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# Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances

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**Résumé**

In this paper, we show that some spectral properties of Anosov diffeomorphisms can be obtained by semi-classical analysis. In particular the Ruelle resonances which are eigenvalues of the Ruelle transfer operator acting in suitable anisotropic Sobolev spaces and which govern the decay of dynamical correlations, can be treated as the quantum resonances of open quantum systems in the Aguilar-Baslev-Combes theory or the more recent Helffer-Sjöstrand phase-space theory [1].

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## 1 Introduction

An Anosov diffeomorphism $f$ on a compact manifold $M$ is characterized by the fact that under iterations, every trajectory has hyperbolic instability, which means that two points that are close to each other will be separated exponentially fast under the dynamics in the future or in the past. As a consequence the behavior of individual trajectories looks like unpredictable or "chaotic". Instead of looking at individual trajectories, it is then more natural to study a set of trajectories, or equivalently the transport of functions (or

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densities) under the map. One is led to study the so-called Ruelle transfer operator $\hat{F}$ defined by $\hat{F}\varphi = \varphi \circ f$ with $\varphi \in C^{\infty}(M)$. The spectral decomposition of this operator provides objects invariant under the dynamics and therefore informs us on the long time behavior of the dynamics, such as ergodicity, mixing, decay of correlations, central limit theorem ...(see [2, chap. VII],[3, 4, 5]).

The main subject of this paper is the spectral properties of the transfer operator $\hat{F}$. The approach we propose is based on an elementary but crucial observation which itself relies on the hypothesis of hyperbolicity: high Fourier modes of $\hat{F}^n\varphi$ go towards infinity as $n \to \infty$ or $n \to -\infty$. In other words, the variations of the function $\varphi$ evolve towards finer and finer scales as $n \to \pm\infty$, and as a consequence the "information" about the initial function $\varphi$ disappear from the macroscopic scale (the observation scale). This is the mechanism responsible for chaotic behavior, and in particular for the decay of dynamical correlation functions. In the 70's, David Ruelle has initiated a fruitful theory called thermodynamic formalism [6, 7, 8] where he studied the transfer operator $\hat{F}$ and defined the Ruelle resonances which govern the exponential decay of the dynamical correlation functions. This approach has recently been improved considerably in the works of M. Blank, S. Gouëzel, G. Keller, C. Liverani [9, 10, 11] and V. Baladi and M. Tsujii [12, 13] (see [13] for some historical remarks) where the authors demonstrate that the Ruelle resonances are the discrete spectrum of the transfer operator in suitably defined functional spaces.

From a mathematical point of view, the escape of the function $\hat{F}^n\varphi$ towards high Fourier modes we are interested in, is similar to the escape of a quantum wave function towards infinity in space occurring in open quantum systems. In such systems, studied since a long time because of their relevance to spontaneous emission of light in atoms [14] or radioactive decay in nuclei, physicists and mathematicians have elaborated concepts and techniques. In the 70's, J. Aguilar, E. Balslev, J.M. Combes, B. Simon and others developed a mathematical theory for quantum resonances, which has been improved after by many authors in the 80's, in particular B. Helffer and J. Sjöstrand [1, 15].

The aim of this paper is to show that some results of C. Liverani et al. [9, 10, 11, 16], V. Baladi et al. [12, 13] concerning the definition and properties of Ruelle resonances fit perfectly well within the semi-classical approach of quantum resonances in phase space developed in [1, 15, 17]. The results we present are not new, but we want to show the relevance of the semi-classical analysis to the theory of hyperbolic dynamical systems.

Semi-classical analysis (or equivalently microlocal analysis) has been developed for the study of partial differential equations in the regime of small wave-length or equivalently, high Fourier mode regime [18, 19, 20]. As we explained above, the very definition of hyperbolic dynamics involves high Fourier modes and this implies that semi-classical analysis should be a "natural" approach for its understanding. The idea of relevance of semi-classical analysis in the context of hyperbolic dynamics has been presented and used in [21] in a simpler framework (i.e. real analytical maps on the torus). In this paper we present the later approach in wider generality.

In semi-classical analysis we distinguish two kinds of operators, the pseudo-differential operators (PDO) and the Fourier integral operators (FIO). To each PDO $\hat{P}$ is associated a function $P = \sigma\left(\hat{P}\right)$ on the cotangent bundle $T^*M$, called its symbol. To each FIO $\hat{F}$ is associated a symplectic map $F$ on $T^*M$. In semi-classical analysis, we manipulate the symbols instead of the operators, and powerful theorems transcribe properties of the symbols in terms of properties of the operators (for example spectral properties).

In our context, the Ruelle transfer operator $\hat{F}$ is a FIO whose associated symplectic map denoted $F : T^*M \to T^*M$, is the lift of $f^{-1}$ (the inverse of the Anosov diffeomorphism). This is presented in Section 2.

In Section 3, we study the dynamics of $F$. It appears that in $T^*M$, the trajectories of $F$ are non compact, except for the maximal compact invariant subspace, the section $\xi = 0$, and this is related in an essential way to the discreteness of the spectrum of the operator $\hat{F}$ obtained after. We construct an "escape function" $A_m$ on the cotangent space $T^*M$, which decreases strictly along the non-compact trajectories of $F$, in a controlled manner. Since it decreases in the unstable direction and increases in the stable direction, the escape function $A_m$ belongs to a class of symbols with variable order, defined in Appendix A. We defined an associated invertible PDO denoted by $\hat{A}_m$. We also define the anisotropic Sobolev space associated to $\hat{A}_m$ in the standard manner: $H^m \stackrel{\text{def}}{=} \hat{A}_m^{-1}\left(L^2(M)\right)$.

In Section 4, we show in Theorem 2 that the operator $\hat{F}$ acting on the anisotropic Sobolev space $H^m$ has a discrete spectrum outside an a disk of radius $\varepsilon_m$ (which can be made arbitrary small). The discrete spectrum does not depend on the choice of $A_m$ and defines the Ruelle resonances. This is the main result of this paper. This theorem has already been obtained by various authors with different degrees of generalities [10, 11, 12, 16], but the proof we present here is different as it uses in a simple way three major Theorems of semi-classical analysis: the "Composition Theorem for PDO", the "Egorov's Theorem for transport" and the "$L^2$ continuity Theorem".

As an application of this approach, we derive expressions for dynamical correlation functions in Section 5. In Section 6, we propose a new proof for a theorem of D. Anosov which states that an Anosov diffeomorphism preserving a smooth measure is mixing. In Section 7, we show that a semi-classical truncation of the operator $\hat{F}$ gives the Ruelle resonance spectrum. This latter result is useful for numerical computations.

In appendix A, we provide a self-contained presentation of semi-classical results adapted for this article.

The case of uniformly expanding maps can be considered with a similar approach. However it would be mostly simplified by the fact that the escape function $A_m$ would have constant order $m$, and the associated Sobolev space $H^m$ are usual (non anisotropic) Sobolev spaces.

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# 2 The model of hyperbolic map

Let $M$ be a smooth compact connected manifold. Let $f : M \to M$ be a $C^\infty$ Anosov diffeomorphism. We recall the definition:

**Definition 1.** *(see [5] page 263) A diffeomorphism $f : M \to M$ is **Anosov** (or uniformly hyperbolic) if there exists a Riemannian metric $g_0$, an $f$-invariant* orthogonal *decomposition of $TM$:*

$$TM = E_u \oplus E_s \tag{1}$$

*and $0 < \theta < 1$, such that for any $x \in M$*

$$\begin{aligned} \left|D_x f\left(v_s\right)\right|_{g_0} &\leq \theta\left|v_s\right|_{g_0}, \qquad \forall v_s \in E_s\left(x\right) \\ \left|D_x f^{-1}\left(v_u\right)\right|_{g_0} &\leq \theta\left|v_u\right|_{g_0}, \qquad \forall v_u \in E_u\left(x\right). \end{aligned} \tag{2}$$

*This means that $E_s$ is the stable foliation and $E_u$ the unstable foliation for positive time.*

**Remarks:**

1. Standard examples are hyperbolic automorphisms of the torus $\mathbb{T}^n$ as well as their $C^1$ small perturbations, thanks to the structural stability theorem (see [5] page 266). The problem of classifying manifolds that admit Anosov diffeomorphisms turned out to be very difficult. The only known examples are infranil manifolds (which contain the torus case) and it is conjectured that they are the only ones [22, p. 16]. Here is a simple example of Anosov diffeomorphism on $(x, y) \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\varepsilon}{2\pi}\sin\left(2\pi\left(2x+y\right)\right) \end{pmatrix} \tag{3}$$

   with $\varepsilon$ small enough[2]. The Ruelle resonances of this map are depicted on figure 6 page 29.

2. The metric $g_0(x)$ (called the Lyapounov metric) and the distributions $E_u(x), E_s(x) \subset T_x M$ are in general only Hölder continuous with respect to $x \in M$ (See [5] chap. 19). For the purpose of semi-classical analysis, one needs a *smooth* metric in order to

[2]This example preserves area $dx \wedge dy$.

construct suitable symbols. In this paper, we will assume that $M$ is endowed with a smooth Riemannian metric $g$ satisfying

$$\frac{1}{c} g_0 \leq g \leq c.g_0 \tag{4}$$

uniformly on $M$, with

$$1 \leq c < \theta^{-\frac{1}{2}}. \tag{5}$$

With $\theta_* \stackrel{\text{def}}{=} c^2\theta$, this implies that $0 < \theta_* < 1$ and for any $v_s \in E_s(x)$ one has estimates similar to Eq.(2), but with the metric $g$:

$$|D_x f(v_s)|_g \leq c\,|D_x f(v_s)|_{g_0} \leq c\theta\,|v_s|_{g_0} \leq c^2\theta\,|v_s|_g = \theta_*\,|v_s|_g \tag{6}$$

Similarly for $v_u \in E_u(x)$, $|D_x f^{-1}(v_u)|_g \leq \theta_*\,|v_u|_g$. Unless specified, we will always work with this metric $g$, which can be obtained from $g_0$ by smoothing[3].

## 2.1 Transfer operators

We denote by $dx = d\mu_{Leb}$ an arbitrary[4] smooth density normalized by $\mu_{Leb}(M) = 1$. Let us define the bounded operator $\hat{F}$ on $L^2(M)$ by[5]

$$\boxed{\hat{F}\varphi \stackrel{\text{def}}{=} \varphi \circ f, \qquad \varphi \in L^2(M)} \tag{7}$$

called the **Ruelle transfer operator** (or Koopman operator).

Let us emphasize that in general $f$ does not preserve any smooth measure, but if $f$ preserves the Lebesgue measure $\mu_{Leb}$ then $\hat{F}$ is unitary in $L^2(M)$.

Let us remark that the $L^2-$adjoint operator $\hat{F}^*$ is given by

$$\left(\hat{F}^*\varphi\right)(y) = \left(\varphi \circ f^{-1}\right)(y)\,\left|D_{f^{-1}(y)}f\right|^{-1}$$

with $\psi, \varphi \in C^\infty(M)$. The adjoint operator $\hat{F}^*$ also called the **Perron-Frobenius operator** is usually considered since it transports densities [11]. Our main result, Corollary 1 page 18, concerns the spectrum of both $\hat{F}$ and $\hat{F}^*$.

[3] If however the metric $g$ is already given, one can always fulfill Eq.(5) by taking some positive power $f^{n_0}$, $n_0 \in \mathbb{N}$, of the Anosov map.

[4] We have chosen a density $dx$ in order to define $L^2(M)$. However this choice does not play any role for the principal results of this paper.

[5] It would have been more natural to consider $\hat{F}\varphi \stackrel{\text{def}}{=} \varphi \circ f^{-1}$ instead, but our choice here follows the paper [21] where we considered expanding maps which are not invertible.

**Other types of transfer operators.** In the context of thermodynamic formalism of dynamical systems introduced by D. Ruelle et al. (see [23] chap. 4, [24] chap. 6), a more general class of transfer operators than Eq.(7) is considered and defined as follow. Let $V \in C^{\infty}(M)$ be a smooth (real or complex) valued function called the potential and let $\hat{F}_V : C^{\infty}(M) \to C^{\infty}(M)$ be defined by

$$\left(\hat{F}_V \varphi\right)(x) := \varphi(f(x))\, e^{V(x)} \tag{8}$$

Compared to the simplest case $V = 0$, given in Eq.(7), the new term $e^V$ is a pseudodifferential operator of order 0 (as defined in the subsequent sections and in Appendix A). Consequently the canonical map $F : T^*M \to T^*M$ associated to $\hat{F}_V$, Eq.(11), is unchanged. This implies that our main results, Theorem 1 page 16, Corollary 1 page 18 and their proof, are the same[6] for this new operator $\hat{F}_V$.

There is even a slightly more general class of transfer operators acting on sections of line bundles[7], for which our results work as well. Since these transfer operators have interesting connections with quantum chaos and geometric quantization (see [25]) we mention them. A general definition proceeds as follow.

**Definition 2.** *Let $L \to M$ be a smooth complex line bundle over $M$. A **transfer operator** $\hat{F}$ associated to the smooth diffeomorphism $f : M \to M$ is a linear map acting on smooth sections, $\hat{F} : C^{\infty}(M; L) \to C^{\infty}(M; L)$, such that for any smooth function $\psi \in C^{\infty}(M)$ and any smooth section $s \in C^{\infty}(M; L)$ one has*

$$\left(\hat{F}(\psi s)\right) = (\psi \circ f) . \left(\hat{F} s\right) \tag{9}$$

*One also requires that for any $x \in M$ and $s \in C^{\infty}(M; L)$,*

$$(s \circ f)(x) \neq 0 \Rightarrow \left(\hat{F} s\right)(x) \neq 0 \tag{10}$$

This definition of transfer operator generalizes Eq.(8), since in the case where $L$ is a trivial line bundle, sections are identified with complex functions thanks to a global non vanishing section $r \in C^{\infty}(M; L)$: any global section $s \in C^{\infty}(M; L)$ can be written $s = \varphi r$ with $\varphi \in C^{\infty}(M)$. Let $e^V \in C^{\infty}(M)$ be defined by $\hat{F}(r) = e^V r$ which is possible from (10). Then (9) gives $\hat{F}(\varphi r) = (\varphi \circ f) e^V r$ which is equivalent to (8).

In this paper we will only consider the simpler expression Eq.(7).

[6] Of course the spectrum of $\hat{F}_V$ depends on $V$ and the value 1 is in general no more an eigenvalue of $\hat{F}_V$. Corollary 2 and 3 page 22 are specific to $\hat{F}_{V=0}$.

[7] This works also for vector bundles.

