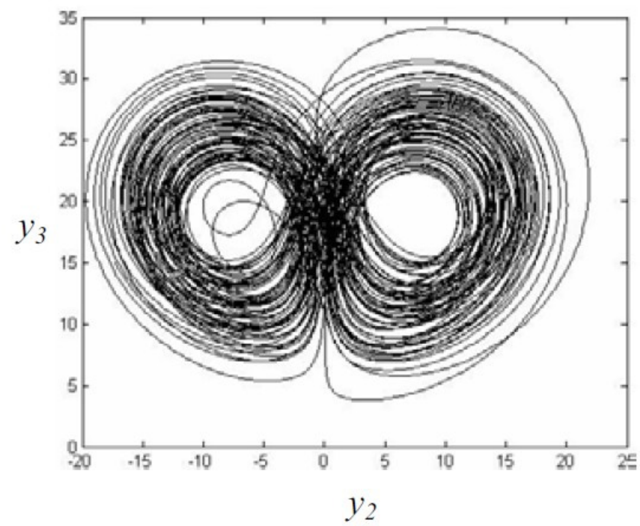

Fig. (3). Phase plot of the new hyperchaotic system in y2 − y3 plane.

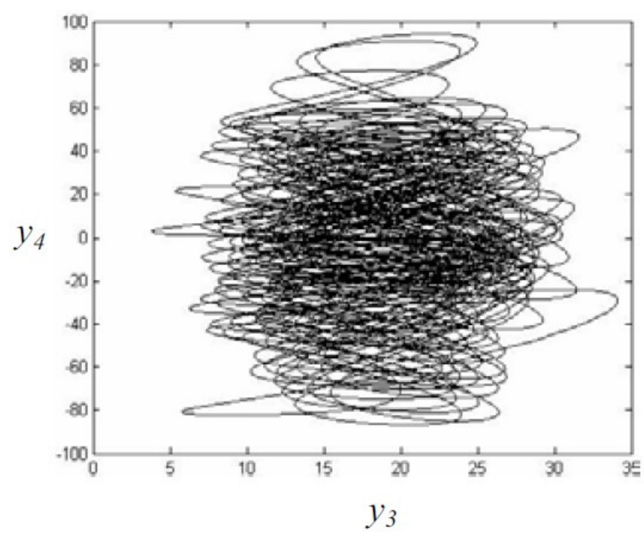

Fig. (4). Phase plot of the new hyperchaotic system in y3 − y4 plane.

Take the Liu system as the drive system, the novel four dimensional fractional order system, as a response system, is given by

$$\begin{cases} D^{\beta} y_1 = a(y_2 - y_1) + y_4 + u_1(t) \\ D^{\beta} y_2 = dy_1 - y_1 y_3 + cy_2 + u_2(t) \\ D^{\beta} y_3 = y_1 y_2 - by_3 + u_3(t) \\ D^{\beta} y_4 = y_2 y_3 + ry_4 + u_4(t) \end{cases}$$

where $u_1(t), u_2(t), u_3(t), u_4(t)$ are the controller functions to design.

For simplicity, we select

$$\phi(x)=[\frac{1}{2}x_1x_2,x_2^2,x_1+x_2+x_3,-x_3]^T$$

Define the feedback gain matrix

$$K=\begin{bmatrix}0&0&0&0\\7&13&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

According to the method, the controller functions as follows:

$$\begin{cases}
u_1=[D^{0.05}-I][a(y_2-y_1)+y_4] \\
+D^{0.05}[\frac{a}{2}x_2(x_2-x_1)+\frac{1}{2}x_1(nx_1-kx_1x_3) \\
+\frac{35}{2}x_1x_2-35x_2{}^2+x_3] \\
u_2=[D^{0.05}-I][dy_1-y_1y_3+cy_2] \\
+D^{0.05}[2x_2(nx_1-kx_1x_3)-7(y_1-0.5x_1x_2) \\
-13(y_2-x_2{}^2)+y_1y_3-\frac{7}{2}x_1x_2-12x_2{}^2] \\
u_3=[D^{0.05}-I][y_1y_2-by_3] \\
+D^{0.05}[a(x_2-x_1)nx_1-kx_1x_3-lx_3+hx_1{}^2) \\
-y_1y_2+3(x_1+x_2+x_3)] \\
u_1=[D^{0.05}-I][y_2y_3+ry_4] \\
+D^{0.05}[lx_3-hx_1{}^2)-(y_4+x_3)-y_2y_3+0.2x_3]
\end{cases}$$

Define the error system:

$$e_1=y_1-\frac{1}{2}x_1x_2,e_2=y_2-x_2^2,$$
$$e_3=y_3-(x_1+x_2+x_3),e_4=y_4+x_3$$

in the numerical simulations, the controller functions $u_1(t)$, $u_2(t),u_3(t),u_4(t)$ are designed as above, the parameters of the fractional order Liu chaotic system as $\alpha=0.9$, $m=10$, $n=40,l=2.5,h=4,k=1$,select the parameters of new four dimensional fraction order system as $\beta=0.95$, $a=35,b=3,c=12,d=7,r=0.2$, the initial values of the drive system and response system are taken as $x_1(0)=7,x_2(0)=-9,x_3(0)=5,$ $y_1(0)=-5,$ $y_2(0)=4,y_3(0)=-6,y_4(0)=8$ respectively. The simulation results are illustrated in (Fig. **5-8**). In these figures, it can be seen that the synchronization error will converge to zerofinally and two different systems are indeed achieving chaos generalized synchronization.

