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A Novel Entropy of Interval Valued Fuzzy Set

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Abstract: The main attribution of this paper is to propose two methods to construct entropy of interval valued fuzzy sets (IVFSs). First, we use an IVFS to construct two FSs and define entropy by similarity measure of the two FSs; then we use an IVFS to construct two IVFSs and also define entropy by similarity measure of the two IVFSs. At the same time, we give two examples to show that our methods are effective.

Keywords: Entropy, interval valued fuzzy set, similarity measure.

1. INTRODUCTION

Fuzzy sets (FS) was first introduced by Zadeh [1] which are used to deal with imprecision and uncertainty information. With the development of the fuzzy sets, many new theories came out, such as the interval valued fuzzy sets [2, 3, 4] and the generalized theory of uncertainty [5]. Later Atanassov [6, 7] extended the fuzzy sets theory to the intuitionistic fuzzy sets (IFSs) which conclude membership function and non-membership function. Subsequently, some experts have applied the FSs and IVFSs theory to many fields. For example, Gorzalczany [8] researched the indicator of inclusion grade for IVFSs and applied to approximate reasoning, Turksen [9] studied the IVFSs in normal forms. Deschrijver [10] made some extentions of interval valued fuzzy sets. Moreover, Bustince [11] pointed out that intuitionistic fuzzy sets and interval valued fuzzy sets are equal in some degree.

It is an important topic to discuss the entropy and similarity measure in fuzzy sets theory. Entropy is a well-known concept in physics, information theory and fuzzy set theory. It describes the degree of uncertainties and fuzziness of the fuzzy sets. Since its appearance, some experts have made great progress in this field. For example, in 1972, in order to describe the fuzziness degree of fuzzy sets, De Luca and Termini [14] proposed some axioms of entropy. Burillo and Bustince [15] extended the definitions of entropy from FSs to the IFSs. Then, Szimidt and Kacprzyk [16] developed a new method to define entropy of IFSs using a geometric interpretation with nonprobabilistic-type. Based on distance, Zhang [17] proposed the new axiomatic definition of entropy for IVFSs and the relationship between entropy and similarity measure. The concept of similarity measure of fuzzy sets is to depict the similarity degree of fuzzy sets. It has many applications, such as fuzzy clustering, image processing, fuzzy reasoning and fuzzy neutral network [5, 13, 17]. Zeng and Guo [25] constructed similarity measure and entropy for IVFSs by normalized distances. Some researchers also have researched the similariy measure and entropy for IVFSs or IVIFSs [16-21] and have made great success in this field.

Consistently with the idea, we want to further study the entropy of IVFSs from the point of similarity measure. First, we use an IVFS to construct two FSs and define entropy by similarity measure of the two FSs; then we use an IVFS to construct two IVFSs and also define entropy by similarity measure of the two IVFSs. At the same time, we give two examples to show that our methods are effective.

The rest of our work is organized as follows. Section 2 gives basic definitions of the FSs and IVFSs. Section 3 constructs the new FSs and IVFS, respectively, and gives entropy of interval valued fuzzy sets. Section 4 gives our conclusion.

2. PRELIMINARIES

Definition 1 [1]. A fuzzy set A in $X=\{x\}$ may be given as

$$A=\{<x,u_A(x)>| x\in X\},$$

where $u_A(x): X\to[0,1]$ is the membership function of x in A.

Definition 2 [27]. We call a mapping $A: X\to[0,1]$ an interval valued fuzzy set in X. For every $A\in$ IVFSs and $x\in X$, $A(x)=[A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, then fuzzy sets $A^-: X\to[0,1]$ and $A^+: X\to[0,1]$ are called a low fuzzy set of A and a upper fuzzy set of A, respectively. For simplicity, we denote $A = [A-, A+]$.

1874-110X/15

Let A,B be two IVFSs, then the following operations are obtained.

$A \subset B$ iff $\forall x \in X$, $A^- \le B^-$ and $A^+ \le B^+$,

$A = B$ iff $\forall x \in X$, $A^- = B^-$ and $A^+ = B^+$,

$$A^C = \left[\left(A^-\right)^C, \left(A^+\right)^C\right] = \left[1-A^+, 1-A^-\right].$$

Definition 3 [15]. A real function E : IVFSs→ [0, 1] is called an entropy on IVFSs, if E satisfies the following properties:

(P1) $E(A)=0$ iff A is a crisp set;

(P2) $E(A)=1$ iff $A^- + A^+ = 1$;

(P3) $E(A) \le E(B)$ if A is less fuzzy than B, *i.e.*, $A^- \le B^-$ and $A^+ \le B^+$ for $B^- + B^+ \le 1$ or $A^- \ge B^-$ and $A^+ \ge B^+$ for $B^- + B^+ \ge 1$;