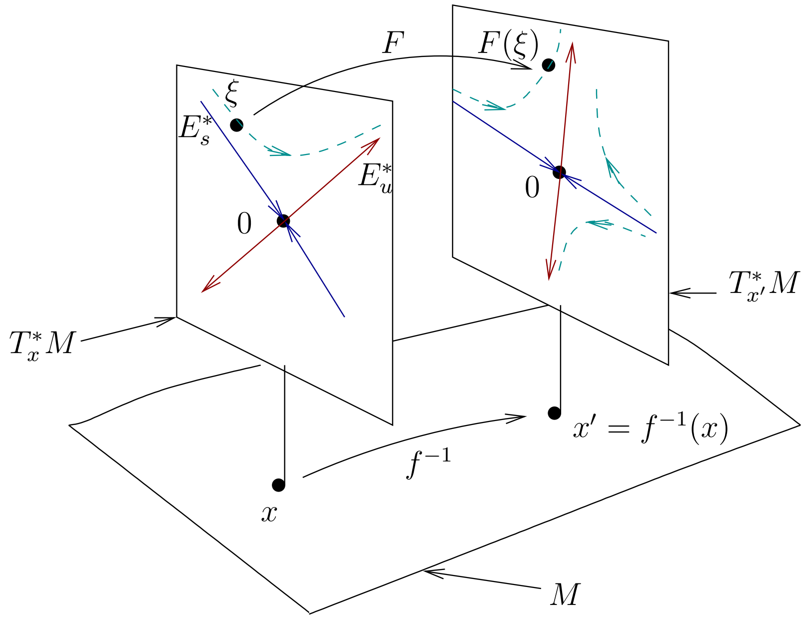


Figure 1: Dynamics of $F$ defined in Eq.(11), on the cotangent space $T^*M$.

## 2.2 Dynamics on the cotangent space

In order to study the spectrum of the operator $\hat{F}$ using semi-classical analysis later on, we need to consider the dynamics induced by $f$ in the cotangent bundle $F : T^*M \to T^*M$, namely the lift of $f^{-1}$. See Figure 1. For any $x \in M$, let $x' = f^{-1}(x)$, and define

$$\begin{array}{cccc} F: & T_x^*M & \to & T_{x'}^*M \\ & \xi & \longmapsto & (D_{x'}f)^t \xi \end{array} \tag{11}$$

In semi-classical analysis, the map $F$ is precisely the canonical map associated to the operator $\hat{F}$ defined in Eq.(7). This appears in Egorov's Theorem 9. It is the lift of $f^{-1}$ rather than $f$ because the support of $\hat{F}\varphi = \varphi \circ f$ is the support of $\varphi$ transported by $f^{-1}$.

Notice that the zero section $\xi = 0$ is a compact set, invariant by the dynamics and its complement contains only unbounded trajectories. This observation is at the origin of the method which leads to the quasi-compacity result obtained in the main Theorem 1.

Let $T^*M = E_s^* \oplus E_u^*$ be the decomposition dual to Eq.(1), i.e. $E_s^*(E_u) = 0$ and $E_u^*(E_s) = 0$.

**Lemma 1.** *The decomposition $T^*M = E_s^* \oplus E_u^*$ is invariant by $F$ and:*

$$\begin{array}{ll} |F(\xi_s)| \leq \theta_* |\xi_s| & \forall \xi_s \in E_s^* \\ |F^{-1}(\xi_u)| \leq \theta_* |\xi_u| & \forall \xi_u \in E_u^*. \end{array} \tag{12}$$

*with $\theta_* = c^2\theta$, $0 < \theta^* < 1$, with $c$ from Eq. (4).*

*Proof.* The distribution $E_s^*$ is invariant by $F$ because for any $\xi_s \in E_s^*$, $v_u \in E_u$, one has $F(\xi_s)(v_u) = \left((Df)^t \xi_s\right)(v_u) = \xi_s(Df(v_u)) = 0$ since $E_u$ is invariant by $Df$. The same holds for $E_u^*$. On the other hand, one gets easily convinced that Eq. (4) implies the same inequalities for the metric on the dual space. Eq.(2) implies that on the dual space, for any $\xi_s \in E_s^*(x)$, $|F(\xi_s)|_{g_0} \leq \theta |\xi_s|_{g_0}$. Then for any $\xi_s \in E_s^*(x)$ one has

$$|F(\xi_s)|_g \leq c\,|F(\xi_s)|_{g_0} \leq c\theta\,|\xi_s|_{g_0} \leq c^2\theta\,|\xi_s|_g$$

and similarly for $\left|F^{-1}(\xi_u)\right|_g \leq c^2\theta\,|\xi_u|\,, \forall \xi_u \in E_u^*$.

□

# 3 The escape function and the anisotropic Sobolev spaces

## 3.1 Construction of the escape function $A_m$ and the pseudodifferential operator $\hat{A}_m$

In this section, we construct a function $A_m$ on the cotangent space which decreases along all the unbounded trajectories of $F$ pictured in Figure 1. It is called an **escape function**. In order to apply semi-classical theorems later on, we make sure that $A_m$ is a suitable symbol. This will allow us to construct a pseudodifferential operator $\hat{A}_m$ from the symbol $A_m$. It turns out that an escape function $A_m(x,\xi)$ suitable for our purposes must have an order $m$ in $\langle\xi\rangle$ which depends itself of the direction $\xi/|\xi|$. This gives rise to general classes of symbols $S_\rho^{m(x,\xi)}$ and PDO's $\Psi_\rho^{m(x,\xi)}$ with variable order $m(x,\xi)$. Their definitions and main properties are summarized in Appendix A.

**Lemma 2.** *Let $u < 0 < s$. There exists a order function $m(x,\xi) \in S_1^0$ taking values in $[u,s]$, with the following properties. For any fixed $x$ and $|\xi| > 1$, $m(x,\xi)$ depends only on the direction $\tilde{\xi} = \xi/|\xi|$ of the cotangent vector. Moreover $m(x,\xi) = s$ (resp. $u$) in a vicinity of the stable direction $E_s^*(x)$ (resp. unstable direction $E_u^*(x)$). See figure 2. The function $m(x,\xi)$ decreases with respect to the map $F$:*

$$\exists R > 0, \quad \forall |\xi| \geq R, \qquad (m \circ F)(x,\xi) - m(x,\xi) \leq 0, \tag{13}$$

*Define*

$$A_m(x,\xi) \stackrel{\text{def}}{=} \langle\xi\rangle^{m(x,\xi)}, \qquad \langle\xi\rangle = \sqrt{1+|\xi|^2}. \tag{14}$$

*which belongs to the class $S_\rho^{m(x,\xi)}$, for any $\frac{1}{2} \leq \rho < 1$. The main property of the symbol $A_m$ is:*

$$\exists R > 0, \quad \forall |\xi| \geq R, \qquad \frac{(A_m \circ F)(x,\xi)}{A_m(x,\xi)} \leq e^{-ca} < 1 \tag{15}$$

*with $a = min(-u,s)$ and $c > 0$ independent of the choice of $u, s$.*

**Remarks:**

- Eq.(15) means that the function $A_m$ decreases strictly and uniformly along the trajectories of $F$ in the cotangent space. We call $A_m$ an **escape function**.
- The constancy of $m$ in the vicinity of the stable/unstable direction allows us to have a smooth order function $m$ despite the foliations $E_s^*(x)$,$E_u^*(x)$ have only Hï¿œelder regularity.
- Inspection of the proof shows that $c$ can be chosen arbitrary close to $\log\left(\frac{1}{\theta_*}\right)$.

The real symbol $A_m$ can be quantized into a pseudodifferential operator $\hat{A}_m$ of variable order $m(x,\xi)$, according to Eq.(57). Then Corollary 4 and Example 1 tell us that we can modify the symbol $A_m$ at a subleading order (i.e. $S_\rho^{m(x,\xi)-(2\rho-1)}$) such that the operator can be assumed to be formally *self-adjoint* and *invertible* on $C^\infty(M)$.

**Proof of Lemma 2.**

**The function** $m$**.** Since the lifted map $F$ defined in Eq.(11) is linear in $\xi$, it defines a map $\tilde{F}$ on the cosphere bundle $S^*M = (T^*M\backslash\{0\})/\mathbb{R}^+$, namely the space of directions

$$\tilde{\xi} := \xi/|\xi|,$$

which is a compact space. See Figure 2. The image of $E_u^*, E_s^* \subset T^*M$ by the projection $T^*M\backslash\{0\} \to S^*M$ are denoted respectively $\tilde{E}_u^*, \tilde{E}_s^* \subset S^*M$. Eq.(12) implies that $\tilde{E}_u^*$ is a uniform attractor for $\tilde{F}$, and $\tilde{E}_s^*$ is a uniform repeller, i.e. $\tilde{F}^n\left(\tilde{\xi}\right)$ converges to $\tilde{E}_u^*$ (respect. $\tilde{E}_s^*$) when $n \to +\infty$ (respect. $n \to -\infty$).

Let $u < 0 < s$. Let $m_0 \in C^\infty(S^*M;[u,s])$ with $m_0 = s > 0$ in a neighborhood $\tilde{N}_s$ of $\tilde{E}_s^*$ and $m_0 = u < 0$ in a neighborhood $\tilde{N}_u$ of $\tilde{E}_u^*$. We also assume that

$$(\tilde{\xi} \in \tilde{N}_s \Rightarrow \tilde{F}^{-1}\left(\tilde{\xi}\right) \in \tilde{N}_s) \text{ and } (\tilde{\xi} \in \tilde{N}_u \Rightarrow \tilde{F}\left(\tilde{\xi}\right) \in \tilde{N}_u) \tag{16}$$

See Figure 3.

Let $N \in \mathbb{N}$ and define $\tilde{m} \in C^\infty(S^*M;[u,s])$ by

$$\tilde{m} := \frac{1}{2N}\sum_{n=-N}^{N-1} m_0 \circ \tilde{F}^n \tag{17}$$

Then

$$\tilde{m} \circ \tilde{F} - \tilde{m} = \frac{1}{2N}\left(m_0 \circ \tilde{F}^N - m_0 \circ \tilde{F}^{-N}\right) \tag{18}$$

We will show now that

$$\forall \tilde{\xi} \in S^*M \qquad \tilde{m}\left(\tilde{F}\left(\tilde{\xi}\right)\right) - \tilde{m}\left(\tilde{\xi}\right) \leq 0. \tag{19}$$

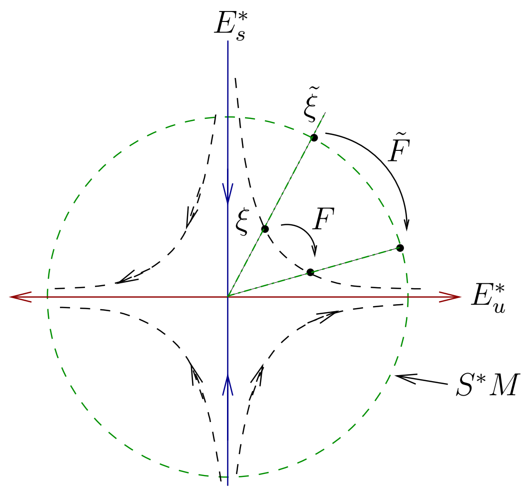


Figure 2: The map $F$ on the cotangent space $T^*M$ and the induced map $\tilde{F}$ on the cosphere bundle $S^*M$.

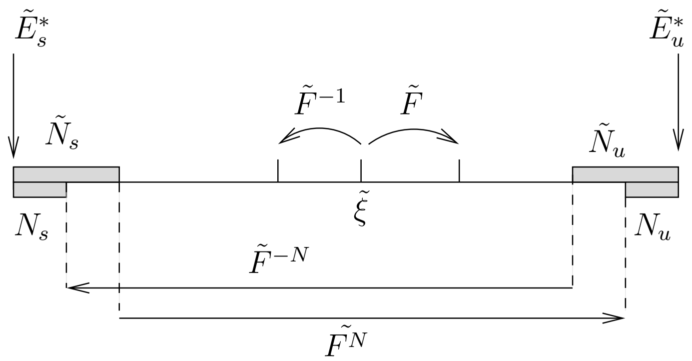


Figure 3: The horizontal axis is a schematic picture of $S^*M$ and this shows the construction and properties of the sets $N_s$ and $N_u$.

Let $N_s := S^*M \setminus \tilde{F}^{-N}\left(\tilde{N}_u\right)$ and $N_u := S^*M \setminus \tilde{F}^{N}\left(\tilde{N}_s\right)$. Then $m_0\left(\tilde{F}^N\left(\tilde{\xi}\right)\right) = u$ for $\tilde{\xi} \notin N_s$, and similarly $m_0\left(\tilde{F}^{-N}\left(\tilde{\xi}\right)\right) = s$ for $\tilde{\xi} \notin N_u$.

For $N$ large enough one has $N_s \subset \tilde{N}_s$, $N_u \subset \tilde{N}_u$ and $N_s \cap N_u = \emptyset$. Therefore,

- if $\tilde{\xi} \in N_s$ then $\tilde{\xi} \notin N_u$ and $m_0\left(\tilde{F}^{-N}\left(\tilde{\xi}\right)\right) = s \geq m_0\left(\tilde{F}^{N}\left(\tilde{\xi}\right)\right)$ and (18) gives $\tilde{m}\left(\tilde{F}\left(\tilde{\xi}\right)\right) - \tilde{m}\left(\tilde{\xi}\right) \leq 0$.
- Similarly if $\tilde{\xi} \in N_u$ then $\tilde{\xi} \notin N_s$ and $m_0\left(\tilde{F}^{N}\left(\tilde{\xi}\right)\right) = u \leq m_0\left(\tilde{F}^{-N}\left(\tilde{\xi}\right)\right)$ thus $\tilde{m}\left(\tilde{F}\left(\tilde{\xi}\right)\right) - \tilde{m}\left(\tilde{\xi}\right) \leq 0$.
- Finally if $\tilde{\xi} \notin (N_u \cup N_s)$ then $m_0\left(\tilde{F}^{N}\left(\tilde{\xi}\right)\right) - m_0\left(\tilde{F}^{-N}\left(\tilde{\xi}\right)\right) = (u-s) < 0$ and therefore
$$\forall \tilde{\xi} \notin (N_u \cup N_s) \qquad \tilde{m}\left(\tilde{F}\left(\tilde{\xi}\right)\right) - \tilde{m}\left(\tilde{\xi}\right) = \frac{1}{2N}(u-s) < 0 \tag{20}$$

We have shown Eq.(19).

We construct a smooth function $m$ on $T^*M$ satisfying

$$\begin{aligned} m(x,\xi) &= \tilde{m}\left(\tilde{\xi}\right), \qquad \text{if } |\xi| > 1, \\ &= 0 \qquad \text{if } |\xi| < 1/2 \end{aligned}$$

Then (19) implies Eq.(13).

From Eq.(16) one deduces that the set

$$\mathcal{S}\left(\tilde{\xi}\right) := \left\{ n \in \mathbb{Z} \quad / \quad \tilde{F}^n\left(\tilde{\xi}\right) \notin (N_s \cup N_u) \right\}$$

is connected. Moreover the cardinal of this set is uniformly bounded:

$$\exists \mathcal{N} \in \mathbb{N} \quad \forall \tilde{\xi} \in S^*M \qquad \sharp \mathcal{S}\left(\tilde{\xi}\right) \leq \mathcal{N}$$

From this we deduce that

- if $\tilde{\xi} \in N_s$ then $\tilde{F}^N\left(\tilde{\xi}\right) \notin \tilde{N}_u$ but also $\tilde{F}^n\left(\tilde{\xi}\right) \notin \tilde{N}_u$ for $n \leq N$ and even $\tilde{F}^n\left(\tilde{\xi}\right) \in \tilde{N}_s$ for $n \leq N - \mathcal{N}$. Thus (17) gives
$$\tilde{m}\left(\tilde{F}\left(\tilde{\xi}\right)\right) \geq \left(1 - \frac{\mathcal{N}+1}{2N}\right)s + \frac{\mathcal{N}+1}{2N}u \geq \frac{s}{2} \tag{21}$$
where the last inequality holds for $N$ large enough.
- If $\tilde{\xi} \in N_u$ one shows similarly that
$$\tilde{m}\left(\tilde{\xi}\right) \leq \left(1 - \frac{\mathcal{N}}{2N}\right)u + \frac{\mathcal{N}}{2N} \leq \frac{u}{2} \tag{22}$$
where the last inequality holds for $N$ large enough.