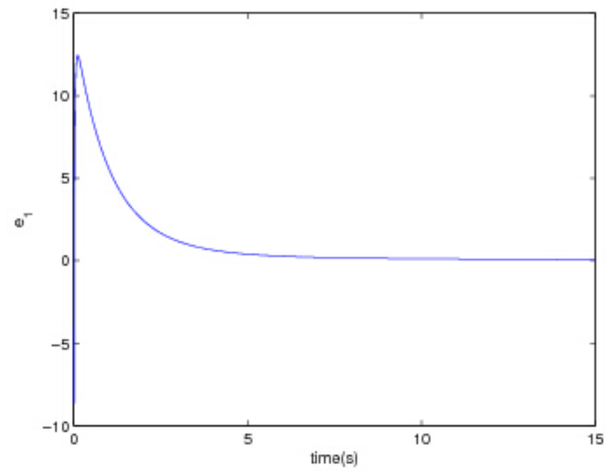

Fig. (5). Error e1 state of the drive system and the response system.

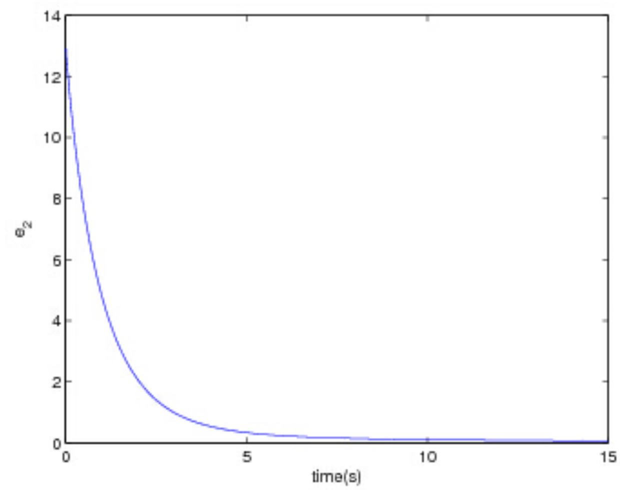

Fig. (6). Error e2 state of the drive system and the response system.

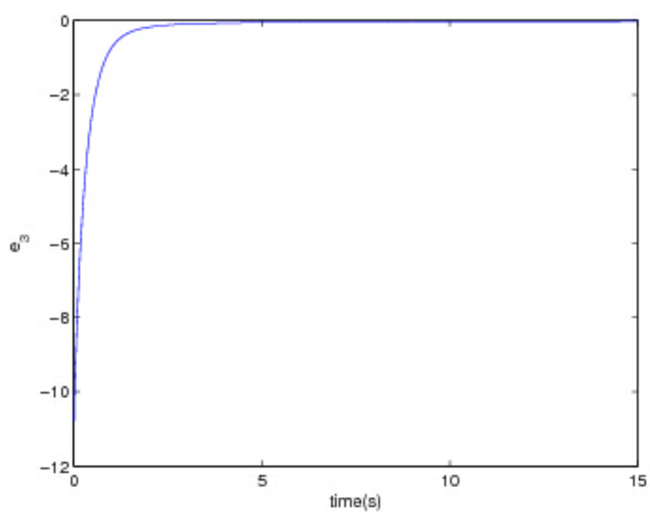

Fig. (7). Error e3 state of the drive system and the response system.

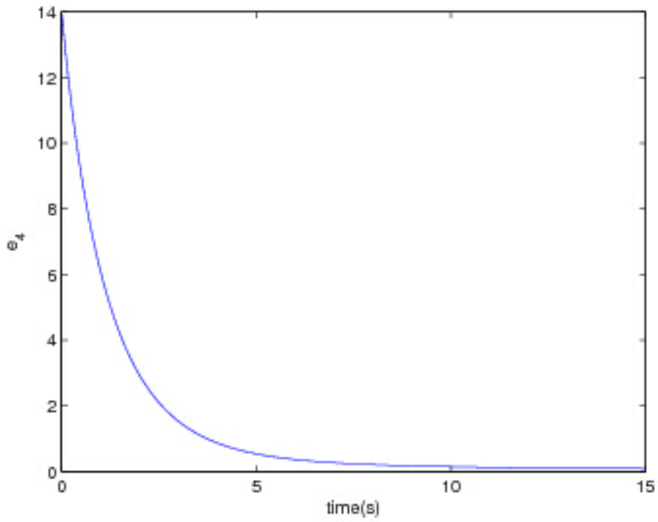

Fig. (8). Error e4 state of the drive system and the response system.

CONCLUSION

The synchronization of fractional order chaotic systems with unequal orders is a hot topic problem. This letter presents generalized synchronization of different fractional orders chaotic systems with non-identical orders and extend the form of the generalized synchronization between the identical orders chaotic systems. Based on the basic properties of fractional calculus and the stability theory of fractional order systems, a general method is obtained. Furthermore, numerical simulations illustrate the effectiveness of the proposed scheme.

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CONFLICT OF INTEREST

The author confirms that this article content has no conflict of interest.

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