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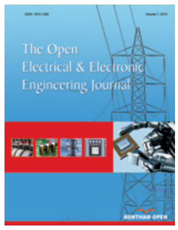

RESEARCH ARTICLE

Stability Analysis for Stochastic Markovian Jumping Neural Networks with Leakage Delay

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Abstract: The stability problem for a class of stochastic neural networks with Markovian jump parameters and leakage delay is addressed in this study. The sufficient condition to ensure an exponentially stable stochastic neural networks system is presented and proven with Lyapunov functional theory, stochastic stability technique and linear matrix inequality method. The effect of leakage delay on the stability of the neural networks system is discussed and numerical examples are provided to show the correctness and effectiveness of the research results .

Keywords: Markovian jumping, Exponentially stable, Linear matrix inequality (LMI), Neural networks, Time-varying delay, Leakage delay.

1. INTRODUCTION

In the past decades, neural networks systems have elicited much attention because of its massive potential application in many fields, such as pattern classification, reconstruction of moving images, and combinatorial optimization, *etc*. In the real applications, ensuring the stability of the equilibrium point of the designed neural networks is an important task and has been a popular topic. Time delay and stochastic disturbance are well known as the two main factors that affect the stability of neural networks. Many new results have been obtained from stability analyses of stochastic neural networks with different types of time delays [1 - 8].

Recently, a special delay called leakage or forgetting delay has been investigated widely since its existence in real systems was discovered. This special delay affects the stability of time delay systems. Many exciting research results have been reported recently [9 - 13]. As pointed out in [9, 10], neural networks with leakage delay is a class of important networks. In [10], the global exponential stability of complex-valued neural networks with leakage delay was investigated with complex-valued linear matrix inequality technique, by establishing a novel stability lemma . Then, by using stochastic analysis theory and matrix inequalities technique, the exponential stability of a kind of stochastic neural networks with leakage terms are studied in [11]. The global μ-stability results for complex-valued neural networks with leakage time delay and unbounded time-varying delays were obtained in [12] through the free weighting matrix method and stability theory. By using the properties of M-matrix, the properties of the fuzzy logic operator, the eigenvalue of the spectral radius of nonnegative matrices and delay differential inequality, a class of fuzzy cellular neural networks with time delay in the leakage term and impulsive perturbations was investigated in [13]. However, the leakage delay studied above was constant. Subsequently, the research on leakage delay was extended to time-varying [14, 15]. In [14], with Lyapunov method, a triple Lyapunov-Krasovskii functional term was employed to study the robust stability of discrete-time uncertain neural networks with leakage time-varying delay.

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Stochastic model plays an important role in the branches of economics, industry and science. A particular field of interest is stochastic system with Markovian jumping parameters. The Markovian jump system shows an advantage in modeling these dynamic systems presented above, and much progress has been made in stability analysis, impulsive response and state estimation of stochastic neural networks with Markovian jumping parameters, [16 - 23] and references therein.

Motivated by the discussion above, this study investigates the exponential stability of a class of stochastic neural networks with time-varying and leakage delays. By employing a suitable Lyapunov functional and introducing a new inequality technique, the sufficient condition to render the system exponentially stable is obtained by solving a set of strict linear matrix inequalities. Examples and simulations are presented to show the effectiveness of the proposed methods. The mutual effect between discrete and leakage delays as well as the derivative of time-delay are discussed. The experimental analysis reveals that the effect of leakage delay existing in neural networks on stability can not be disregarded.

2. PROBLEM FORMULATION

Let r_t, $t \geq 0$ be a right-continuous Markov chain defined on a complete probability space (Ω,F,P) and take discrete values in a finite state space $S = \{1, 2, ..., N\}$ with generator given by $\Pi = (\pi_{ij})N\times N$:

$$P\{r(t+\Delta) = j|r(t)=i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta), & i \neq j \\ 1+\pi_{ij}\Delta + o(\Delta), & i=j. \end{cases} \tag{1}$$

where $\Delta > 0$, $\pi_{ij} \geq 0$ is the transition rate from i to j while $\pi_{ii} = -\sum_{j\neq i}\pi_{ij}$.

Consider the following stochastic neural network with time-varying delay:

$$\begin{cases} dx(t) = [-A(r_t)x(t-\delta(t)) + W_0(r_t)f(x(t)) \\ \qquad + W_1(r_t)f(x(t-\tau(t))]\ dt \\ \qquad + [C(r_t)x(t)+D(r_t)x(t-\tau(t)) \\ \qquad + W_2(r_t)f(x(t)) + W_3(r_t)f(x(t-\tau(t)))]d\omega(t) \\ x(t) = \zeta(t),\ \forall t \in [-\rho,0], \end{cases} \tag{2}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector of the neural network associated with n neurons, $\omega(t)$ is an m-dimensional Brownian motion defined on a probability space (Ω,F,P), which is assumed to satisfy $E\{d\omega(t)\} = 0$, $E\{d\omega^2(t)\} = dt$. $A(r_t) = diag\{a_1,...,a_n\}$ is a diagonal matrix with positive entries $a_i > 0$ $(i = 1, 2,..., n)$, $f(x(t)) = [f(x_1(t)),..., f(x_n(t))]^T \in \mathbb{R}^n$ denotes the neuron activation function and $W(r_t)$, $W_1(r_t)$, $W_2(r_t)$, $W_3(r_t)$ are the connection weight matrices and the delayed connection weight matrices, respectively. $C(r_t)$ and $D(r_t)$ are known real constant matrices with compatible dimensions. The $\delta(t)$, $\tau(t)$ denotes leakage term and the transmission delay, respectively, which satisfying:

$$0 \leq \delta(t) \leq \delta,\ 0 \leq \tau(t) \leq \tau,\ \dot{\tau}(t) \leq \mu < 1, \\ \dot{\delta}(t) \leq \rho_\delta < 1 \tag{3}$$

$\rho = max\{\delta, \tau\}$ and μ is some positive scalar. $\zeta(t)$ is real-valued continuous initial condition on $[-\rho, 0]$. Throughout the paper, we assume that $\omega(t)$ and $r(t)$ are independent.