**The symbol $A_m$.** Let

$$A_m(x,\xi) := \langle\xi\rangle^{m(x,\xi)}$$

with $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. $A_m$ belongs to the class $S_\rho^{m(x,\xi)}$, for any $\frac{1}{2} \leq \rho < 1$ from Lemma 6. We will show now the uniform escape estimate Eq.(15).

For $|\xi|$ large enough one has

$$\log A_m(x,\xi) = \tilde{m}\left(\tilde{\xi}\right)\ln\langle\xi\rangle, \qquad \log\left(A_m \circ F\right)(x,\xi) = \tilde{m}\left(\tilde{F}\left(\tilde{\xi}\right)\right)\ln\langle F(\xi)\rangle$$

- If $\tilde{\xi} \notin (N_s \cup N_u)$ then (20) gives $\tilde{m}\left(\tilde{F}\left(\tilde{\xi}\right)\right) = \tilde{m}\left(\tilde{\xi}\right) + \frac{1}{2N}(u-s)$. One also has $\log\langle F(\xi)\rangle = \log\langle\xi\rangle + \mathcal{O}(1)$. Therefore

$$\begin{aligned}
\tilde{m}\left(\tilde{F}\left(\tilde{\xi}\right)\right)\ln\langle F(\xi)\rangle - \tilde{m}\left(\tilde{\xi}\right)\ln\langle\xi\rangle &= \left(\tilde{m}\left(\tilde{\xi}\right) + \frac{1}{2N}(u-s)\right)\left(\log\langle\xi\rangle + \mathcal{O}(1)\right) - \tilde{m}\left(\tilde{\xi}\right)\ln\langle\xi\rangle \\
&= \tilde{m}\left(\tilde{\xi}\right)\mathcal{O}(1) + \frac{1}{2N}(u-s)\log\langle\xi\rangle \\
&\leq -c\min(s,-u)
\end{aligned}$$

  with $c > 0$ and if $|\xi|$ is large enough.

- If $\tilde{\xi} \in N_s$ (neighborhood of the stable direction) then $|F(\xi)| \leq \frac{1}{C'}|\xi|$, with $C' > 1$, so $0 < \ln\langle F(\xi)\rangle \leq \ln\langle\xi\rangle - \ln C$, with $C > 1$ (close to $C'$). And using (21),(19)

$$\begin{aligned}
\tilde{m}\left(\tilde{F}\left(\tilde{\xi}\right)\right)\ln\langle F(\xi)\rangle &\leq \tilde{m}\left(\tilde{F}\left(\tilde{\xi}\right)\right)\left(\ln\langle\xi\rangle - \ln C\right) \\
&\leq \tilde{m}\left(\tilde{\xi}\right)\ln\langle\xi\rangle - \frac{s}{2}\ln C
\end{aligned}$$

- If $\tilde{\xi} \in N_u$ (neighborhood of the unstable direction) then $|F(\xi)| \geq C'|\xi|$, with $C' > 1$, so $\ln\langle F(\xi)\rangle \geq \ln\langle\xi\rangle + \ln C$ with $C > 1$ (close to $C'$). And using (21),(19),

$$\begin{aligned}
\tilde{m}\left(\tilde{F}\left(\tilde{\xi}\right)\right)\ln\langle F(\xi)\rangle &\leq \tilde{m}\left(\tilde{\xi}\right)\left(\ln\langle\xi\rangle + \ln C\right) \\
&\leq \tilde{m}\left(\tilde{\xi}\right)\ln\langle\xi\rangle + \frac{1}{2}u\ln C
\end{aligned}$$

In conclusion, for any $x$ and $|\xi|$ large enough, there exists $c > 0$ independent of $u, s$ such that

$$\log\left(A_m \circ F\right)(x,\xi) \leq \log A_m(x,\xi) - c\min(s,-u)$$

We have obtained the uniform escape estimate Eq.(15) and finished the proof of Lemma 2.

## 3.2 The anisotropic Sobolev spaces

A particular feature of the self-adjoint and invertible PDO $\hat{A}_m \in \Psi_\rho^{m(x,\xi)}$ introduced above is that its symbol $A_m$ has a non-isotropic behavior with respect to $\xi \in T_x^* M$. It is a PDO with maximum order $a = \min(|u|, s)$, but with variable order $m(x, \xi) \in [u, s]$, with $u < 0 < s$. For large $|\xi|$, the symbol $A_m$ increases in the stable direction $\xi \in E_s^*(x)$ as $A_m(\xi) \sim |\xi|^s$ and decreases in the unstable direction $\xi \in E_u^*(x)$ as $A_m(\xi) \sim 1/|\xi|^{|u|}$. In this section we consider a slightly more general space of functions $m$ and do not require any relation with the dynamics of $F$: we just assume that $m$ is in $C^\infty(T^*M)$ and is a function of $\left(x, \frac{\xi}{|\xi|}\right)$ for $|\xi|$ large enough. Therefore $m \in S_1^0$ is an order function.

We define the **anisotropic Sobolev space** to be the space of distributions (included in $\mathcal{D}'(M)$):

$$\boxed{H^m \stackrel{\text{def}}{=} \hat{A}_m^{-1}\left(L^2(M)\right)} \tag{23}$$

**Remarks:**

- This definition is very similar to the standard definition of Sobolev spaces on $\mathbb{R}^d$ ([26] p.271):
$$H^s \stackrel{\text{def}}{=} Op\left(\langle\xi\rangle^s\right)^{-1}\left(L^2\left(\mathbb{R}^d\right)\right)$$
except for anisotropy with respect to $\xi$. Equivalent definitions of anisotropic Sobolev spaces have been given in [27] and also by V. Baladi et M. Tsujii in [13] for the specific purpose of hyperbolic dynamics.

- Notice that $\varphi \in H^m \Leftrightarrow \hat{A}_m \varphi \in L^2(M)$, so roughly speaking, in the case of the function $m$ defined in Lemma 2, it means that the Fourier transform $\hat{\varphi}(\xi)$ performed in a vicinity of $x \in M$, increases less than $|\xi_s|^{-s-d/2}$ for $\xi_s \in E_s^*(x)$ and less than $|\xi_u|^{-u-d/2}$ for $\xi_u \in E_u^*(x)$, with $d = \dim(M)$. We can say that if $u < 0 < s$, then $\varphi$ is regular in the stable direction and irregular in the unstable direction.

**Some simple properties of the anisotropic Sobolev spaces $H^m$:**

1. $H^m$ is a Hilbert space with the scalar product
$$(\varphi_1, \varphi_2)_{H^m} \stackrel{\text{def}}{=} \left(\hat{A}_m \varphi_1, \hat{A}_m \varphi_2\right)_{L^2(M)}, \qquad \varphi_1, \varphi_2 \in H^m$$
and the map
$$\hat{A}_m : \left(H^m, (.,.)_{H^m}\right) \to \left(L^2(M), (.,.)_{L^2}\right) \tag{24}$$
is unitary.

2. On $L^2(M)$, $\mathrm{Dom}\left(\hat{A}_m\right) = H^m \cap L^2(M)$ and $\mathrm{Dom}\left(\hat{A}_m^{-1}\right) = H^{-m} \cap L^2(M)$.

3. There are embedding relations as for usual Sobolev spaces. First
$$H^{\max(m)} \subset H^m \subset H^{\min(m)} \tag{25}$$
If $m' \geq m$ then
$$H^{m'} \subset H^m \tag{26}$$
and $H^{m'}$ is dense in $H^m$.

4. If $\varphi \in H^m$ and $g \in C^\infty(M)$ then
$$g\varphi \in H^m \tag{27}$$
and moreover, the map $\varphi \to g\varphi$ is continuous $H^m \to H^m$.

5. Let
$$H^{-m} \stackrel{\text{def}}{=} \hat{A}_m \left(L^2(M)\right).$$
The spaces $H^m$ and $H^{-m}$ are dual in the following sense: if $\psi \in H^m, \varphi \in H^{-m}$, we note
$$\overline{\psi}(\varphi) = \varphi\left(\overline{\psi}\right) = (\psi, \varphi)_{H^m \times H^{-m}} \stackrel{\text{def}}{=} \left(\hat{A}_m \psi, \hat{A}_m^{-1} \varphi\right)_{L^2(M)}. \tag{28}$$
Then
$$\left|(\psi, \varphi)_{H^m \times H^{-m}}\right| \leq \|\psi\|_{H^m} \|\varphi\|_{H^{-m}}. \tag{29}$$

6. If $\psi \in H^m \cap L^2(M)$ and $\varphi \in H^{-m} \cap L^2(M)$ then
$$(\psi, \varphi)_{H^m \times H^{-m}} = (\psi, \varphi)_{L^2(M)} \tag{30}$$
Since the dual bracket coincides with the $L^2$ scalar product, we will drop the indices in the sequel of the paper, and write:
$$(\psi, \varphi) \stackrel{\text{def}}{=} (\psi, \varphi)_{H^m, H^{-m}}$$

7. If $\psi \in H^m$, $\varphi \in H^{-m}$ and $g \in C^\infty$, then:
$$(g\psi, \varphi) = (\psi, \overline{g}\varphi) \tag{31}$$

*Proof.* The proofs of properties 1 to 4 follow directly from those of $\hat{A}_m$.

5.$|(\psi, \varphi)| = \left|\left(\hat{A}_m \psi, \hat{A}_m^{-1} \varphi\right)_{L^2(M)}\right| \leq \left\|\hat{A}_m \psi\right\|_{L^2} \left\|\hat{A}_m^{-1} \varphi\right\|_{L^2} = \|\psi\|_{H^m} \|\varphi\|_{H^{-m}}$.

6.$(\psi, \varphi) = \left(\hat{A}_m \psi, \hat{A}_m^{-1} \varphi\right)_{L^2(M)} = (\psi, \varphi)_{L^2(M)}$ by self adjointness of $\hat{A}_m$.

7. Let $\mathcal{M}_g$ denotes multiplication $g \in C^\infty$. The operator $\hat{B}_g = \hat{A}_m \mathcal{M}_g \hat{A}_m^{-1}$ is bounded in $L^2(M)$. Moreover, one has $\hat{B}_g^* = \hat{A}_m^{-1} \mathcal{M}_{\overline{g}} \hat{A}_m$ since $\hat{A}_m$ is self-adjoint. We deduce that
$$\begin{aligned}(g\psi, \varphi) &= \left(\left(\hat{A}_m \mathcal{M}_g \hat{A}_m^{-1}\right) \hat{A}_m \psi, \hat{A}_m^{-1} \varphi\right)_{L^2} = \left(\hat{A}_m \psi, \left(\hat{A}_m^{-1} \mathcal{M}_{\overline{g}} \hat{A}_m\right) \hat{A}_m^{-1} \varphi\right)_{L^2} \\ &= (\psi, \overline{g}\varphi).\end{aligned}$$

□

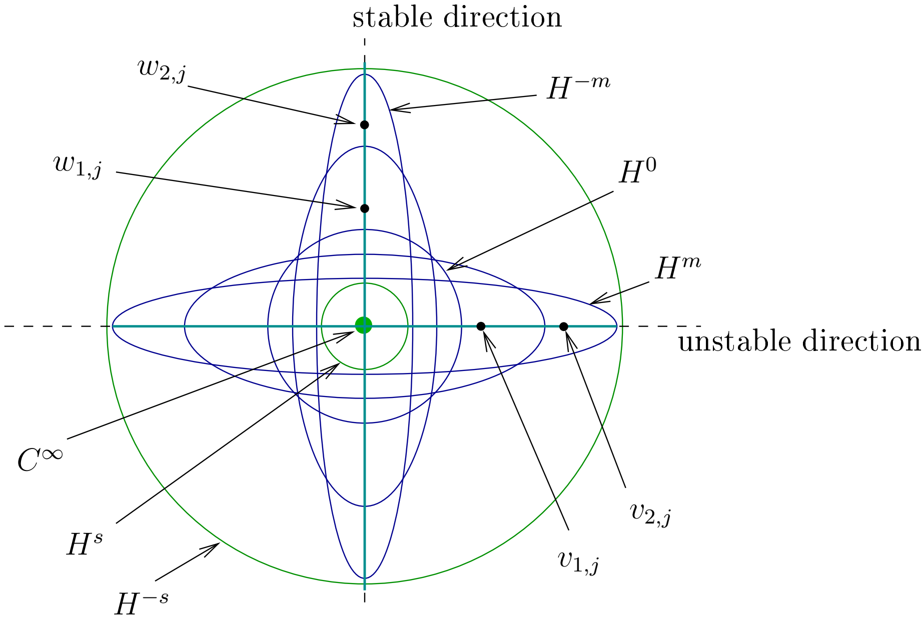


Figure 4: Schematic representation of the anisotropic Sobolev spaces $H^m$ and their embedding relations Eq.(25) with the usual Sobolev spaces $H^s$, $s \in \mathbb{R}$. The eigen-distributions $v_{i,j}, w_{i,j}$ of the operator $\hat{F}$ which appear in Corollary 1 are also represented.

# 4 Spectrum of resonances

We give now the main result of this paper. Its proof relies on semi-classical analysis and is inspired by the study of resonances in open quantum systems [1, 28]. It is also inspired from a previous work [21] performed within a simple and illuminating model, namely analytical hyperbolic map on the torus. In some sense it shows a close analogy between Ruelle resonances and quantum resonances. The essential point of this approach is to find the discrete spectrum of resonances of the operator $\hat{F}$ in the Sobolev space of distributions $H^m$, thanks to a conjugacy by the escape operator $\hat{A}_m$ defined in Section 3.1. First observe that the operator $\hat{F}$ defined in Eq.(7) extends by duality to the distribution space $\mathcal{D}'(M)$ by

$$\hat{F}(\alpha)(\varphi) = \alpha\left(\hat{F}^*(\varphi)\right)$$

where $\alpha \in \mathcal{D}'(M)$, $\varphi \in C^\infty(M)$ and $\hat{F}^*$ is the $L^2$−adjoint operator. One checks easily that for $\psi, \varphi \in L^2(M)$, $\hat{F}^*$ is given by $\left(\hat{F}^*\varphi\right)(y) = \left(\varphi \circ f^{-1}\right)(y)\left|D_{f^{-1}(y)}f\right|^{-1}$.