(P4) $E(A) = E\left(A^C\right)$

Definition 4 [23]. A real function S: IVFSs×IVFSs→ [0, 1] is called similarity measure of interval valued fuzzy sets, if S satisfies the following properties:

(N1) $S\left(A, A^C\right) = 0$ if A is a crisp set;

(N2) $S(A,B) = 1 \Leftrightarrow A = B$;

(N3) $S(A,B) = S(B,A)$;

(N4) for all $A,B,C \in \text{IVFSs}$, if $A \subseteq B \subseteq C$, then

$S(A,C) \le S(A,B)$, $S(A,C) \le S(B,C)$.

Specially, if interval valued fuzzy sets A and B become fuzzy sets, then $S(A,B)$ is similarity measure of fuzzy sets.

3. ENTROPY OF INTERVAL VALUED FUZZY SET

Let $A = [A^-, A^+]$ be an IVFS, we construct two fuzzy sets M^-, M^+ based on A.

$$M^- = \frac{1 + \left(A^- + A^+ - 1\right)^2}{2}, M^+ = \frac{1 - \left|A^- + A^+ - 1\right|}{2} \tag{1}$$

Theorem 1. Suppose $S\left(M^-, M^+\right)$ is the similarity of M^-, M^+, then $E(A) = S\left(M^-, M^+\right)$.

Proof. (P1) If A is a crisp set, then for every $x \in X$, we have $A^- = A^+ = 1$ or $A^- = A^+ = 0$, it means that A^- and A^+ are crisp sets and $M^- = M^+ = 1$, therefore, $E(A) = S\left(M^-, \left(M^+\right)^C\right) = 0$.

(P2) Known by the definition of similarity measure of fuzzy sets, we have $E(A) = S\left(M^-, M^+\right) = 1 \Leftrightarrow M^- = M^+ \Leftrightarrow A^- + A^+ = 1$.

(P3) Since $A^- \le B^-$ and $A^+ \le B^+$ with $B^- + B^+ \le 1$ implies that

$$M_A^- \le M_B^- \le M_B^+ \le M_A^+.$$

From the definition of similarity measure of fuzzy sets, we have

$$S\left(M_A^-, M_A^+\right) \le S\left(M_A^-, M_B^+\right) \le S\left(M_B^-, M_B^+\right),$$

then we successfully prove $E(A) \le E(B)$.

With the same reason, when $A^- \ge B^-$ and $A^+ \ge B^+$ with $B^- + B^+ \ge 1$, we can prove

$$E(A) \le E(B).$$

(P4) Because $A^C = [\left(A^-\right)^C, \left(A^+\right)^C] = [1 - A^+, 1 - A^-]$, we have

$M_{A^C}^- = M^-, M_{A^C}^+ = M^+$. From the definition of entropy and similarity measure of FSs, we get

$$E(A) = S\left(M^-, M^+\right) = S\left(M^+, M^-\right) = S\left(M_{A^C}^-, M_{A^C}^+\right) = E\left(A^C\right)$$

Hence, we complete the proof of Theorem 1.

Remark 1 We define entropy of IVFS A by similarity measure of two fuzzy sets M^-, M^+ which are constructed by A. The construction is the extention of Zeng [26]. In Zeng's, the new FSs were A^-, A^+. While in ours, the new FSs are M^-, M^+ which are the functions of A^-, A^+ They both satisfy the definition of entropy of IVFSs.

Let $A = \{x \mid u_A(x)\}$, $B = \{x \mid u_B(x)\}$ be two FSs, and the Euclidean distance proposed by Deschrijver [10] is as follows:

$$d(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left|u_A(x_i) - u_B(x_i)\right|^2} \tag{2}$$

and the similarity measure of two fuzzy sets A;B is as follows:

$$S(A,B) = 1 - d(A,B) \tag{3}$$

Example 1. Let $A = [0.2, 0.4]$ and $B = [0.2, 0.4]$ be two IVFSs, now we utilize (1), (2), (3) to obtain the entropy of A,B.

For A, $M_A^- = 0.58$, $M_A^+ = 0.3$ and $S\left(M_A^-, M_A^+\right) = 0.72$, then we get $E\left(A\right) = 0.72$.

For B, $M_B^- = 0.52$, $M_B^+ = 0.4$ and $S\left(M_A^-, M_A^+\right) = 0.88$, then we get $E\left(B\right) = 0.88$.

We obtain $E\left(A\right) < E\left(B\right)$. For IVFSs, the longer the interval value is, the fuzzier the set is, so the result is consistent with our idea.