For simplicity, in the sequel, for each $r_t = i \in S$, $A(r_t)$, $W(r_t)$, $W_1(r_t)$ are denoted by A_i, W_i, W_{1i} and so on. Therefore, the system (2) can be rewritten as:

$$\begin{cases} dx(t) = [-A_i x(t-\delta(t)) + W_{0i}f(x(t)) \\ \qquad + W_{1i}f(x(t-\tau(t)))]\ dt + [C_i x(t) \\ \qquad + D_i x(t-\tau(t)) + W_{2i}f(x(t)) \\ \qquad + W_{3i}f(x(t-\tau(t)))]d\omega(t) \\ x(t) = \zeta(t),\ \forall t \in [-\rho,0], \end{cases} \tag{4}$$

Assumption 1. For $i \in \{1,2,..., n\}$, $\forall$ x, y, $\in$, $\mathbb{R}$, $x \neq y$, the neuron activation function $f_i(\cdot)$ is continuous, bounded and satisfies:

$$[\mathrm{f}(x)-\mathrm{f}(y)-\Sigma_1(x-y)]^{\mathrm{T}} \times [\mathrm{f}(x)-\mathrm{f}(y)-\Sigma_2(x-y)]\ 0< \tag{5}$$

where Σ_1 and Σ_2 are some constant matrices.

Remark 1. In this study, Assumption 1 is based on neuron activation function, which is called sector-bounded neuron activation function [12]. As pointed out in [12], when $\Sigma_1 = \Sigma_2 = -\Sigma$, the condition (5) becomes:

$$[\mathrm{f}(x)-\mathrm{f}(y)]^{T} \times [\mathrm{f}(x)-\mathrm{f}(y)] \leq (x-y)^{\mathrm{T}}\, \Sigma^{\mathrm{T}}\, \Sigma(x-y) \tag{6}$$

The condition is less restrictive than the descriptions on both the sigmond activation functions [9, 10] and the Lipschitz-type activation functions.

At first, we derive the following Lemmas which will be used frequently in the proof of our main results.

Lemma 1 [4]. For any constant symmetric positive defined matrix $J \in R^{m\times m}$, scalar η and the vector function ν : [0, η]→ $\mathbb{R}^{m}$, the following inequality holds:

$$\eta\int_0^{\eta} \nu^{\mathrm{T}}(s)\mathrm{J}\nu(s)\mathrm{d}s \geq (\int_0^{\eta}\nu(s)\mathrm{d}s)^{\mathrm{T}}\,\mathrm{J}(\int_0^{\eta}\nu(s)\mathrm{d}s).$$

Lemma 2 [24]. Assume that x(t), φ(t), g(t) satisfy the stochastic differential equation

$$\mathrm{d}x(\mathrm{t})=\varphi(\mathrm{t})\mathrm{dt}+\mathrm{g}(\mathrm{t})\mathrm{d}\omega(\mathrm{t}), \tag{7}$$

where ω(t) is a Brownian motion. For any constant matrix $Z \geq 0$ and scalar $h > 0$, the following integration holds:

$$\begin{aligned} &-\mathrm{h}\int_{\mathrm{t-h}}^{\mathrm{t}} \varphi^{T}(\mathrm{s})Z\varphi(\mathrm{s})\mathrm{ds} \\ &\leq \begin{bmatrix} x(\mathrm{t}) \\ x(\mathrm{t-h}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -Z & Z \\ Z & -Z \end{bmatrix} \begin{bmatrix} x(\mathrm{t}) \\ x(\mathrm{t-h}) \end{bmatrix} \\ &+ \begin{bmatrix} x(\mathrm{t}) \\ x(\mathrm{t-h}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Z \\ -Z \end{bmatrix} \int_{\mathrm{t-h}}^{\mathrm{t}} g(\mathrm{s})\mathrm{d}\omega(\mathrm{t}) \end{aligned} \tag{8}$$

Definition 1. The stochastic neural networks system (2) is said to be exponential stable in the mean square sense, if there exist positive scalars α, β such that:

$$\mathrm{E}\{\| x(\psi,\mathrm{r}(0)\|^2\} \leq \alpha e^{-\beta\mathrm{t}}\mathrm{E}\{\sup_{-\tau\leq s\leq 0} \|\psi(s)\|^2\}. \tag{9}$$

3. MAIN RESULTS

In this section, the exponential stability of system (4) is developed by Theorem 1.