**Theorem 1.** *Let $m$ be a function which satisfies the hypothesis of Lemma 2. $\hat{F}$ leaves the anisotropic Sobolev space $H^m$ globally invariant. The operator*

$$\hat{F} : H^m \to H^m$$

*is a bounded operator and can be written*

$$\hat{F} = \hat{r}_m + \hat{k}_m \tag{32}$$

*where $\hat{k}_m$ is a compact operator and $\|\hat{r}_m\| \leq \varepsilon_m = \left\|\hat{F}\right\|_{L^2} e^{-ca}$ with constants $c, a > 0$ defined in Lemma 2. Consequently, the essential spectral radius is smaller than $\varepsilon_m$, which means that $\hat{F}$ has a discrete spectrum $\lambda_i$ outside the circle of radius $\varepsilon_m$.*

**Remarks :**

- $\left\|\hat{F}\right\|_{L^2}$ depends on the choice of the density $dx$. We could have $\left\|\hat{F}\right\|_{L^2}$ closer to 1 by another choice of $dx$.
- Notice that 1 is an eigenvalue of $\hat{F}$, with constant eigenfunction. We will see in Corollary 2 that the spectral radius of $\hat{F}$ is one, i.e. that there are no eigenvalues outside the unit circle.
- For future purpose, let $\varepsilon > 0$ and $\mathcal{O}_\varepsilon$ denotes the set of order functions $m$ which satisfy the hypothesis of Lemma 2, and such that $\varepsilon_m = \left\|\hat{F}\right\|_{L^2} e^{-ca} < \varepsilon$:

$$\mathcal{O}_\varepsilon \stackrel{\text{def}}{=} \left\{ m \quad / \, \varepsilon_m = \left\|\hat{F}\right\|_{L^2} e^{-ca} < \varepsilon \right\} \tag{33}$$

The set $\mathcal{O}_\varepsilon$ is non empty.

*Proof.* We use the unitary map between $H^m$ and $L^2(M)$ given in Eq.(23), and consider $\hat{Q}_m \stackrel{\text{def}}{=} \hat{A}_m \hat{F} \hat{A}_m^{-1} : L^2(M) \to L^2(M)$, defined on a dense domain, and which is unitary equivalent to $\hat{F} : H^m \to H^m$:

$$\begin{array}{ccc} L^2(M) & \stackrel{\hat{Q}_m}{\to} & L^2(M) \\ \downarrow \hat{A}_m^{-1} & \circlearrowleft & \downarrow \hat{A}_m^{-1} \\ H^m & \stackrel{\hat{F}}{\to} & H^m \end{array}$$

Instead of working with $\hat{Q}_m$ directly, it is more convenient to consider

$$\hat{P}_m \stackrel{\text{def}}{=} \hat{F}^{-1} \hat{Q}_m = \left(\hat{F}^{-1} \hat{A}_m \hat{F}\right) \hat{A}_m^{-1}.$$

It follows from *Egorov's Theorem* 9, that the product $\hat{F}^{-1}\hat{A}_m\hat{F}$ is a PDO in $\Psi_\rho^{m\circ F(x,\xi)}$ whose symbol is $A_m \circ F$ modulo subleading terms in $S_\rho^{m\circ F(x,\xi)-(2\rho-1)}$. On the other hand, *the composition Theorem 8 for PDO* tells that $\hat{P}_m$ is a PDO in $\Psi_\rho^{m\circ F(x,\xi)-m(x,\xi)}$ whose symbol is $P_m = \frac{A_m \circ F}{A_m}$ modulo subleading corrections in $S_\rho^{m\circ F(x,\xi)-m(x,\xi)-(2\rho-1)}$. From the construction of the escape function $A_m$, Eq.(13) insures that $\hat{P}_m \in \Psi_\rho^0$. On the other hand Eq.(15) gives

$$\limsup P_m \leq e^{-a.c}.$$

This allows us to apply the *Lemma 14 of $L^2$-continuity* and obtain that for any $\varepsilon > 0$, $\hat{P}_m$ decomposes as

$$\hat{P}_m = \hat{p}_\varepsilon + \hat{k}_\varepsilon$$

with $\hat{k}_\varepsilon \in \Psi^{-\infty}$ a smoothing operator and $\|\hat{p}_\varepsilon\| \leq e^{-a.c} + \varepsilon$. Finally, we multiply on the left by $\hat{F}$ to obtain

$$\hat{Q}_m = \hat{F}\hat{p}_\varepsilon + \hat{F}\hat{k}_\varepsilon.$$

The second term is smoothing, hence compact, while the first one has an operator norm bounded by $(e^{-ac} + \varepsilon)\left\|\hat{F}\right\| = Ce^{-ac}$, with any $C > \left\|\hat{F}\right\|$ and the choice $\varepsilon = e^{-ac}\left(\frac{C-\|\hat{F}\|}{\|\hat{F}\|}\right)$. We have shown the claimed spectral results for $\hat{Q}_m : L^2(M) \to L^2(M)$ and therefore for $\hat{F} : H^m \to H^m$.

$\square$

**Corollary 1.** *Let $\varepsilon > 0$ and let $m \in \mathcal{O}_\varepsilon$ be an order function as defined in Eq.(33). If we denote by $\pi$ the spectral projector associated to $\hat{F} : H^m \to H^m$ outside the disk of radius $\varepsilon$, and $\hat{K} \stackrel{\text{def}}{=} \hat{\pi}\hat{F}$, $\hat{R} \stackrel{\text{def}}{=} (1 - \hat{\pi})\hat{F}$, then we have a spectral decomposition*

$$\hat{F} = \hat{K} + \hat{R}, \qquad \hat{K}\hat{R} = \hat{R}\hat{K} = 0 \tag{34}$$

*and*

1. *The spectral radius of $\hat{R}$ is smaller than $\varepsilon$.*

2. *$\hat{K}$ has finite rank. Its spectrum has generalized eigenvalues $\lambda_i$ (counting multiplicity) called the **Ruelle resonances**, with $\varepsilon < |\lambda_i|$. The general Jordan decomposition of $\hat{K}$ can be written*

$$\hat{K} = \sum_{i \geq 0, |\lambda_i| > \varepsilon} \left( \lambda_i \sum_{j=1}^{d_i} v_{i,j} \otimes w_{i,j} + \sum_{j=1}^{d_i - 1} v_{i,j} \otimes w_{i,j+1} \right) \tag{35}$$

*with $d_i$ the dimension of the Jordan block associated to the eigenvalue $\lambda_i$, with $v_{i,j} \in H^m$, $w_{i,j} \in H^{-m}$ ($w_{i,j}$ is viewed as a linear form on $H^m$ with the duality Eq.(28)). They satisfy $w_{i,j}(v_{k,l}) = \delta_{ik}\delta_{jl}$.*

3. *The distributions $v_{i,j}, w_{i,j}$ and the corresponding eigenvalues $\lambda_i$ do not depend on the choice of $m$, but are intrinsic to the operator $\hat{F}$:*

$$v_{i,j} \in \left( \bigcap_{m \in \mathcal{O}_{|\lambda_i|}} H^m \right), \qquad w_{i,j} \in \left( \bigcap_{m \in \mathcal{O}_{|\lambda_i|}} H^{-m} \right). \tag{36}$$

*In other words, $v_{i,j}$ are smooth in every direction except in the unstable direction which contains their wave front ([29] p.27):*

$$WF(v_{i,j}) \subset E_u^*.$$

*Similarly $w_{i,j}$ are smooth except in the stable direction which contains their wave front:*

$$WF(w_{i,j}) \subset E_s^*.$$

*The resolvent $\left(z - \hat{F}\right)^{-1}$ has a meromorphic extension from $C^\infty(M)$ to $\mathcal{D}'(M)$, whose poles are the $\lambda_i$.*

*Proof.* Points 1 and 2 are immediate consequences of Theorem 1. The projector $\pi$ can be obtained from an integral of the resolvent $\hat{R}(z) = \left(z - \hat{F}\right)^{-1} : H^m \to H^m$ on a circular contour of radius $\varepsilon$.

We will prove now point 3 namely that the spectrum and eigen-distribution do not depend on $m$. Let $\varepsilon > 0$, and $m, m' \in \mathcal{O}_\varepsilon$ (defined in Eq.(33)), and suppose first that $m' \geq m$. From Eq.(26), one has $H^{m'} \subset H^m$. Let $\hat{F}_m$ (resp. $\hat{F}_{m'}$) denotes the restriction of $\hat{F}$ to the distribution space $H^m$ (resp. $H^{m'}$). From Theorem 1, both $\hat{F}_m$ and $\hat{F}_{m'}$ are bounded operators and have essential spectrum radius less than $\varepsilon$. For $|z|$ large enough, the resolvents of $\hat{F}_m$ and $\hat{F}_{m'}$ are equal on $H^{m'}$ because one can write

$$\hat{R}_{m'}(z) = \left(z - \hat{F}_{m'}\right)^{-1} = \frac{1}{z}\left(1 + \sum_{n=1}^{\infty}\left(\frac{\hat{F}_{m'}}{z}\right)^n\right) = \hat{R}_m(z)$$

since $\hat{F}_{m'} = \hat{F}_m$ on $H^{m'}$ and the sum is convergent. By meromorphic continuation, the resolvents also coincide for $|z| > \varepsilon$. By explicit contour integral of the resolvent on a circle of radius $\varepsilon$, one deduces that the corresponding finite rank operators $\hat{K}_{m'}$ and $\hat{K}_m$ are equal on $H^{m'}$. But since $H^{m'}$ is dense in $H^m$, one deduces that there are no eigenspaces of $\hat{K}_m$ outside $H^{m'}$. Therefore the eigen-distributions $v_{i,j}$ belongs to $H^{m'}$.

Now let $m, m'' \in \mathcal{O}_\varepsilon$ be any two order functions (with no mutual inclusion). From the explicit construction given in the proof of Lemma 2, one can find $m' \in \mathcal{O}_\varepsilon$, such that $m' \geq m$ and $m' \geq m''$. Then the above argument shows that $v_{i,j} \in H^{m'} \subset \left(H^m \cap H^{m''}\right)$.

Similar (but dual) arguments for the operator $\hat{F}^* : H^{-m} \to H^{-m}$ show that its eigenvectors $w_{i,j}$ are in $\left(H^{-m} \cap H^{-m''}\right)$. We have obtained Eq.(36).

Since $C^\infty \subset H^m$ for any $\varepsilon > 0$, and any $m \in \mathcal{O}_\varepsilon$, and $H^m \subset \mathcal{D}'(M)$, we deduce that $\left(z - \hat{F}\right)^{-1} : C^\infty(M) \to \mathcal{D}'(M)$ admits a meromorphic extension on $\mathbb{C} \setminus \{0\}$.

□

# 5 Asymptotic expansion for dynamical correlation functions

One usually obtains much information on $\hat{F}$ and the dynamical system $f$ through the study of dynamical correlation functions.

**Definition 3.** *For $\psi_1, \psi_2 \in L^2(M)$ and $n \in \mathbb{Z}$, define the **Lebesgue dynamical correlation function***

$$C^{Leb}_{\psi_2,\psi_1}(n) \stackrel{\text{def}}{=} \left(\psi_2, \hat{F}^n \psi_1\right) = \int \overline{\psi}_2(x)\, \psi_1(f^n(x))\, d\mu_{Leb} \tag{37}$$

**Definition 4.** *The diffeomorphism $f$ is called* ***Lebesgue-mixing*** *if there exists an invariant measure*[8] $\mu_{srb}$ *called the* ***Sinai-Ruelle-Bowen (SRB) measure*** *such that for any $\psi_1, \psi_2 \in C^\infty(M)$,*

$$C^{Leb}_{\psi_2,\psi_1}(n) \underset{n\to\infty}{\longrightarrow} \left(\int \overline{\psi}_2 d\mu_{Leb}\right)\left(\int \psi_1 d\mu_{srb}\right) \tag{38}$$

**Definition 5.** *For $\psi_1, \psi_2 \in L^2(M)$ and $n \in \mathbb{Z}$, define the* ***SRB dynamical correlation function***

$$C^{srb}_{\psi_2,\psi_1}(n) \stackrel{\text{def}}{=} \int \overline{\psi}_2(x)\,\psi_1(f^n(x))\,d\mu_{srb} \tag{39}$$

We show in Theorem 3 that Lebesgue-mixing implies **SRB-mixing**, i.e.

$$C^{srb}_{\psi_2,\psi_1}(n) \underset{n\to\infty}{\longrightarrow} \left(\int \overline{\psi}_2 d\mu_{srb}\right)\left(\int \psi_1 d\mu_{srb}\right) \tag{40}$$

which is also the usual definition of mixing.

Let us mention the following conjecture ([5] p. 575, foot-note (2), or [30] p. 7.)

**Conjecture 1.** *A smooth Anosov diffeomorphism $f : M \to M$ on a connected compact manifold $M$ is Lebesgue-mixing.*

In the particular case where $f$ preserves a smooth measure $dx$ (so $\hat{F}$ is unitary on $L^2(M)$), this has been proved by Anosov in his PhD-thesis [31] (see [32] Theorem 6.3.1). In Section 6 we provide a different proof entirely based on the semi-classical approach developed in this paper.

We will assume Lebesgue-mixing in Theorem 3.

Theorems 2 and 3 below have been obtained before with various degrees of generality.

## 5.1 The Lebesgue correlation function

The Ruelle resonances $\lambda_i$ and associated distributions $v_{i,j}, w_{i,j}$ have been defined in Corollary 1.

**Theorem 2.** *For any $\psi_1, \psi_2 \in C^\infty(M)$, $\varepsilon > 0$ such that $\varepsilon \neq |\lambda_i|, \forall i$, and $n \geq 1$, one has*

$$C^{Leb}_{\psi_2,\psi_1}(n) = \sum_{i\geq 0, |\lambda_i|>\epsilon} \sum_{k=0}^{min(n,d_i-1)} C_n^k \lambda_i^{n-k} \sum_{j=1}^{d_i-k} v_{i,j}\left(\overline{\psi_2}\right) w_{i,j+k}(\psi_1) + \|\psi_1\|_{H^m} \|\psi_2\|_{H^{-m}} O_\varepsilon(\varepsilon^n). \tag{41}$$

*with any $m \in \mathcal{O}_\varepsilon$ (defined in Eq.(33)) and $C_n^k := \frac{n!}{(n-k)!k!}$.*

**Remarks:**

- More generally Eq.(41) still holds for $\psi_1 \in H^m$ and $\psi_2 \in H^{-m}$, with $m \in \mathcal{O}_\varepsilon$.
- The right hand side of Eq.(41) is complicated by the possible presence of "Jordan blocks". In the case where the spectrum $\lambda_i$ is simple ($\lambda_i \neq \lambda_j$) it reads more simply

$$C^{Leb}_{\psi_2,\psi_1}(n) = \sum_{i\geq 0, |\lambda_i|>\epsilon} \lambda_i^n v_i\left(\overline{\psi_2}\right) w_i(\psi_1) + O_\varepsilon(\varepsilon^n).$$

*Proof.* Theorem 2 is deduced from Corollary 1. For any $\varepsilon > 0$, let $m \in \mathcal{O}_\varepsilon$. For any $n \geq 0$ we have $\hat{F}^n = \hat{K}^n + \hat{R}^n$ and $\left\|\hat{R}^n\right\|_{H^m} = O_\varepsilon(\varepsilon^n)$. If $\psi_1 \in H^m$, $\psi_2 \in H^{-m}$ then we use Eq.(29) to write

$$\begin{aligned} C^{Leb}_{\psi_2,\psi_1}(n) &= \left(\psi_2, \hat{F}^n \psi_1\right) \\ &= \left(\psi_2, \hat{K}^n \psi_1\right) + \left(\psi_2, \hat{R}_a^n \psi_1\right) \\ &= \left(\psi_2, \hat{K}^n \psi_1\right) + \|\psi_2\|_{H^m} \|\psi_1\|_{H^{-m}} O_\varepsilon(\varepsilon^n) \end{aligned} \tag{42}$$

Using the Jordan Block decomposition of $\hat{K}$, Eq.(35), and Eq.(28), we have

$$\begin{aligned} \left(\psi_2, \hat{K}^n \psi_1\right) &= \sum_{i\geq 0, |\lambda_i|>\varepsilon} \sum_{k=0}^{min(n,d_i-1)} C_n^k \lambda_i^{n-k} \sum_{j=1}^{d_i-k} (\psi_2, v_{i,j}) w_{i,j+k}(\psi_1), & (43) \\ &= \sum_{i\geq 0, |\lambda_i|>\varepsilon} \sum_{k=0}^{min(n,d_i-1)} C_n^k \lambda_i^{n-k} \sum_{j=1}^{d_i-k} v_{i,j}\left(\overline{\psi_2}\right) w_{i,j+k}(\psi_1) & (44) \end{aligned}$$

We have obtained Eq.(41).