Next we develop another method to construct the entropy of interval valued fuzzy set which is on account of similarity measure of interval valued fuzzy sets.

For an IVFS A, for every x in X, we construct two IVFSs T_A, R_A and denote them $T_A = [T_A^-, T_A^+]$, $R_A = [R_A^-, R_A^+]$, where

$$T_A^- = 1 - \left(\frac{A^- + A^+}{2}\right)^2, \; T_A^+ = 1 - \frac{A^- + A^+}{2} \tag{4}$$

$$R_A^- = \left(\frac{A^- + A^+}{2}\right)^2, \; R_A^+ = \frac{A^- + A^+}{2} \tag{5}$$

Theorem2. Suppose $S\left(T_A, R_A\right)$ is the similarity of IVFSs T_A, R_A, and E is entropy of IVFS, then $E\left(A\right) = S\left(T_A, R_A\right)$.

Proof. (P1) If A is a crisp set, then for every $x \in X$, we have $A^- = A^+ = 0$ or $A^- = A^+ = 1$, it means that $T_A^- = T_A^+ = 1$, $R_A^- = R_A^+ = 0$ or $T_A^- = T_A^+ = 0$, $R_A^- = R_A^+ = 1$ we get $T_A = [1,1]$, $R_A = [0,0]$ or $T_A = [0,0]$, $R_A = [1,1]$, therefore, $E\left(A\right) = S\left(T_A, R_A\right) = 0$.

(P2) Known by the definition of similarity measure of fuzzy sets, we have

$$E\left(A\right) = S\left(T_A, R_A\right) = 1 \Leftrightarrow T_A = R_A \Leftrightarrow A^- + A^+ = 1.$$

(P3) Since $A^- \le B^-$ and $A^+ \le B^+$ with $B^- + B^+ \le 1$ which implies that

$$R_A \subseteq R_B \subseteq T_B \subseteq T_A.$$

From the definition of similarity measure of IVFSs, we have

$$S\left(T_A, R_A\right) \le S\left(T_B, R_A\right) \le S\left(T_B, R_B\right),$$

then we successfully prove $E\left(A\right) \le E\left(B\right)$.

With the same reason, when $A^- \ge B^-$ and $A^+ \ge B^+$ with $B^- + B^+ \ge 1$, we can prove

$$E\left(A\right) \le E\left(B\right).$$

(P4) From the definition of T_A, R_A, we get

$$T_A = T_{A^C}, R_A = R_{A^C} \text{ and } S\left(T_A, R_A\right) = S\left(T_{A^C}, R_{A^C}\right),$$

thus

$$E\left(A\right) = E\left(A^C\right).$$

Hence, we complete the proof of Theorem 2.

Remark 2 The construction of the two IVFSs is also based on the IVFS A. In our construction, it is more simple in form and more easier to calculate than Zeng [23]. It provides a new method to calculate the entropy of IVFS.

Let $A = \left[A^-, A^+\right]$, $B = \left[B^-, B^+\right]$ be two IVFSs, and the Euclidean distance of two IVFSs proposed by Atanassov [7] is as follows:

$$Q\left(A,B\right) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left(\left|A^-\left(x_i\right) - B^-\left(x_i\right)\right|^2 + \left|A^+\left(x_i\right) - B^+\left(x_i\right)\right|^2\right)} \tag{6}$$

and the similarity measure of two fuzzy sets A;B is as follows:

$$S\left(A,B\right) = 1 - Q\left(A,B\right) \tag{7}$$

Example 2. Let $A = [0.2, 0.4]$ and $B = [0.2, 0.4]$ be two IVFSs, now we utilize (4)-- (7) to obtain the entropy of A, B.

For A, $T_A^- = 0.91, T_A^+ = 0.7$ and $R_A^- = 0.09$, $R_A^+ = 0.3$ then we get $S\left(T_A, R_A\right) = 0.2512$, $E\left(A\right) = 0.2512$.

For B, $T_B^- = 0.84, T_B^+ = 0.6$ and $R_B^- = 0.16$, $R_B^+ = 0.4$, then we get

$$S\left(T_B, R_B\right) = 0.4988, \; E\left(B\right) = 0.4988.$$

We obtain $E\left(A\right) < E\left(B\right)$, the result is consistent with example 1.

CONCLUSION

In this paper, we comment on the axiomatic definitions of entropy of IVFSs and proposed a new method to construct new FSs and IVFSs based on the IVFSs which are more simple in form and more easier to calculate. Furthermore, we obtained the entropy of IVFSs by the similarity measure of two FSs and IVFSs, repectively. Then we gave two examples to show that our methods are effective. The new methods developed and enriched the entropy of FSs and IVFSs.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

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