Theorem 1. For given positive scalars δ, μ, ρ_σ and τ, the Markovian jumping stochastic neural network system (4) is exponential mean square stable, if there exist symmetric positive definite matrices P_i, $i \in S$, $R_k(k = 1,2,3,4,5)$ and two diagonal matrices $F_1 > 0$, $F_2 > 0$ such that the following linear matrix inequality (LMI) holds:

$$\Pi = \begin{bmatrix} \Pi^{11} & \Pi^{12} & \Pi^{13} & \Pi^{14} \\ * & -P_i & 0 & 0 \\ * & * & -R_5 & 0 \\ * & * & * & Q \end{bmatrix} < 0, \tag{10}$$

where

$$\Pi^{11} = \begin{bmatrix} \Pi_{11} & 0 & \Pi_{13} & R_5 & \Pi_{15} & P_i W_{1i} \\ * & \Pi_{22} & \Pi_{23} & 0 & 0 & 0 \\ * & * & \Pi_{33} & 0 & \Pi_{34} & -A_i^T P_i W_{1i} \\ * & * & * & \Pi_{44} & 0 & -\lambda_{2i} F_2 \\ * & * & * & * & \Pi_{55} & 0 \\ * & * & * & * & * & \Pi_{66} \end{bmatrix},$$

$$\begin{aligned} &\Pi^{12} = [\tau P_i C_i \quad 0 \quad 0 \quad \tau P_i D_i \quad \tau P_i W_{2i} \quad \tau P_i W_{3i}]^T, \\ &\Pi^{13} = [0- \quad R_5 A_i \quad 0 \quad 0 \quad R_5 W_{0i} \quad R_5 W_{1i}]^T, \\ &\Pi^{14} = [A_i^T P_i \sqrt{\rho_\delta} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, \end{aligned}$$

$$\begin{aligned} {}_{11} &= -P_i A_i - P_i A_i^T + R_1 + R_2 + \delta^2 R_3 \\ &\quad - R_5 + \sum_{j=1}^{N} \pi_{ij} P_j - \lambda_{1i} F_1, \\ I_{13} &= -\sum_{j=1}^{N} \pi_{ij} P_j + A_i^T P_i A_i, \\ \Pi_{15} &= P_i W_{0i} - \lambda_{1i} F_2, \\ \Pi_{35} &= -A_i^T P_i W_{0i}, \\ \Pi_{55} &= -\lambda_{1i} + R_4, \\ &= -\lambda_{2i} - (1-\mu) R_4, \\ \Pi_{22} &= Q\rho_\delta - (1-\rho_\delta) R_1, \\ \Pi_{23} &= A_i^T P_i A_i \rho_\delta, \\ \Pi_{33} &= -R_3 + \sum_{j=1}^{N} P \pi_{ij} A_i^T {}_i A_i, \\ \Pi_{44} &= -(1-\mu) R_2 - R_5 - \lambda_{2i} F_1. \end{aligned}$$

Proof: For simplicity, we let:

$$\begin{aligned} \varphi(t) = &-A_i x(t-\delta(t)) + W_{0i} f(x(t)) \\ &+ W_{1i} f(x(t-\tau(t))) \end{aligned} \tag{11}$$

$$\begin{aligned} g(t) = &C_i x(t) + D_i x(t-\tau(t)) + W_{2i} f(x(t)) \\ &+ W_{3i} f(x(t-\tau(t))) \end{aligned} \tag{12}$$

Then the system (4) can be rewritten as:

$$dx(t) = \varphi(t)dt + g(t)d\omega(t) \tag{13}$$

We choose the following Lyapunov-Krasovskii functional candidate as:

$$V(x(t),t) = \sum_{N=1}^{6} V_N(x(t),t) \tag{14}$$

where

$$V_1(x(t),t,i) = [x(t) - A_i\int_{t-\delta(t)}^{t} x(s)ds]^T \times P[x(t) - A_i\int_{t-\delta(t)}^{t} x(s)ds],$$

$$V_2(x(t),t,i) = \int_{t-\delta(t)}^{t} x^T(s)R_1x(s)ds,$$

$$V_3(x(t),t,i) = \int_{t-\tau(t)}^{t} x^T(s)R_2x(s)ds\ ,$$

$$V_4(x(t),t,i) = \delta\int_{-\delta}^{0}\int_{t+\theta}^{t} x^T(s)R_3x^T(s)dsd\theta,$$

$$V_5(x(t),t,i) = \int_{t-\tau(t)}^{t} f^T(x(s))R_4f(x(s))ds,$$

$$V_6(x(t),t,i) = \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} \varphi^T(s)R_5\varphi^T(s)dsd\theta.$$

Following Itô differential rule, the stochastic differential of $dV(x(t), t, i)$ with respect to t along the system (4) is obtained by:

$$dV(x(t),t,i) = \sum_{N=1}^{6} \mathcal{L}V_N(x(t),t)dt + 2x^T(t)P_ig(t)d\omega(t), \tag{15}$$