$\square$

**Corollary 2.** *For any $i \geq 0$, $|\lambda_i| \leq 1$. Therefore the spectral radius of the operator $\hat{F} : H^m \to H^m$ is one. If an eigenvalue is on the unit circle, $|\lambda_i| = 1$, then $d_i = 1$, i.e. it has no Jordan block.*

*Proof.* Since $\hat{F}\varphi = \varphi \circ f$, it is clear that $\lambda_0 = 1$ is an eigenvalue for the constant function. For all $n$, and $\psi_1, \psi_2 \in C^\infty(M)$, one has $\left(\psi_2, \hat{F}^n \psi_1\right) = \int \overline{\psi_2}(x)\, \psi_1(f^n(x))\, d\mu_{Leb}$ hence

$$\left|C^{Leb}_{\psi_1,\psi_2}(n)\right| = \left|\left(\psi_2, \hat{F}^n \psi_1\right)\right| \leq |\psi_2|_{C^0} |\psi_1|_{C^0} \operatorname{Vol}(M) \tag{45}$$

is bounded uniformly with respect to $n$.

Suppose that $\lambda_i > 1$. Since $C^\infty(M)$ is dense in $H^m$ and $H^{-m}$, there exists $\psi_1, \psi_2 \in C^\infty(M)$ such that $v_{i,1}\left(\overline{\psi_2}\right) \neq 0$ and $w_{i,1}(\psi_1) \neq 0$. Then $\lambda_i^n v_{i,1}\left(\overline{\psi_2}\right) w_{i,1}(\psi_1)$ would diverge for $n \to \infty$, and Eq.(41) implies that $C^{Leb}_{\psi_2,\psi_1}(n)$ would diverge also, in contradiction with Eq.(45).

Similarly, suppose that $|\lambda_i| = 1$, but $d_i \geq 2$. There exists $\psi_1, \psi_2 \in C^\infty(M)$ such that $v_{i,j}\left(\overline{\psi_2}\right) \neq 0$ and $w_{i,j}(\psi_1) \neq 0$. Then the term $k = d_i - 1$ in Eq.(41) which contains $C_n^k$ diverges as $n^{d_i-1}$ for $n \to \infty$. Eq.(41) implies that $C^{Leb}_{\psi_2,\psi_1}(n)$ would diverge also, in contradiction with Eq.(45).

□

**Corollary 3.** *The following two propositions are equivalent:*

1. *$f$ is Lebesgue-mixing.*
2. *$\lambda_0 = 1$ is simple, $v_0 = Leb$ is the Lebesgue measure and $w_0 = \mu_{srb}$ the SRB measure. The other eigenvalues satisfy $|\lambda_i| < 1$, $i \geq 1$.*

Therefore:

$$C^{Leb}_{\psi_2,\psi_1}(n) \underset{n\to\infty}{\longrightarrow} v_0\left(\overline{\psi_2}\right) w_0(\psi_1)$$

**Remarks:**

- It turns out that the SRB correlation function $C^{srb}_{\psi_2,\psi_1}(n)$ admits an asymptotic expansion similar to Eq.(41), see Theorem 3 below.
- Without the Lebesgue-mixing assumption and if Conjecture 1 is wrong, there may be a finite number of eigenvalues on the unit circle.

*Proof.* The Lebesgue-mixing assumption implies that $C^{Leb}_{\psi_1,\psi_2}(n)$ converges for $n \to \infty$. The constant function $v_0 = 1$ is obviously an eigenfunction of $\hat{F}$ with eigenvalue $\lambda_0 = 1$. There are no other eigenvalues on the unit circle otherwise from Eq.(41), $C^{Leb}_{\psi_1,\psi_2}(n)$ would not converge for $n \to \infty$. We obtain $C^{Leb}_{\psi_1,\psi_2}(n) = \left(\psi_2, \hat{F}^n\psi_1\right) \underset{n\to\infty}{\to} v_0\left(\overline{\psi_2}\right) w_0(\psi_1)$ with $w_0 = \mu_{srb}$ from Definition 4. But Eq.(45) also implies that $|w_0(\psi_1)| \leq C\,|\psi_1|_{C^0}$. Therefore $w_0$ is distribution of order 0, hence defines a measure.

□

## 5.2 The SRB correlation function

We have shown above that $\mu_{srb} = w_0 \in H^{-m}$ for any $m \in \mathcal{O}_\varepsilon$ and $\varepsilon < 1$. Notice that from Eq.(27), we have $w_0\psi_2 \in H^{-m}$ for any $\psi_2 \in C^\infty(M)$. Since $v_{i,j} \in H^m$ then Eq.(28) implies that $v_{i,j}(w_0\psi_2) = (v_{ij}, w_0\psi_2)$ makes sense.

**Theorem 3.** *Assume that $f$ is Lebesgue-mixing. If $\psi_1, \psi_2 \in C^\infty(M)$ then the asymptotic behavior of the SRB correlation function Eq.(40) is given by:*

$$\begin{aligned} C^{srb}_{\psi_2,\psi_1}(n) &= w_0\left(\overline{\psi_2}\right) w_0(\psi_1) \\ &\quad + \sum_{i\geq 0, |\lambda_i|>\epsilon} \sum_{k=0}^{min(n,d_i-1)} C_n^k \lambda_i^{n-k} \sum_{j=1}^{d_i-k} v_{i,j}\left(\overline{w_0\psi_2}\right) w_{i,j}(\psi_1) + O_\varepsilon(\varepsilon^n)\,. \end{aligned} \tag{46}$$

*In particular $f$ is SRB-mixing :*

$$C^{srb}_{\psi_2,\psi_1}(n) \underset{n\to\infty}{\longrightarrow} w_0\left(\overline{\psi_2}\right) w_0(\psi_1) = \left(\int \overline{\psi}_2 d\mu_{srb}\right)\left(\int \psi_1 d\mu_{srb}\right)$$

*and the convergence is exponentially fast.*

*Proof.* Using Eq.(28) and Eq.(31), we start with an equivalent expression for the SRB correlation function:

$$C^{srb}_{\psi_2,\psi_1}(n) = \int \overline{\psi}_2(x)\,\psi_1(f^n(x))\,d\mu_{srb} = \left(w_0, \overline{\psi}_2\hat{F}^n\psi_1\right) = \left(w_0\psi_2, \hat{F}^n\psi_1\right)$$

Then as in Eq.(42), we use the decomposition Eq.(35) and deduce Eq.(46), since $v_0\left(\overline{w_0\psi_2}\right) w_0(\psi_1) = w_0\left(\overline{\psi_2}\right) w_0(\psi_1)$.

□

# 6 Mixing of Anosov maps preserving a smooth measure

In the particular case where $f$ preserves a smooth measure $dx$, ergodicity of $f$ and therefore mixing, has been proved by Anosov in his PhD thesis [31] (see [32] Theorem 6.3.1). In this section we provide a different proof entirely based on the semi-classical approach developed in this paper.

**Theorem 4.** *Suppose that $f$ preserves a smooth measure $dx$. Then on the unit circle, there is no Ruelle resonance, except 1 with multiplicity one (equivalently $f$ is Lebesgue-mixing from Corollary 3).*

*Proof. of Theorem 4.* From Corollary 1 and Corollary 2 an eigenvalue $\lambda = e^{i\theta}$ on the unit circle would have no Jordan Block and would correspond to an eigen-vector $u \in H^m$, $\hat{F}u = e^{i\theta}u$. Lemma 4 and Lemma 3 below imply that $u$ is a constant function and $\lambda = 1$.

□

The following Lemma contains the global aspect of the problem.

**Lemma 3.** *If* [9] *$u \in C^1(M)$ and $\hat{F}u = \lambda u$, with $|\lambda| = 1$ then $u$ is a constant function, and $\lambda = 1$.*

*Proof.* Let us assume that $u \in C^1(M)$, with $\hat{F}u = \lambda u$, $|\lambda| = 1$. Let $u_n := \hat{F}^n u = u \circ f^n$. One has $u_n = \lambda^n u$, therefore $|du_n|_\infty = |du|_\infty < \infty$ is bounded uniformly with respect to $n$, since $M$ is compact. On the other hand $(du_n)_{f^{-n}(x)} = (Df^n)^{\mathbf{t}}_{f^{-n}x}(du)_x$. Suppose that there exists $x \in M$ such that $du_x \neq 0$. If $du_x \notin E_s^*(x)$ (stable direction) then $\left|(du_n)_{f^{-n}(x)}\right|$ diverge when $n \to +\infty$. If $du_x \in E_s^*(x)$ then $\left|(du_n)_{f^{-n}(x)}\right|$ diverge when $n \to -\infty$. This contradicts $|du_n|_\infty < \infty$, therefore $du = 0$. $M$ is connected therefore $u$ is constant.

□

**Lemma 4.** *Assume there exists a positive smooth density $dx$ on $M$ which is invariant under the map $f$. Let $u \in H^m$, $\hat{F}u = e^{i\theta}u$. Then $u \in C^\infty(M)$.*

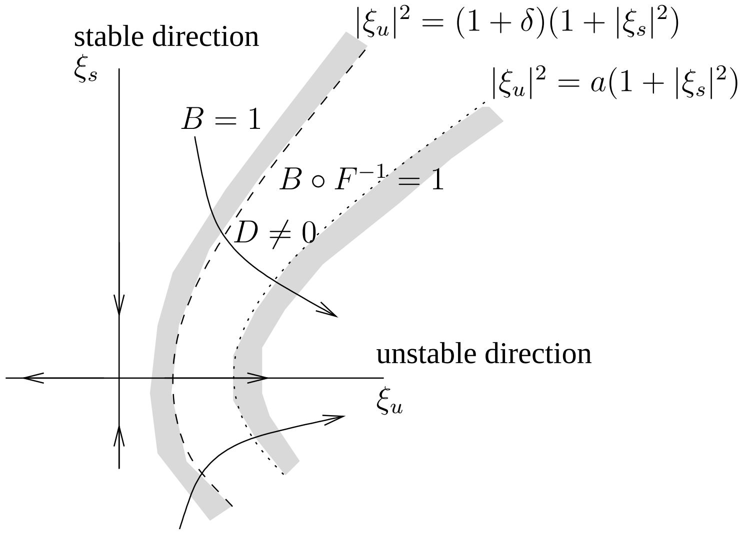


Figure 5: A distribution $u \in H^m$ is regular in the stable direction. If $\hat{F}u = e^{i\theta}u$, the idea of the proof of Lemma 48 is to propagate this regularity under the map $F$ towards the unstable direction. For that purpose, we propagate the symbol $B$ and establish in (56) that $u$ is semi-classically negligible in a zone where $D \neq 0$.

*Proof.* We shall make use of some $h$-pseudodifferential calculus[10]. Assume there exists a positive smooth density $dx$ on $M$ which is invariant under the map $f$. Then $\hat{F} : L^2(M, dx) \to L^2(M, dx)$ is unitary. The idea of the proof is to use Corollary 1 which states that $u$ is $C^\infty$ in every direction except the unstable direction, and use the unitary of $\hat{F}$, to deduce that $u$ is $C^\infty$ also in the unstable direction (by propagation).

It is easy to see that there exist symbols

$$0 \leq B(x,\xi), C(x,\xi) \in S^0(T^*M),$$

such that

$$1 = B^2 + C^2, \tag{47}$$

and $B(x,\xi) = 1$ on the set

$$\|\xi_u\|^2 \leq (1-\delta)(1+\|\xi_s\|^2) \tag{48}$$

with support in

$$\|\xi_u\|^2 \leq (1+\delta)(1+\|\xi_s\|^2), \tag{49}$$

where we choose $\delta > 0$ sufficiently small. See Figure 5.

Then $B \circ F^{-1} \in S^0$ is equal to 1 on a set

$$\|\xi_u\|^2 \leq a(1+\|\xi_s\|^2) \tag{50}$$

and has its support in a set

$$\|\xi_u\|^2 \leq b(1+\|\xi_s\|^2), \tag{51}$$

[10]In particular $\xi$ is quantized into the operator $hD_x = -ih\partial/\partial x$ while for ordinary PDO, $\xi$ is quantized into $D_x = -i\partial/\partial x$.

where $1 < a < b$ are independent of $\delta$ when $\delta > 0$ is small enough. Now we can construct corresponding $h$-pseudodifferential operators $\hat{B}, \hat{C}$ such that

$$1 = \hat{B}^2 + \hat{C}^2 + K, \tag{52}$$

where $K$ is negligible in the sense that

$$K = \mathcal{O}(h^N) : H^{-N} \to H^N, \forall N \in \mathbb{N}, \tag{53}$$

and such that the symbol of $\hat{B}$ is equal to $B$ modulo $hS^{-1}$, and modulo $h^\infty S^{-\infty}$ it is equal to 1 on the set (48) and has its support in the set (49). It follows from Egorov Theorem that $\hat{F}\hat{B}\hat{F}^{-1} = \hat{F}\hat{B}\hat{F}^*$ has the corresponding properties with the sets (48), (49) replaced by (50), (51).

We can find a self-adjoint $h$-pseudodifferential operator $\hat{D}$ with symbol of class $S^0$, such that

$$(\hat{F}\hat{B}\hat{F}^*)^2 - \hat{B}^2 = \hat{D}^2 + L, \tag{54}$$

where $L$ is negligible as in (53). In fact, in the region (48) we can take $\hat{D} = 0$ and when we further approach the unstable directions we first have $\hat{F}\hat{B}\hat{F}^* = 1$ micro-locally, so that the left hand side in (54) is $\equiv 1 - \hat{B}^2 \equiv \hat{C}^2$, so that we can take $\hat{D} = \hat{C}$. Even closer to the unstable directions, we get outside the support of $\hat{B}$ and we can take $\hat{D} = \hat{F}\hat{B}\hat{F}^*$.