where

$$\begin{aligned}
&\mathcal{L}V_1(x(t),t,i)\\
&=2\left[x(t)-A_i\int_{t-\delta(t)}^{t}x(s)ds\right]^T P_i\bar{\varphi}(t)\\
&\quad+g^T(t)P_ig(t)\\
&\quad+\sum_{j=1}^{N}\pi_{ij}\left[x(t)-A_i\int_{t-\delta(t)}^{t}x(s)ds\right]^T\\
&\quad\times P_j\left[x(t)-A_i\int_{t-\delta(t)}^{t}x(s)ds\right]\\
&\le -2x^T(t)P_iA_ix(t)+2x^T(t)P_iW_{0i}f(t)\\
&\quad+2x^T(t)P_iW_{1i}f(t-\tau(t))\\
&\quad+2\left[\int_{t-\delta(t)}^{t}x(s)ds\right]^T A_i^TP_iA_ix(t)\\
&\quad-2\left[\int_{t-\delta(t)}^{t}x(s)ds\right]^T A_i^TP_iW_{0i}f(t)\\
&\quad-2\left[\int_{t-\delta(t)}^{t}x(s)ds\right]^T A_i^TP_iW_{1i}f(t-\tau(t))\\
&\quad+2\left[\int_{t-\delta(t)}^{t}x(s)ds\right]^T A_i^TP_iA_ix(t-\delta(t))\rho_\delta\\
&\quad+x^T(t)P_iA_iQ^{-1}A_i^TP_ix(t)\rho_\delta\\
&\quad+x^T(t-\delta(t))Qx(t-\delta(t))\rho_\delta+g^T(t)P_ig(t)
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\bar{\varphi}(t) = -A_i x(t) - \rho_\delta A_i x(t-\delta(t)) \\
&\quad + W_{0i} f(t) + W_{1i} f(t-\tau(t)) \\
&-2x^T(t) P_i A_i x^T(t-\delta(t)) \overset{g}{\delta}(t) \\
&\quad \le x^T(t) \quad {}^{-1} \quad {}^{T} \quad x(t)\rho_\delta \\
&+ x^T(t-\delta(t)) Q x(t-\delta(t)) \rho_\delta ,
\end{aligned},$$

$$\begin{aligned}
\mathcal{L}V_2(x(t),t,i) &= x^T(t) R_1(x(t)) \\
&\quad -(1\text{-}\rho_\delta) x^T(t-\delta(t)) R_1 x(t-\delta(t))
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\mathcal{L}V_3(x(t),t,i) \\
&= x^T(t) R_2(x(t)) - (1 - \&(t)) x^T(t-\tau(t)) \\
&\times R_2 x(t-\tau(t)) \\
&\le x^T(t) R_2(x(t)) \\
&\quad -(1-\mu) x^T(t-\tau(t)) R_2 x(t-\tau(t))
\end{aligned} \tag{18}$$

$$\begin{aligned}
\mathcal{L}V_4(x(t),t,i) &\le \delta^2 x^T(t) R_3 x(t) \\
&\quad \delta \int_{t\text{-}\delta(t)}^{t} x^T(s) R_3 x(s) ds
\end{aligned} \tag{19}$$

$$\begin{aligned}
\mathcal{L}V_5(x(t),t,i) &\le f^T(x(t)) R_4 f(x(t)) \\
&\quad (1-\mu) f^T(x(t-\tau(t))) \\
&\quad \times R_4 f(x(t-\tau(t)))
\end{aligned} \tag{20}$$

$$\begin{aligned}
\mathcal{L}V_6(x(t),t,i) &\le \tau^2 \varphi^T(t) R_5 \varphi(t) \\
&\quad -\tau \int_{t\text{-}\tau}^{t} \varphi^T(s) R_5 \varphi(s) ds
\end{aligned} \tag{21}$$

Applying Lemma 1 to (19) and Lemma 2 to (21), we can obtain:

$$\begin{aligned}
&-\delta \int_{t\text{-}\delta(t)}^{t} x^T(s) R_3 x(s) ds \\
&\quad \le -\int_{t\text{-}\delta(t)}^{t} x^T(s) ds R_3 \int_{t\text{-}\delta(t)}^{t} x^T(s) ds
\end{aligned} \tag{22}$$

$$\begin{aligned}
&-\tau \int_{t\text{-}\tau}^{t} \varphi^T(s) R_5 \varphi(s) ds \\
&\le -\tau(t) \int_{t\text{-}\tau(t)}^{t} \varphi^T Z \varphi(s) ds \\
&\le \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix}^T \begin{bmatrix} -R_5 & R_5 \\ R_5 & -R_5 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix}. \\
&+ \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix}^T \begin{bmatrix} R_5 \\ -R_5 \end{bmatrix} \int_{t\text{-}\tau(t)}^{t} g(s) d\omega(t)
\end{aligned} \tag{23}$$

On the other hand, it can be deduced from Assumption 1 that for $i = 1,2,..., n$,

$$\begin{aligned}&\left(f_i\left(x_i(t)\right)-\Sigma_1 x_i(t)\right)^T\left(f_i\left(x_i(t)\right)-\Sigma_2 x_i(t)\right)\le 0,\\&\left(f_i\left(x_i\left(t-\tau(t)\right)\right)-\Sigma_1 x_i\left(t-\tau(t)\right)\right)^T(f_i\left(x_i\left(t-\tau(t)\right)\right)\\&-\Sigma_2 x_i\left(t-\tau(t)\right))\le 0,\end{aligned} \tag{24}$$

Then there exist scalars $\lambda_{1i} > 0$, $\lambda_{2i} > 0$, diagonal matrices $F_1 \geq 0$, $F_2 \geq 0$ and $\Sigma_i(i = 1, 2)$ such that the following inequalities hold:

$$0\le-\lambda_{1i}\begin{bmatrix}x(t)\\f\left(x(t)\right)\end{bmatrix}^T\begin{bmatrix}F_1 & F_2\\ * & I\end{bmatrix}\begin{bmatrix}x(t)\\f\left(x(t)\right)\end{bmatrix} \tag{25}$$

$$\begin{aligned}0\le-\lambda_{2i}&\begin{bmatrix}x\left(t-\tau(t)\right)\\f\left(x(t)-\tau(t)\right)\end{bmatrix}^T\begin{bmatrix}F_1 & F_2\\ * & I\end{bmatrix}\\&\times\begin{bmatrix}x\left(t-\tau(t)\right)\\f\left(x\left(t-\tau(t)\right)\right)\end{bmatrix}\end{aligned} \tag{26}$$

where

$$F_1=\frac{\Sigma_1^T\Sigma_1+\Sigma_2^T\Sigma_1}{2},\ F_2=-\frac{\Sigma_1^T+\Sigma_2^T}{2} \tag{27}$$

By substituting (16)-(23) into (15), and adding the right sides of (25)-(26) to the right side of (15), we can obtain:

$$\begin{aligned}\mathcal{L}V\left(x(t),t,i\right)&=\sum_{N=1}^{6}\mathcal{L}V_N\left(x(t),t,i\right)dt\\&\le\xi^T(t)\Pi\xi(t)+\theta(t)\end{aligned} \tag{28}$$

where $\Pi = \Pi^{11} + \Pi^{12} P_i^{-1} (\Pi^{12})^T + \Pi^{13} R_5^{-1} (\Pi^{13})^T$,

$$\begin{aligned}\xi^T(t)=[&x^T(t)\ \ x^T\left(t-\delta(t)\right)\ \ \int_{t-\delta(t)}^{t}x^T(s)ds\\&x^T\left(t-\tau(t)\right)\ \ f^T\left(x(t)\right)\ \ f^T\left(x\left(t-\tau(t)\right)\right)]\end{aligned}.$$

$$\theta(t)=\begin{bmatrix}x(t)\\x\left(t-\tau(t)\right)\end{bmatrix}^T\begin{bmatrix}R_5\\-R_5\end{bmatrix}\int_{t-\tau(t)}^{t}g(s)d\omega(t).$$

By taking expectations on both sides of (28), we can obtain:

$$E\left\{\mathcal{L}V_i\left(x(t),t,i\right)\right\}\le-\beta E\left\{x^T(t)x(t)\right\} \tag{29}$$

where

$$\beta=\min_{i\in S}\left\{\lambda_{min}\left(-\Pi\right)\right\}..$$

Then, by taking expectations on both side of (16) and integrating from 0 to T, we can obtain:

$$\begin{aligned}&E\left\{V\left(x(t),T,i\right)\right\}-V\left(x(0),0,r_0\right)\\&\le-\beta\int_0^T E\left\{x^T(t)x(t)\right\}dt\end{aligned} \tag{30}$$

At the same time, it follows from (14) that:

$$E\left\{V\left(x(t),t,i\right)\right\}\ge bE\left\{x(t)^T x(t)\right\} \tag{31}$$

where

$$\mathrm{b} = \min_{\mathrm{i} \in \mathrm{S}} \left\{ \lambda_{\min} \left(\mathrm{P_i} \right) \right\} > 0 \quad .$$

Therefore, from (30)-(31), the following inequality can be obtained:

$$\begin{aligned} \mathrm{E}\left\{ \left(x(\mathrm{T})^{\mathrm{T}} x(\mathrm{T}) \right) \right\} &\le \mathrm{b}^{-1} \mathrm{V}\left(x(0), 0, \mathrm{r}_0 \right) \\ &\quad - \mathrm{b}^{-1} \beta \int_0^{\mathrm{T}} \mathrm{E}\left\{ x^{\mathrm{T}}(\mathrm{t}) x(\mathrm{t}) \right\} \mathrm{dt} \end{aligned} \tag{32}$$

Applying Gronwall-Bellman lemma to the inequality (30), we can obtain:

$$\mathrm{E}\left\{ \left(x(\mathrm{T})^{\mathrm{T}} x(\mathrm{T}) \right) \right\} \le \mathrm{b}^{-1} \mathrm{V}\left(x(0), 0, \mathrm{r}_0 \right) \exp\left(-\beta \mathrm{b}^{-1} \mathrm{T} \right)$$

Noting that there exists a scalar $\alpha > 0$ such that:

$$\mathrm{b}^{-1} \mathrm{V}\left(x(0), 0, \mathrm{r}_0 \right) \le \alpha \sup_{-\mathrm{p} \le \theta \le 0} \left| x(\theta) \right|^2 \tag{33}$$

Then by (9), we can draw the conclusion that the system (4) is exponentially mean square stable, thus completing the proof.

When leakage delay is constant, that is, $\delta(\mathrm{t}) = \delta$, the system (4) can be rewritten as follows:

$$\begin{cases} \mathrm{d}x(\mathrm{t}) = [-\mathrm{A_i} x(\mathrm{t}-\delta) + \mathrm{W_{0i}} \mathrm{f}(x(\mathrm{t})) \\ \qquad + \mathrm{W_{1i}} \mathrm{f}(x(\mathrm{t}-\tau(\mathrm{t})))]\, \mathrm{dt} \\ \qquad + [\mathrm{C_i} x(\mathrm{t}) + \mathrm{D_i} x(\mathrm{t}-\tau(\mathrm{t})) \\ \qquad + \mathrm{W_{2i}} \mathrm{f}(x(\mathrm{t})) + \mathrm{W_{3i}} \mathrm{f}(x(\mathrm{t}-\tau(\mathrm{t})))] \mathrm{d}\omega(\mathrm{t}) \\ x(\mathrm{t}) = \zeta(\mathrm{t}), \ \forall \mathrm{t} \in [-\rho, 0], \end{cases} \tag{34}$$

Then we can get the following Theorem 2:

Theorem 2. For given positive scalars δ, μ and τ, the Markovian jumping stochastic neural network system (34) is exponential mean square stable, if there exist symmetric positive definite matrices P_i, R_j(j = 1,2,3,4,5) and two diagonal matrices $F_1 > 0$, $F_2 > 0$ such that the following linear matrix inequality (LMI) holds:

$$\Pi = \begin{bmatrix} \Pi^{11} & \Pi^{12} & \Pi^{13} \\ * & \mathrm{P_i} & 0 \\ * & * & -\mathrm{R_5} \end{bmatrix} < 0, \qquad ,$$

where

$$\Pi^{11} = \begin{bmatrix} \Pi_{11} & 0 & \Pi_{13} & \mathrm{R_5} & \Pi_{15} & \mathrm{P_i W_{1i}} \\ * & \Pi_{22} & 0 & 0 & 0 & 0 \\ * & * & \Pi_{33} & 0 & \Pi_{34} & -\mathrm{A_i}^{\mathrm{T}} \mathrm{P_i W_{1i}} \\ * & * & * & \Pi_{44} & 0 & -\lambda_{2i} \mathrm{F_2} \\ * & * & * & * & \Pi_{55} & 0 \\ * & * & * & * & * & \Pi_{66} \end{bmatrix},$$

$$\begin{aligned} \Pi^{12} &= [\tau \mathrm{P_i C_i} \quad 0 \quad 0 \quad \tau \mathrm{P_i D_i} \quad \tau \mathrm{P_i W_{2i}} \quad \tau \mathrm{P_i W_{3i}}]^{\mathrm{T}}, \\ \Pi^{13} &= [0 - \quad \mathrm{R_5 A_i} \quad 0 \quad 0 \quad \mathrm{R_5 W_{0i}} \quad \mathrm{R_5 W_{1i}}]^{\mathrm{T}}, \end{aligned}$$

$$\begin{aligned} \Pi_{11} &= -\mathrm{P_i A_i} - \mathrm{P_i A_i}^{\mathrm{T}} + \mathrm{R_1} + \mathrm{R_2} + \delta^2 \mathrm{R_3} \\ &\quad - \mathrm{R_5} + \sum_{j=1}^{N} \pi_{ij} \mathrm{P_j} - \lambda_{1i} \mathrm{F_1} \qquad , \\ \mathrm{I}_{13} &= -\sum_{j=1}^{N} \pi_{ij} \mathrm{P_j} + \mathrm{A_i}^{\mathrm{T}} \mathrm{P_i A_i}, \\ \Pi_{15} &= \mathrm{P_i W_{0i}} - \lambda_{1i} \mathrm{F_2}, \\ \Pi_{22} &= \ \mathrm{R_1}, \\ \Pi_{33} &= -\mathrm{R_3} + \sum \mathrm{P}_{j=1}^{N} \pi_{ij} \mathrm{A_i}^{\mathrm{T}} \ {}_{i} \mathrm{A_i}, \\ \Pi_{35} &= -\mathrm{A_i}^{\mathrm{T}} \mathrm{P_i W_{0i}}, \\ \Pi_{44} &= -(1-\mu) \mathrm{R_2} - \mathrm{R_5} - \lambda_{2i} \mathrm{F_1} \\ \Pi_{55} &= -\lambda_{1i} \ + \mathrm{R_4}, \\ \Pi_{66} &= -\lambda_{2i} - (1-\mu) \mathrm{R_4}, \end{aligned} \qquad ,$$

4. NUMERICAL EXAMPLES

In this section, two numerical examples with simulation results are provided to demonstrate the effectiveness of the proposed approaches.

Example 1. Consider a two-neuron stochastic neural networks system (4) with the following parameters:

Mode 1

$$A_1=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},\ W_{01}=\begin{bmatrix}0.2 & 0.3\\ 0.1 & 0.5\end{bmatrix}\ W_{11}=\begin{bmatrix}0.3 & 0.2\\ -0.2 & -0.3\end{bmatrix},\ W_{21}=\begin{bmatrix}0.2 & -0.4\\ 0.1 & 0.3\end{bmatrix},W_{31}=\begin{bmatrix}0.1 & -0.3\\ -0.2 & 0.2\end{bmatrix},\ C_1=\begin{bmatrix}0.4 & -0.1\\ 0.2 & -0.2\end{bmatrix}\ D_1=\begin{bmatrix}0.2 & -0.3\\ 0.2 & 0.2\end{bmatrix}.$$

Mode 2

$$A_2=\begin{bmatrix}2 & 0\\ 0 & 1.3\end{bmatrix},\ W_{02}=\begin{bmatrix}0.4 & 0.5\\ 0.3 & 0.2\end{bmatrix},\ W_{12}=\begin{bmatrix}0.1 & 0.3\\ -0.2 & -0.2\end{bmatrix},\ W_{22}=\begin{bmatrix}-0.2 & 0.2\\ -0.3 & 0.1\end{bmatrix},\ W_{32}=\begin{bmatrix}-0.3 & 0.1\\ -0.2 & 0.4\end{bmatrix},\ C_2=\begin{bmatrix}0.2 & -0.3\\ 0.2 & 0.2\end{bmatrix}\ D_2=\begin{bmatrix}0.3 & 0.1\\ 0.2 & -0.3\end{bmatrix}.$$

Assume that the Markov process governing the mode switching generates:

$$\Pi=\begin{bmatrix}-0.3 & 0.3\\ 0.7 & -0.7\end{bmatrix}$$

Take the neuron activation functions as follows: $f(x) = 0.5(|x + 1| - |x - 1|)$.