Now, let $u \in H^m$ be as in the proposition and write

$$\hat{B}u = e^{i\theta}\hat{B}\hat{F}^{-1}u.$$

Thanks to the properties of $\hat{B}$ and $m$, this quantity belongs to $L^2$, and using the unitarity of $\hat{F}$, we get

$$\|\hat{B}u\|^2 = \|\hat{F}\hat{B}\hat{F}^{-1}u\|^2.$$

Combining this with (54), we get

$$0 = \|\hat{F}\hat{B}\hat{F}^{-1}u\|^2 - \|\hat{B}u\|^2 = \|\hat{D}u\|^2 + (Lu|u), \tag{55}$$

and since $L$ is negligible,

$$\|\hat{D}u\| = \mathcal{O}(h^\infty). \tag{56}$$

Since $\hat{D}$ is semi-classically elliptic in the region

$$(1+\delta)(1 + \|\xi_s\|^2) \le \|\xi_u\|^2 \le a(1 + \|\xi_s\|^2),$$

we see that $u$ is micro-locally $\mathcal{O}(h^\infty)$ in the region (replacing $\xi \to h\xi$)

$$(1+\delta)(\frac{1}{h^2} + \|\xi_s\|^2) \le \|\xi_u\|^2 \le a(\frac{1}{h^2} + \|\xi_s\|^2),$$

and letting $h \to 0$, we see that $u$ has no wave-front set in a conical neighborhood of $E_u^*$. Since we already know that $\mathrm{WF}(u) \subset E_u^*$, we conclude that $u \in C^\infty$, and this ends the proof of Lemma 4. □

# 7 Truncation and numerical calculation of the resonance spectrum

Let $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be a $C^\infty$ function such that $\chi(x) = 1$, if $x \leq 1$, and $\chi(x) = 0$, if $x \geq 2$. For $r > 0$, let the function $\chi_r$ on $T^*M$ be defined by $\chi_r(x,\xi) = \chi(|\xi|/r)$. Let the **truncation operator** be:

$$\hat{\chi_r} \stackrel{\text{def}}{=} Op(\chi_r)$$

Notice that $\hat{\chi_r}$ is a smoothing operator which truncates large components in $\xi$. In [21] section 2.1.4 and references therein, we interpret $\hat{\chi_r}$ as a "noisy operator" with a noise of amplitude $1/r$.

**Theorem 5.** $\left(\hat{F}\hat{\chi_r}\right)$ *is a smoothing operator. For any $\varepsilon > 0$ the spectrum of $\left(\hat{F}\hat{\chi_r}\right)$ in $L^2(M)$ outside the disk of radius $\varepsilon$, converges for $r \to \infty$, towards the spectrum of Ruelle resonances $(\lambda_i)_i$, counting multiplicities. The eigenspaces converge towards the eigen-distributions.*

**Remarks:**

1. Theorem 5 gives a practical way to compute numerically the resonance spectrum: one expresses the operator $\hat{F}$ in a discrete basis of $L^2(M)$, truncates it smoothly (according to the operator $\hat{\chi_r}$), and diagonalizes the resulting matrix numerically. See Figure 6.

2. Theorem 5 gives also a simple way to establish a relation between Ruelle resonances $\lambda_i$ and the periodic points of the map $f$, via dynamical zeta functions, see e.g. [11, 33]. On one hand the Atiyah-Bott fixed point formula ([34] corollary 5.4 p.393), gives for any $n \geq 1$,

$$\mathbf{Tr}\left(\left(\hat{F}\hat{\chi_r}\right)^n\right) \underset{r\to\infty}{\longrightarrow} \sum_{x \in Fix(f^n)} \frac{1}{|\det(1 - D_x f^n)|}$$

   and on the other hand, the zeros of the dynamical zeta function

$$d(z) = \exp\left(-\sum_{n\geq 1} \frac{z^n}{n} \mathbf{Tr}\left(\left(\hat{F}\hat{\chi_r}\right)^n\right)\right) = \mathbf{det}\left(1 - z\left(\hat{F}\hat{\chi_r}\right)\right)$$

   converge towards $(1/\lambda_i)_i$.

*Proof.* Let $\varepsilon > 0$, and $m \in \mathcal{O}_{\varepsilon/2}$. From Eq.(32), the operator $\hat{F} : H^m \to H^m$ can be written $\hat{F} = \hat{r} + \hat{k}$, with $\|\hat{r}\| \leq \varepsilon/2$, and $\hat{k}$ compact. In $H^m$, the operator $\hat{\chi}_r \underset{r\to\infty}{\longrightarrow} Id$ converges strongly, and $\|\hat{\chi_r}\| \leq C_r \underset{r\to\infty}{\longrightarrow} 1$, in particular $\|\hat{\chi_r}\| \leq 2$ for $r$ large enough. For $|z| > \varepsilon$ write

$$\hat{F} - z = \hat{r} - z + \hat{k} = (\hat{r} - z)\left(1 + (\hat{r} - z)^{-1}\hat{k}\right)$$

$$\left(\hat{F} - z\right)^{-1} = \left(1 + (\hat{r} - z)^{-1}\hat{k}\right)^{-1}(\hat{r} - z)^{-1}$$

when the first factor on the right is well defined. Similarly

$$\left(\hat{F}\hat{\chi}_r - z\right)^{-1} = \left(1 + (\hat{r}\hat{\chi}_r - z)^{-1}\hat{k}\hat{\chi}_r\right)^{-1}(\hat{r}\hat{\chi}_r - z)^{-1}.$$

Let $z \in \mathbb{C} \setminus \sigma\left(\hat{F}\right)$ and $|z| > \varepsilon$. Then $(\hat{r}\hat{\chi}_r - z)^{-1}$ is well defined since $\|\hat{r}\hat{\chi}_r\| \leq \varepsilon$. Moreover $(\hat{r}\hat{\chi}_r - z)^{-1} \underset{r\to\infty}{\longrightarrow} (\hat{r} - z)^{-1}$ strongly and $(\hat{r}\hat{\chi}_r - z)^{-1}\hat{k}\hat{\chi}_r \underset{r\to\infty}{\longrightarrow} (\hat{r} - z)^{-1}\hat{k}$ in norm since $\hat{k}$ is compact. Therefore $\left(\hat{F}\hat{\chi}_r - z\right)^{-1} \underset{r\to\infty}{\longrightarrow} \left(\hat{F} - z\right)^{-1}$ converges strongly. Now the finite rank operator $\hat{K}$ in the spectral decomposition Eq.(34) can be obtained by a contour integral of the resolvent:

$$\hat{\pi} = \frac{1}{2\pi i}\oint_{|z|=\varepsilon}\left(z - \hat{F}\right)^{-1} dz, \qquad \hat{K} = \hat{\pi}\hat{F}$$

Similarly the spectrum of $\left(\hat{F}\hat{\chi}_r\right)$ outside the disk of radius $\varepsilon$ is the spectrum of the finite rank operator $\hat{K}_r$ given by:

$$\hat{\pi}_r \stackrel{\text{def}}{=} \frac{1}{2\pi i}\oint_{|z|=\varepsilon}\left(z - \hat{F}\hat{\chi}_r\right)^{-1} dz, \qquad \hat{K}_r = \hat{\pi}_r\left(\hat{F}\hat{\chi}_r\right).$$

The operator $\hat{K}_r \underset{r\to\infty}{\longrightarrow} \hat{K}$ converges strongly and therefore in norm since it has finite rank (see Th. 9.19 p.98 in [28]). We deduce Theorem 5.

□

# 8 Conclusion and perspectives

In this paper we have proposed a semi-classical approach for spectral properties of Anosov dynamical systems. We discuss now some possible perspectives for this work.

First, we have treated the case of hyperbolic diffeomorphisms $f$. The case of expansive maps which are not invertible is more simple, and the method we propose works equally well, but need some adaptations. For example the transfer operator $\hat{F}$ has the same definition Eq.(7), but the associated canonical map $F$ on $T^*M$ is now multivalued (its graph on $T^*M \times T^*M$ is well defined).

The case of partially hyperbolic systems (and in particular hyperbolic flows) where there is a neutral direction is very interesting, and there are important recent results concerning

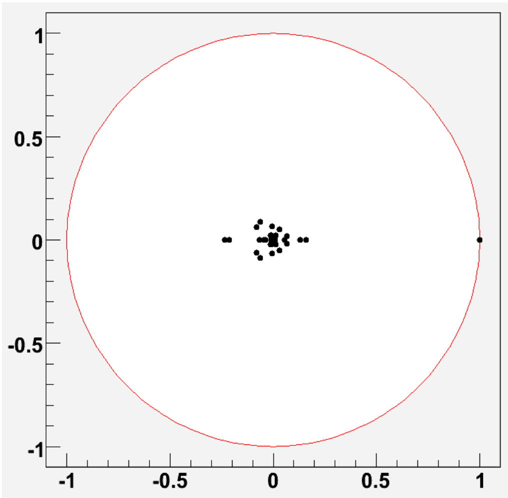


Figure 6: Ruelle resonances $\lambda_i$ obtained numerically for the model Eq.(3), with $\varepsilon = 0.5$.

their spectral properties [35, 36]. For the same reasons explained in the introduction, we think that a semi-classical approach is natural and hopefully fruitful for these systems too.

Finally let us mention the open question mentioned in the conjecture 1. One can wonder if a semi-classical approach similar to Section 6 could help towards the resolution of this.

# A Pseudodifferential operators with variable order

## A.1 Preliminary remarks

### A.1.1 Semi-classical analysis

In this appendix, we provide a self-contained series of analytic tools for studying pseudodifferential operators with slightly more general classes of symbols than usual ones, namely symbols with variable order. All the results we give come from the standard semi-classical analysis but for our special symbol classes, and are given mainly without proof. We refer to [37, 38, 29] for the standard results in semi-classical analysis. Note that the idea of using symbols with variable order is not new. See for example [39, 40, 27].

Semi-classical analysis is a rich theory which gives a sense to pseudodifferential operators $\hat{P}$ of the form

$$\varphi \rightarrow \hat{P}(\varphi)(x) = \int_{\mathbb{R}^{2d}} e^{i\xi(x-y)} P(x,\xi)\,\varphi(y)\,dyd\xi \tag{57}$$

where $P(x,\xi)$ is a smooth function called the **symbol** of $\hat{P}$ and belonging to some appropriate class of functions satisfying certain regularity conditions at infinity. These classes yield to a powerful **symbol calculus**, i.e., a tool for extracting information in an asymptotic way about the operator $\hat{P}$ by means of its symbol. The main results are basically the following :

- Composition of PDO's. Given two PDO's $\hat{P}$ and $\hat{Q}$, the product $\hat{A} = \hat{P}\hat{Q}$ is also a PDO and its symbol is given to leading order in $\xi$ by the product of the symbols $A = PQ$.
- Sobolev continuity. First defined on $C^\infty(\mathbb{R}^n)$, pseudodifferential operators are shown to be continuous between certain Sobolev spaces. Moreover, the corresponding operator norm is estimated by some norms of the derivatives of the symbol.
- Ellipticity, parametrix. A condition called **ellipticity** imposed on a symbol is enough to insure that the corresponding operator is invertible up to a regularizing operator. The "almost-inverse" is called a parametrix.

### A.1.2 Symbols with constant order

The typical class of symbols one considers is the set $S^m_{\rho,\delta} \subset C^\infty(\mathbb{R}^{2n})$ of smooth functions $P(x,\xi)$ which satisfy the following estimates. For any compact subset $K \subset \mathbb{R}^n$ and any multi-index $\alpha, \beta \in \mathbb{N}^n$, there is a constant $C_{K,\alpha,\beta}$ such that

$$\left|\partial^\alpha_\xi \partial^\beta_x P(x,\xi)\right| \le C_{K,\alpha,\beta} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{58}$$

for any $(x,\xi) \in K \times \mathbb{R}^n$. Here $\langle\xi\rangle$ means $\sqrt{1+|\xi|^2}$ and we use the standard multi-indices notation $\partial^\alpha_x f = \frac{\partial^{\alpha_i} f}{\partial x_i^{\alpha_i}} \dots \frac{\partial^{\alpha_n} f}{\partial x_n^{\alpha_n}}$. The number $m \in \mathbb{R}$ is called the (**constant**) **order** of the symbol. These classes of symbols, introduced by Hörmander [41], are quite general and allow one nevertheless to develop a symbol calculus, provided the constants $\rho, \delta$ fulfill certain conditions. The corresponding class of PDO's (Formula 57) is denoted by $\Psi^m_{\rho,\delta}$. The operator $\hat{P}$, denoted sometimes also by $P(x,D)$, is the (**left**) **quantization** of $P$.

Nevertheless, as explained in the introduction, we need to consider symbols with an order $m$ which is no longer constant but depends on the variables $(x,\xi)$. It turns out that these classes are contained in some Hörmander classes (with order equal to $\limsup m$) but one needs to keep track of the fact that the order is variable in order to develop a more general notion of ellipticity, for symbols which would not be elliptic in the usual sense. This will be explained in the rest of this appendix which is devoted to the symbol calculus for symbols with variable order. But before, we need one additional remark about the theory of PDO's on manifolds.

### A.1.3 Pseudodifferential operators on manifold

The usual way of defining PDO's on a para-compact Hausdorff manifold $M$ is to make a partition of the unity of $M$ and use in each chart the semi-classical analysis on $\mathbb{R}^n$. Defined in this way, the symbol of an operator depends unfortunately on the charts. But it turns out that, provided $\frac{1}{2} < \rho \leq 1$ and $\rho + \delta \geq 1$, there is an element of the quotient space $S^m_{\rho,\delta} / S^{m-(\rho-\delta)}_{\rho,\delta}$ called the **principal symbol** which is well-defined independently of the charts. Any member of this equivalence class is a function on the cotangent bundle $T^*M$, also called a principal symbol. For technical reasons it is common and convenient to assume $\rho = 1 - \delta$ and $\rho > \frac{1}{2}$, and to denote $S^m_\rho = S^m_{\rho,\delta}$. We will follow from now on this convention.

It is well-known that many results of semi-classical analysis on a manifold are given in terms of principal symbols. On the other hand, when we consider symbols with a non-constant order function $m(x,\xi) \in T^*M$, we need to manipulate carefully the concept of principal symbol, since there are two different notions depending on whether we consider our symbol to have a variable order $m(x,\xi)$ or a constant order equal to $\limsup m$.

To avoid possibly confusing considerations about principal symbols, we will use a very convenient quantization scheme, developed in [42], which works on Riemannian manifolds $(X, g)$ and provides a notion of **total symbol**. It is defined as follows. First, we fix a cut-off function $\chi \in C^\infty(TM, [0,1])$ which equals 1 on a neighborhood of the zero section $0_{TM}$ and is supported in a neighborhood $W \subset TM$ of $0_{TM}$ in which the exponential map defines a diffeomorphism onto an open neighborhood of the diagonal in $M \times M$. Then, for any $u \in C^\infty(M)$ we define $u_\chi \in C^\infty(TM)$ its semi-classical lift on $TM$ by

$$u_\chi(x,v) = \begin{cases} \chi(v)\, u(\exp_x v) & \text{for } (x,v) \in W \\ 0 & \text{else.} \end{cases}.$$

Finally, for any symbol $p \in C^\infty(T^*M)$ one defines the operator $\hat{P}$ by

$$u \to \hat{P}(u)(x) = \int_{T^*_x X} p(x,\xi)\, \widetilde{u_\chi}(x,\xi)\, d\xi \tag{59}$$

where $\widetilde{f}$ denotes the Fourier transform of a function $f(x,v) \in C^\infty(TM)$ with respect to the $v$ variable, i.e.,

$$\widetilde{f}(x,\xi) = \frac{1}{(2\pi)^n} \int_{T_x X} e^{-i\langle \xi | v \rangle} f(x,v)\, dv.$$

It is shown in [42] that this construction gives rise to a notion of total symbol (called there **normal symbol**) well-defined independently of the cut-off function $\chi$ up to an element of $S^{-\infty}_\rho$. The classes of symbols are defined in the usual way : Fix $m \in \mathbb{R}$ and $\rho > \frac{1}{2}$. Then, a function $p \in C^\infty(T^*M)$ belongs to the class $S^m_\rho$ if in any trivialization $(x,\xi) : T^*M|_U \to \mathbb{R}^{2n}$, for any compact $K \subset U$ and any multi-indices $\alpha, \beta \in \mathbb{N}^n$, there is a constant $C_{K,\alpha,\beta}$ such that

$$\left|\partial^\alpha_\xi \partial^\beta_x p(x,\xi)\right| \leq C_{K,\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+(1-\rho)|\beta|}$$

on $T^*M|_U$. Now the function $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ is defined in term of the norm $|\xi|^2 = g_x(\xi,\xi)$ with the scalar product on $T^*M$ denoted by the same letter $g$.

## A.2 Symbols with variable orders

As usual, *la constante positive $C$, qui est le fidèle compagnon de l'analyste, pourra varier d'une formule à l'autre*[11].

### A.2.1 Definition and main basic properties of $S_\rho^{m(x,\xi)}$

As explained before, we want to develop a symbolic calculus for PDO's whose symbol has an order $m$ which depends on the point $(x,\xi) \in T^*M$. We first need to explain which functions are acceptable as order functions.

**Definition 6.** *An* ***order function*** $m(x,\xi)$ *is an element of* $S_r^0$*, for some* $\frac{1}{2} < r < 1$ *which is also bounded at infinity in* $\xi$*, i.e.,*

$$\sup_{x,\xi \in T^*M} |m(x,\xi)| < \infty.$$

We now define the class of symbols of variable order exactly in the same way as symbols with constant order. We will use the slightly abusive notation $S_\rho^{m(x,\xi)}$ to emphasize the fact that $m(x,\xi)$ is not constant.

**Definition 7.** *Let* $m(x,\xi) \in S_r^0$ *be an order function and* $\frac{1}{2} < \rho < 1$*. A function* $p \in C^\infty(T^*M)$ *belongs to the class* $S_\rho^{m(x,\xi)}$ *if in any trivialization* $(x,\xi) : T^*M|_U \to \mathbb{R}^{2n}$*, for any compact* $K \subset U$ *and any multi-indices* $\alpha, \beta \in \mathbb{N}^n$*, there is a constant* $C_{K,\alpha,\beta}$ *such that*

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \leq C_{K,\alpha,\beta} \langle\xi\rangle^{m(x,\xi)-\rho|\alpha|+(1-\rho)|\beta|}$$

*for any* $(x,\xi) \in T^*M|_U$*.*

As with the usual calculus, for any symbol $p \in S_\rho^{m(x,\xi)}$ and any multi-indices $\alpha, \beta$ we have

$$\partial_\xi^\alpha \partial_x^\beta p \in S_\rho^{m(x,\xi)-\rho|\alpha|+(1-\rho)|\beta|}.$$

This means that $\partial_\xi^\alpha \partial_x^\beta p$ has an order function given by $m(x,\xi) - \rho|\alpha| + (1-\rho)|\beta|$. Similarly, for any two symbols $p \in S_\rho^{m(x,\xi)}$ and $q \in S_{\rho'}^{m'(x,\xi)}$, the point-wise product $p(x,\xi)\, q(x,\xi)$ belongs to $S_{\rho''}^{m(x,\xi)+m'(x,\xi)}$ with $\rho'' = \min(\rho, \rho')$.

Another simple but important property of variable order symbols, is that they belong in fact to some Hörmander class. This follows from the following fact.

[11] A. Unterberger [40]

**Lemma 5.** *For any two non-constant order $m(x,\xi)$ and $m'(x,\xi)$ satisfying $m(x,\xi) \leq m'(x,\xi)$ on $T^*M$ and any $\rho \geq \rho'$, the following holds*

$$S_\rho^{m(x,\xi)} \subset S_{\rho'}^{m'(x,\xi)}.$$

*In particular, one has*

$$S_\rho^{m(x,\xi)} \subset S_{\rho'}^{\sup m} \text{ and } S_\rho^{m(x,\xi)} \subset S_{\rho'}^{\varepsilon+\limsup m} \text{ for any } \varepsilon > 0.$$

*Proof.* Let $p \in S_\rho^{m(x,\xi)}$ be a symbol. For any $\alpha, \beta \in \mathbb{N}^n$, any compact $K \subset M$, the first inclusion comes simply from

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \leq C \langle\xi\rangle^{m(x,\xi)-\rho|\alpha|+(1-\rho)|\beta|} \leq C \langle\xi\rangle^{m'(x,\xi)-\rho'|\alpha|+(1-\rho')|\beta|}.$$

On the other hand, for large enough $\langle\xi\rangle$, $m$ is bounded by $\limsup m + \varepsilon$ for any $\varepsilon > 0$. This provides the second estimate.

□

This property implies in particular that we can use the same class of residual symbols as for Hörmander symbols, namely

$$S^{-\infty} := \bigcap_{m<0} S_\rho^m$$

which is independent of $\rho$. For convenience, we also introduce the class of symbols of any order

$$S_\rho^\infty := \bigcup_{m>0} S_\rho^m.$$

#### A.2.2 Canonical examples

We show now that the natural candidate $\langle\xi\rangle^{m(x,\xi)}$ is indeed a suitable symbol.

**Lemma 6.** *Let $m \in S_\rho^0$ be an order function. The smooth function $p(x,\xi) = \langle\xi\rangle^{m(x,\xi)}$ belongs to $S_{\rho-\varepsilon}^{m(x,\xi)}$ for any $\varepsilon > 0$.*

*Proof.* We will prove that for any $\alpha, \beta$, we have

$$\partial_\xi^\alpha \partial_x^\beta p(x,\xi) = q(x,\xi)\langle\xi\rangle^{m(x,\xi)} \text{ with } q \in S_\rho^{(-\rho+\varepsilon)|\alpha|+(1-\rho+\varepsilon)|\beta|} \tag{60}$$

for any $\varepsilon > 0$. First of all, this is true for first order derivatives. Indeed, for $|\beta| = 1$ we compute the derivative :

$$\partial_x^\beta p(x,\xi) = \left(\partial_x^\beta(\ln(\langle\xi\rangle) m(x,\xi))\right)\langle\xi\rangle^{m(x,\xi)} =: q(x,\xi)\langle\xi\rangle^{m(x,\xi)}.$$

The appearance of logarithmic terms is actually the worse that can happen when differentiating a symbol with variable order, but it is easily controlled. First, $\ln(\langle\xi\rangle)$ is bounded by $\langle\xi\rangle^\varepsilon$ for $\varepsilon > 0$ arbitrarily small. Moreover, the logarithm disappears as soon as we take at least one derivative in $x$ or in $\xi$. Namely, whenever $(\alpha,\beta) \neq (0,0)$ one has $\partial_\xi^\alpha \partial_x^\beta(\ln(\langle\xi\rangle)) \in S_1^{-|\alpha|}$. This shows that $\ln(\langle\xi\rangle) \in S_1^\varepsilon$ for any $\varepsilon > 0$. This means that $\ln(\langle\xi\rangle) m(x,\xi) \in S_1^\varepsilon.S_\rho^0 = S_\rho^\varepsilon$. Then the derivative yields to $q(x,\xi) \in S_\rho^{\varepsilon+1-\rho}$. Similarly, for $|\alpha| = 1$ one shows that $\partial_\xi^\alpha p(x,\xi) = q(x,\xi)\langle\xi\rangle^{m(x,\xi)}$ with $q \in S_\rho^{\varepsilon-\rho}$ for any $\varepsilon > 0$. Let us prove by iteration that Equation (60) holds in general. Suppose it holds for all $\alpha, \beta$ satisfying $|\alpha+\beta| \leq N$ for some $N \in \mathbb{N}$. Then, any $(\alpha', \beta')$ with $|\alpha+\beta| = N+1$ has the form $(\alpha+a, \beta+b)$ with $|a+b| = 1$, i.e., either $(|a|,|b|) = (1,0)$ or $(|a|,|b|) = (0,1)$. In the first case, we want to compute

$$\partial_\xi^a \partial_\xi^\alpha \partial_x^\beta p(x,\xi) = \left(\partial_\xi^a q(x,\xi)\right)\langle\xi\rangle^{m(x,\xi)} + q(x,\xi)\,\partial_\xi^a\left(\langle\xi\rangle^{m(x,\xi)}\right).$$

By assumption, the first term is in $S_\rho^{(-\rho+\varepsilon)|\alpha|+(1-\rho+\varepsilon)|\beta|-\rho}.\langle\xi\rangle^{m(x,\xi)}$ and the second one is in

$$S_\rho^{(-\rho+\varepsilon)|\alpha|+(1-\rho+\varepsilon)|\beta|}.S_\rho^{-\rho+\varepsilon}.\langle\xi\rangle^{m(x,\xi)}.$$

Together, this provides

$$\langle\xi\rangle^{-m(x,\xi)}\,\partial_\xi^a \partial_\xi^\alpha \partial_x^\beta p(x,\xi) \in S_\rho^{(-\rho+\varepsilon)|\alpha+a|+(1-\rho+\varepsilon)|\beta|}.$$

Similarly, we would find

$$\langle\xi\rangle^{m(x,\xi)}\,\partial_x^b \partial_\xi^\alpha \partial_x^\beta p(x,\xi) \in S_\rho^{(-\rho+\varepsilon)|\alpha|+(1-\rho+\varepsilon)|\beta+b|}$$

for $|b| = 1$. This proves Formula (60) for any $\alpha, \beta$ satisfying $|\alpha+\beta| \leq N+1$ and the formula for all $\alpha, \beta$ is proved by induction. Then, we deduce that

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \leq \langle\xi\rangle^{m(x,\xi)-(\rho-\varepsilon)|\alpha|+(1-\rho+\varepsilon)|\beta|}$$

and thus $p(x,\xi) \in S_{\rho-\varepsilon}^{m(x,\xi)}$ for any $\varepsilon > 0$.

□

### A.2.3 Action of diffeomorphisms

For any diffeomorphism $\phi : M \to M$, we denote by $\phi_* : T^*M \to T^*M$ the lift on the cotangent bundle defined by $\phi_*(x,\xi) = \left(\phi(x), \left((D_x\phi)^{-1}\right)^t\right)$.

**Lemma 7.** *Let $p \in S_\rho^{m(x,\xi)}$ be a symbol with non-constant order $m(x,\xi)$ and $\phi : M \to M$ a diffeomorphism. Then, the composition $p \circ \phi_*$ belongs to $S_\rho^{m\circ\phi_*}$.*

*Proof.* Set $a = p \circ \phi_*$. For simplicity, we will write $\xi^a \circ \phi_*$ for the $\xi^a$ component of $\phi_*(x,\xi)$. We will prove by iteration that for any order function $m(x,\xi)$, any symbol $p \in S_\rho^m$ and any $\alpha, \beta \in \mathbb{N}^d$, one has

$$\partial_\xi^\alpha \partial_x^\beta (p \circ \phi_*) = q_{\alpha,\beta} \circ \phi_* \text{ where } q_{\alpha,\beta} \in S_\rho^{m-\rho|\alpha|+(1-\rho)|\beta|}, \tag{61}$$

where we write $m = m(x,\xi)$ for shortness. First of all, this is trivially true for $\alpha = \beta = 0$. Then, we suppose it is true for all $\alpha, \beta \in \mathbb{N}^d$, with $|\alpha + \beta| \le N$. If we compute the derivative $\partial_\xi^{\alpha+a}\partial_x^\beta$ of $p \circ \phi_*$ with $|a| = 1$, we obtain simply

$$\begin{aligned}\partial_\xi^{\alpha+a}\partial_x^\beta (p \circ \phi_*) &= \partial_\xi^a (q_{\alpha,\beta} \circ \phi_*) \\ &= \sum_{|a'|=1} \partial_\xi^{a'} (q_{\alpha,\beta}) \circ \phi_* . \partial_\xi^a \left(\xi^{a'} \circ \phi_*\right).\end{aligned}$$

The terms $\partial_\xi^{a'}(q_{\alpha,\beta})$ live in $S_\rho^{m-\rho|\alpha+a|+(1-\rho)|\beta|}$ by assumption whereas the second term in the product belongs to $S_1^0$. This means that $\partial_\xi^{\alpha+a}\partial_x^\beta (p \circ \phi_*) = q_{\alpha+a,\beta} \circ \phi_*$ where the symbol

$$q_{\alpha+a,\beta} = \sum_{|a'|=1} \partial_\xi^{a'} (q_{\alpha,\beta}) . \partial_\xi^a \left(\xi^{a'} \circ \phi_*\right) \circ \phi_*^{-1}$$

belongs to $S_\rho^{m-\rho|\alpha+a|+(1-\rho)|\beta|}$. We consider now the derivative $\partial_\xi^\alpha \partial_x^{\beta+b}$ with $|b| = 1$. The computation is slightly more complicated, since there is an $x$-dependence in the $\xi$ component of $\phi_*(x,\xi)$. Namely,

$$\begin{aligned}\partial_\xi^\alpha \partial_x^{\beta+b} (p \circ \phi_*) &= \sum_{|b'|=1} \partial_x^{b'} (q_{\alpha,\beta}) \circ \phi_* . \partial_x^b \left(\phi^{b'}(x)\right) \\ &\quad + \sum_{|a'|=1} \partial_\xi^{a'} (q_{\alpha,\beta}) \circ \phi_* . \partial_x^b \left(\xi^{a'} \circ \phi_*\right).\end{aligned}$$

In the first sum, the terms $\partial_x^{b'}(q_{\alpha,\beta})$ belong to $S_\rho^{m-\rho|\alpha|+(1-\rho)|\beta+b|}$ and $\partial_x^b\left(\phi^{b'}(x)\right)$ is in $S_1^0$. On the other hand, in the second sum we have $\partial_\xi^{a'}(q_{\alpha,\beta}) \in S_\rho^{m-\rho|\alpha+a|+(1-\rho)|\beta|}$ but the second one in the product is in $S_1^1$. This means that the second sum brings a power $(1-\rho)$ of $\xi$. All together, we obtain $\partial_\xi^\alpha \partial_x^{\beta+b}(p \circ \phi_*) = q_{\alpha,\beta+b} \circ \phi_*$ with $q_{\alpha,\beta+b} \in S_\rho^{m-\rho|\alpha+a|+(1-\rho)|\beta|+1-\rho}$. This proves therefore by induction Formula (61) for all $\alpha, \beta \in \mathbb{N}^d$.