Following Assumption 1, we obtain, $F_1 = diag\ \{0, 0\}$, $F_2 = diag\ \{0.05, 0.05\}$. In this example, we set $\mu = 0.3$, $\tau = 0.8$ and $\delta = 0.26$, by virtue of the Matlab LMI Control box, solving the LMI (10), the feasible solution can be obtained as follows:

$$P_1=\begin{bmatrix}159.8315 & -38.4769\\ -38.4769 & 189.4208\end{bmatrix},\ P_2=\begin{bmatrix}4.0602 & 7.6295\\ 7.6295 & 20.9717\end{bmatrix}\ R_1=\begin{bmatrix}0.0938 & -0.0084\\ -0.0084 & 0.1335\end{bmatrix},\ R_2=\begin{bmatrix}0.1858 & -0.0037\\ -0.0037 & 0.1539\end{bmatrix}\ R_3=\begin{bmatrix}0.1205 & -0.0186\\ -0.0186 & 189.4208\end{bmatrix},R_4=\begin{bmatrix}0.0938 & -0.0084\\ -0.0084 & 0.1335\end{bmatrix}$$

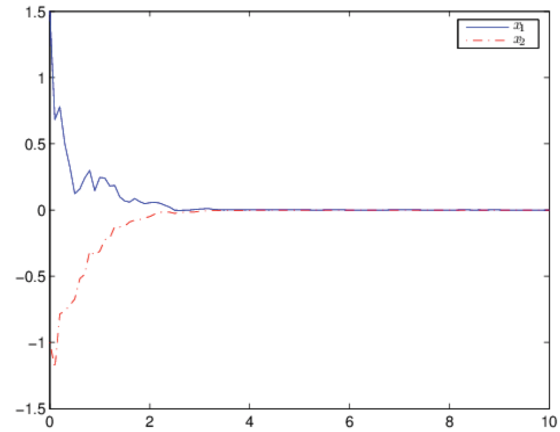

Fig. (1). Response of the state x(t).

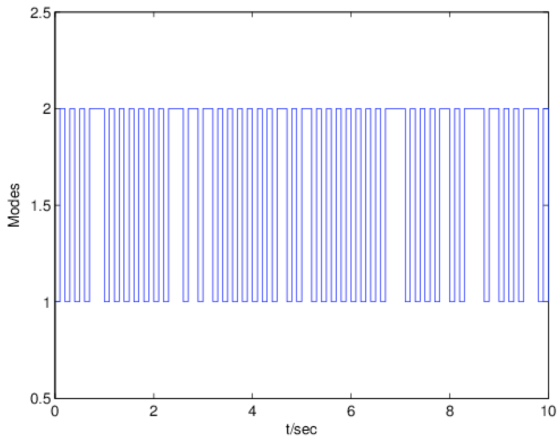

Fig. (2). Markovian jumping mode of x(t).

The simulation results of the state response of system (4) are plotted in Figs. (**1**) and (**2**), where the initial condition $x(0) = [1.5; -1]$, Fig. (**1**) shows the state response of system (4), and Fig. (**2**) depicts the switching modes. The figures illustrate that when two Markov processes are under control, the stochastic neural networks system with leakage delay is stable.

The upper bounds of delays ρσ, δ and τ guaranteeing the stability of system (4) are listed from Tables **1** to **3**, where - signifies that LMI (10) exhibits no feasible solution. Table **1** shows the maximum allowable upper bound δ for different values of ρσ, which indicating that the bound of the derivative of the leakage time-varying is effective and plays an important role in obtaining feasible results.

Table 1. Allowable upper bounds of δ with different value of ρ_σ, $\mu = 0.1$, and $\tau = 0.5$.

ρσ	0	0.02	0.06	0.1	0.2	0.3	0.4
δ	0.3619	0.2970	0.2408	0.1991	0.1176	0.0519	-

Table 2. Allowable upper bounds of τ for different value of δ, $\gamma = 0.1$, and $\tau\delta = 0.01$, $\mu = 0.5$.

δ	0	0.05	0.1	0.15	0.2
τ	0.3867	0.2887	0.2415	0.1292	-

Table **2** indicates that when fixing the value of ε, ρσ and γ, the allowable upper value of τ is effected by δ, especially when $\delta = 0.2$, the feasible solution cannot be obtained.

When ρ_σ is a non-zero constant, the allowable upper bounds of τ for different values of ε are listed in Table **3**.

Table 3. Minimum allowable bounds of τ for different values of ρ_σ and $\delta = 0.2$.

μ	0.1	0.3	0.5	0.9	0.95
τ	0.4778	0.4174	0.3254	0.2727	0.2727

Example 2. Consider a three-neuron two-mode stochastic neural networks with Markovian jump parameters and mixed time delays (2) with the following parameters:

Mode 1

$$A_1=\begin{bmatrix}2&0&0\\0&3.2&0\\0&0&4.3\end{bmatrix},\ W_{01}=\begin{bmatrix}0.2&0.3&0.2\\0.1&0.5&0.4\\0.2&0.1&0.3\end{bmatrix},\ W_{11}=\begin{bmatrix}0.1&-0.1&0.4\\0.2&0.2&0.3\\0.5&0.4&-0.3\end{bmatrix},\ W_{21}=\begin{bmatrix}0.1&0.5&0.2\\0.3&0.25&0.12\\0.2&0.1&0.22\end{bmatrix},\ W_{31}=\begin{bmatrix}0.3&0.22&0.12\\0.2&0.5&0.4\\0.1&0.4&0.3\end{bmatrix},\ C_1=\begin{bmatrix}0.2&0.3&0.1\\0.2&0.4&0.5\\0.5&0.3&0.1\end{bmatrix},\ D_1=\begin{bmatrix}0.4&0.2&0.1\\0.2&0.2&0.8\\0.1&0.3&0.2\end{bmatrix}$$

Mode 2

$$A_2=\begin{bmatrix}3&0&0\\0&1.2&0\\0&0&3.2\end{bmatrix},\ W_{02}=\begin{bmatrix}-0.4&0.2&0.2\\-0.3&0.5&0.2\\-0.3&0.2&0.2\end{bmatrix},\ W_{12}=\begin{bmatrix}3&0&0\\0&1.2&0\\0&0&3.2\end{bmatrix},\ W_{22}=\begin{bmatrix}0.4&0.5&0.1\\0.3&0.2&0.1\\0.3&0.17&0.1\end{bmatrix},\ W_{32}=\begin{bmatrix}0.4&-0.2&0.3\\0.3&0.2&-0.3\\-0.3&0.2&-0.1\end{bmatrix},C_2=\begin{bmatrix}0.1&0.2&-0.1\\-0.2&0.1&0.3\\0.4&0.2&0.1\end{bmatrix},\ D_2=\begin{bmatrix}0.3&-0.2&0.1\\0.2&0.5&-0.3\\-0.2&0.3&0.6\end{bmatrix}.$$

Let the Markov process governing the mode switching has generator:

$$\Pi=\begin{bmatrix}-0.4&0.4\\0.6&-0.6\end{bmatrix}$$

By setting $\mu = 0.1$, $\rho_\sigma = 0.001$, $\tau = 1.2$ the state and switching modes simulation curve of the system (4) are shown in Figs. (**3**) and (**4**), which conform the effectiveness of our results.

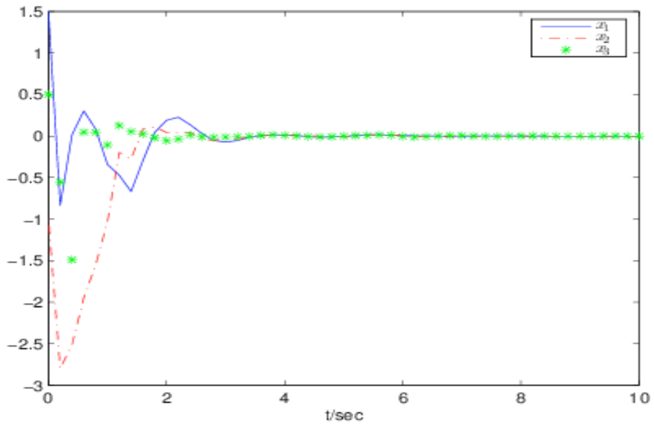

Fig. (3). Response of the state *x*(t).

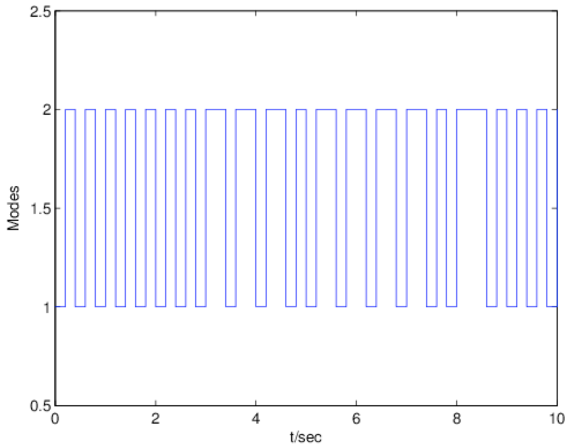

Fig. (4). Markovian jumping mode of *x*(t).

Furthermore, by setting δ = 0.8 and τ = 1.6, we can obtain the state simulation curve in Fig. (**5**). Fig. (**5**) indicates that when leakage delay increases, the system (4) tends to be unstable.

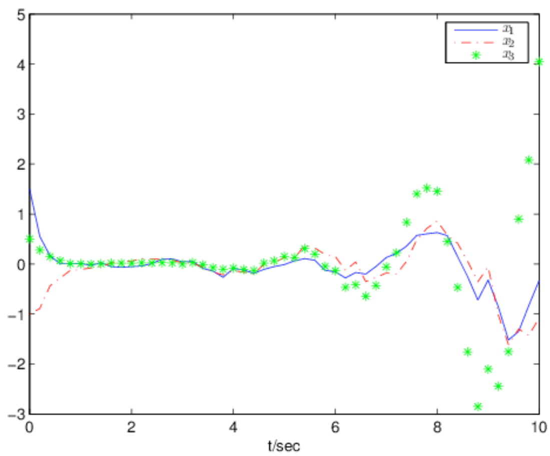

Fig. (5). Response of the state *x*(t).

SUMMARY

We studied the stability problem for a class of stochastic neural networks with Markovian jump parameters and leakage delay in this study. By employing a proper Lyapunov functional, combined with the stochastic stability analysis method, the stability criterion is derived to ensure the developed system is exponentially mean square stable based on LMIs. Finally, we discussed the effect of leakage delay on stability of neural networks system. Numerical examples were provided to testify the rightness and effectiveness of the research results.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

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