Now, the lemma follows easily form this formula. Indeed, for any $\alpha, \beta \in \mathbb{N}^d$ one has

$$\left|\partial_\xi^\alpha \partial_x^\beta (p \circ \phi_*)\right| \le C \langle \xi \circ \phi_* \rangle^{m\circ\phi_* - \rho|\alpha| + (1-\rho)|\beta|}.$$

On the other hand, since both $D_x\phi$ and $(D_x\phi)^{-1}$ are uniformly bounded on $M$, it follows that there is a constant $C>1$ such that $\frac{1}{C}\langle\xi\rangle \leq \langle\xi\circ\phi_*\rangle \leq C\langle\xi\rangle$. This implies that

$$\left|\partial_\xi^\alpha \partial_x^\beta (p\circ\phi_*)\right| \leq C\langle\xi\rangle^{m\circ\phi_*-\rho|\alpha|+(1-\rho)|\beta|}.$$

$\square$

## A.3 PDO with variable order

Given an order function $m(x,\xi)$ and a symbol $p(x,\xi)$ in the class $S_\rho^{m(x,\xi)}$, Formula (59) provides an operator from $C_0^\infty(M)$ to $C^\infty(M)$, which is actually continuous. By duality, it is also continuous from $\mathcal{E}'$ to $\mathcal{D}'$. The class $\Psi_\rho^{m(x,\xi)}$ of PDO's is then the set of operators of the form (59) modulo a **smoothing** operator, i.e. an operator which sends $\mathcal{E}'$ into $C^\infty(M)$ continuously. We denote by $\Psi^{-\infty} = \bigcap_{n>0}\Psi_\rho^{-n}$ the class of smoothing operators and also the class $\Psi_\rho^\infty = \bigcup_{n>0}\Psi_\rho^n$ of all PDOs of type $\rho$ which contains $\Psi_\rho^{m(x,\xi)}$ for any variable order $m(x,\xi)$. Notice that given an operator $\hat{p}\in\Psi_\rho^\infty$, its symbol is well-defined only up to an element in $S_\rho^{-\infty}$.

We now review the most important properties of PDO's. The proofs for non-constant order symbols follow in most cases the line of the proofs for usual symbols (see for example [38, 29]) and are omitted for shortness of this paper. In all the sequel, the parameter $\rho$ is always supposed to satisfy $\rho>\frac{1}{2}$. As well, in order to avoid any discussion about properly supported operators, we assume from now on $M$ to be compact, since it will be the case for the application of these tools to Ruelle-Pollicott resonances.

### A.3.1 Asymptotic expansions

Semi-classical analysis is naturally an asymptotic theory. In order to prove the basic theorems about composition, Egorov, ellipticity or functional calculus, one needs to give a sense to formal series like $\sum p_j$, where $\{p_j\}_{j\in\mathbb{N}}$ is a sequence of symbols with decreasing orders. Such a series is most of the time divergent, but it is possible to find a symbol $p$ which is asymptotically equivalent to the series. This is an adaptation of an old result by Borel.

**Theorem 6.** *Let $p_j \in S_\rho^{m_j(x,\xi)}$ be a sequence of symbols with variable order $m_j \in S_r^1$, with $\rho < r \leq 1$, satisfying $m_j \downarrow -\infty$, in the sense that, for all $j \in \mathbb{N}$*

$$\sup_{x,\xi} m_j(x,\xi) \to -\infty \text{ and } m_{j+1}(x,\xi) \leq m_j(x,\xi).$$

*Then, there exists a symbol $p \in S_\rho^{m_0(x,\xi)}$ such that for all $N \geq 0$*

$$p - \sum_{j=0}^{N-1} p_j \in S_\rho^{m_N(x,\xi)}.$$

*The symbol $p$ is unique modulo a residual symbol, i.e., an element of $S^{-\infty}$.*

The proof of this theorem is a straightforward adaptation of the proof for usual symbols, which can be found for example in [18, II, 3] or [38]. This fact implies automatically the corresponding result for asymptotic sums of PDO's. Namely, if $\hat{p}_j \in \Psi_\rho^{m_j(x,\xi)}$ is sequence of PDO with decreasing orders, then there exist an operator $\hat{p} \in \Psi_\rho^{m_0(x,\xi)}$ which satisfies

$$\hat{p} - \sum_{j=0}^{N-1} \hat{p}_j \in \Psi_\rho^{m_N(x,\xi)}$$

for all $N \geq 0$.

### A.3.2 Adjoint and composition

**Theorem 7.** *Let $\hat{p} \in \Psi_\rho^{m(x,\xi)}$ be a PDO with non-constant order symbol $p \in S_\rho^{m(x,\xi)}$. Then the adjoint $\hat{p}^*$ is itself a PDO in $\Psi_\rho^{m(x,\xi)}$ and its symbol $p^* \in S_\rho^{m(x,\xi)}$ satisfies*

$$p^*(x,\xi) - \overline{p(x,\xi)} \in S_\rho^{m(x,\xi)-(2\rho-1)},$$

*where $\bar{}$ denotes the complex conjugate.*

Here, the adjoint means the formal $L^2$-adjoint defined on the same domain $C^\infty(M)$.

**Theorem 8.** *Let $\hat{p} \in \Psi_\rho^{m(x,\xi)}$ and $\hat{q} \in \Psi_\rho^{m'(x,\xi)}$ be two PDO's with non-constant order $m(x,\xi)$ and $m'(x,\xi)$. Then the product $\hat{a} := \hat{p}\hat{q}$ is a PDO in $\Psi_\rho^{m(x,\xi)+m'(x,\xi)}$ and its symbol $a(x,\xi)$ satisfies*

$$a(x,\xi) - p(x,\xi)\, q(x,\xi) \in S_\rho^{m(x,\xi)+m'(x,\xi)-(2\rho-1)}.$$

### A.3.3 Egorov's theorem

Egorov's Theorem describes how PDO's transform under conjugation with a Fourier Integral Operator. We will nevertheless avoid talking about general FIO's and restrict ourselves to the simplest case, namely the composition by a diffeomorphism on $M$, which is sufficient for our purposes. See [29, p.24]

**Theorem 9.** *Let $\hat{p} \in \Psi_\rho^{m(x,\xi)}$ be a PDO with non-constant order $m(x,\xi)$ and $f : M \to M$ a diffeomorphism. Denote by $\hat{F}$ the pull-back operator $\hat{F}(u) = u \circ f$ and by $F : T^*M \to T^*M$ the lift of $f^{-1}$ to the cotangent bundle defined by $F(x,\xi) = \left(f^{-1}(x), (D_x f)^t\right)$. Then, the conjugation $\hat{a} := \hat{F}^{-1}\hat{p}\hat{F}$ belongs to $\Psi_\rho^{m \circ F(x,\xi)}$ and its symbol $a(x,\xi)$ satisfies*

$$a(x,\xi) - p \circ F(x,\xi) \in S_\rho^{m\circ F(x,\xi)-(2\rho-1)}.$$

### A.3.4 Sobolev continuity

It is a well-known fact that on a compact manifold, a PDO of constant order $m$ extends to a continuous operator $H^s \to H^{s-m}$ for all $s$. Thanks to Lemma 5, a PDO with variable order $m(x,\xi)$ extends to a continuous operators $H^s \to H^{s-m^+}$ with $m^+ = \limsup m(x,\xi)$. On the other hand the embeddings $H^s \hookrightarrow H^{s'}$ for $s' < s$ are compact. In particular, smoothing operators are compact in any Sobolev space.

## A.4 Non-isotropic ellipticity

### A.4.1 Variable order ellipticity

We know from the standard theory of PDO's that an operator $\hat{p} \in \Psi_\rho^m$ is invertible modulo $\Psi^{-\infty}$ with an "inverse" in $\Psi_\rho^{-m}$ as soon as its symbol $p$ satisfies an **ellipticity** condition. We now show that the classical definition of ellipticity extends in a natural way to symbols with variable order $m(x,\xi)$. This leads to a more general notion of ellipticity which is proved to be equivalent to the existence of a parametrix $\Psi_\rho^{-m(x,\xi)}$.

**Definition 8.** *A symbol a non-constant order $p \in S_\rho^{m(x,\xi)}$ is called* **elliptic** *if there is a $C > 0$ such that $|p(x,\xi)| \geq \frac{1}{C} \langle \xi \rangle^{m(x,\xi)}$ whenever $\langle \xi \rangle \geq C$. An operator $\hat{p}$ is elliptic if its symbol is elliptic.*

One can easily check that the following statement is equivalent to ellipticity.

**Lemma 8.** *A symbol $p \in S_\rho^{m(x,\xi)}$ is elliptic if and only if there exists a symbol $q \in S_\rho^{-m(x,\xi)}$ such that*

$$p(x,\xi)\, q(x,\xi) - 1 \in S_\rho^{-\infty}.$$

**Example 1.** *For any order function $m(x,\xi)$ the symbol $p(x,\xi) = \langle \xi \rangle^{m(x,\xi)}$ is elliptic and one can choose $q = \langle \xi \rangle^{-m(x,\xi)}$ on the whole of $T^*M$.*

For usual symbols, it is well-known that ellipticity is a phenomenon of the principal symbol. This is also true for variable order symbols, in the following sense.

**Lemma 9.** *Let $p \in S_\rho^{m(x,\xi)}$ be an elliptic symbol. Then any other symbol $q \in S_\rho^{m(x,\xi)}$ satisfying $p - q \in S_\rho^{m(x,\xi)-\varepsilon}$ for some $\varepsilon > 0$ is elliptic as-well.*

*Proof.* Suppose $p = q + s$ with $s \in S_\rho^{m(x,\xi)-\varepsilon}$. Ellipticity of $p$ means that for large enough $\langle \xi \rangle$ one has $|p(x,\xi)| \geq \frac{1}{C} \langle \xi \rangle^{m(x,\xi)}$. On the other hand, for large $\langle \xi \rangle$ one has also $|s(x,\xi)| \leq C \langle \xi \rangle^{m(x,\xi)-\varepsilon}$. Therefore

$$|q(x,\xi)| \geq C \langle \xi \rangle^{m(x,\xi)} \left(1 - C' \langle \xi \rangle^{-\varepsilon}\right) \geq C'' \langle \xi \rangle^{m(x,\xi)}$$

for large $\langle \xi \rangle$, hence $q$ is elliptic.

□

Notice that this notion of ellipticity is more general than the usual one. Indeed, the symbol $\langle \xi \rangle^{m(x,\xi)}$ with an order which takes its values, say between $-1$ and $+1$ is not elliptic in the usual sense. Indeed, the symbol $\langle \xi \rangle^{m(x,\xi)}$ with an order taking its values, say between -1 and +1 is not elliptic in the usual sense, when we view it as a symbol with constant order $\sup m$.

### A.4.2 Parametrix and invertibility

The main point in considering this notion of non-isotropic ellipticity is of course that it is equivalent to the existence of a parametrix, as explained in Theorem 10 below.

**Definition 9.** *Let $\hat{p} \in \Psi_\rho^{m(x,\xi)}$ be any PDO. A **parametrix** of $\hat{p}$ is a PDO $\hat{q} \in \Psi_\rho^{-m(x,\xi)}$ such that*

$$\hat{p}\hat{q} - \mathbb{I} \in \Psi^{-\infty} \text{ and } \hat{q}\hat{p} - \mathbb{I} \in \Psi^{-\infty}.$$

**Theorem 10.** *An operator $\hat{p} \in \Psi_\rho^{m(x,\xi)}$ admits a parametrix if and only if its symbol $p$ is elliptic.*

For this reason, we will say equally that the symbol $p$ or the operator $\hat{p}$ is elliptic.

The construction is standard. We just check that it works as-well in the variable order context. Assume $p \in S_\rho^{m(x,\xi)}$ is elliptic. Lemma 8 implies that there is a $q_0 \in S_\rho^{-m(x,\xi)}$ such that $\hat{p}\hat{q}_0 = \mathbb{I} - \hat{r}$ where $\hat{r} \in \Psi_\rho^{-(2\rho-1)}$ because of Theorem 8. This implies $\hat{r}^j \in \Psi_\rho^{-j(2\rho-1)}$ for all $j \in \mathbb{N}$ and it follows that $\hat{q}_N = \hat{q}_0 \left(\mathbb{I} + \hat{r} + \ldots + \hat{r}^N\right)$ satisfies

$$\hat{p}\hat{q}_{N-1} - \mathbb{I} \in \Psi_\rho^{-N(2\rho-1)}.$$

On the other hand, thanks to the re-summation Theorem 6 we can find a $\hat{q}_R \in \Psi_\rho^{-m(x,\xi)}$ satisfying $\hat{q}_R - \hat{q}_{N-1} \in \Psi_\rho^{-m(x,\xi)-N(2\rho-1)}$ for all $N \in \mathbb{N}$. Therefore we have

$$\hat{p}\hat{q}_R - \mathbb{I} \in \Psi_\rho^{-N(2\rho-1)}$$

for all $N \in \mathbb{N}$, hence $\hat{q}_R$ is a right parametrix for $\hat{p}$. Similarly, we can construct a left parametrix $\hat{q}_L \sim (\mathbb{I} + \hat{s} + \hat{s}^2 + \ldots)\, \hat{q}_0$ with $\hat{s} \in \Psi_\rho^{-(2\rho-1)}$ given by $\hat{p}\hat{q}_0 = \mathbb{I} - \hat{s}$. Finally, the fact that $\hat{q}_L - \hat{q}_R \in \Psi_\rho^{-\infty}$ comes from the observation that both $\hat{q}_L\hat{p}\hat{q}_R - \hat{q}_R$ and $\hat{q}_L\hat{p}\hat{q}_R - \hat{q}_L$ are smoothing. Therefore, say $\hat{q}_L$ is a (both sided) parametrix for $\hat{p}$.

*Proof.* □

The existence of a parametrix has many interesting consequences, such as those listed below. First of all, it is well-known that a standard elliptic PDO (with constant order $m$) is Fredholm $H^s \to H^{s-m}$ for any $s$. For PDO's with variable order, a slightly weaker result holds.

**Lemma 10.** *Let $\hat{p} \in \Psi_\rho^{m(x,\xi)}$ be elliptic. Then the kernel of the operator $\hat{p} : H^s \to H^{s-m^+}$ with $m^+ = \limsup m(x,\xi)$ is finite dimensional and contained in $C^\infty(M)$.*

*Proof.* The key point in this proof is the fact that for any smoothing operator $\hat{r} \in \Psi^{-\infty}$, the operator $\mathbb{I} + \hat{r} : H^s \to H^s$ is Fredholm for any $s$ and its kernel is contained in $C^\infty(M)$. See for example [37, ch. 7] for a proof for $s = 0$ which extends straightforwardly to the case $s \neq 0$.

Now, the ellipticity $\hat{p}$ implies the existence of a left parametrix $\hat{q} \in \Psi_\rho^{-m(x,\xi)}$, i.e. $\hat{q}\hat{p} = \mathbb{I} + \hat{r}$ with $\hat{r} \in \Psi^{-\infty}$. This operator extends to $\hat{q} : H^{s-m^+} \to H^{s-m^+-m^-}$ where $m^- = \limsup(-m(x,\xi))$. It follows that $\ker \hat{p}$ is contained in the kernel of

$$\mathbb{I} + \hat{r}_1 : H^s \to H^s \subset H^{s-m^+-m^-}$$

which is finite dimensional and itself contained in $C^\infty(M)$.

□

This lemma has the consequence that we can make $\hat{p}$ invertible by adding a smoothing operator, as shown in Lemma 12. This is useful in practice, since one often needs to construct PDO's whose symbol satisfies certain properties which are not modified by adding a residual term. One needs first a preliminary result.

**Lemma 11.** *Let $\hat{p} \in \Psi_\rho^{m(x,\xi)}$ be an elliptic and formally self-adjoint operator. Then $\hat{p}$ viewed as an unbounded operator on $L^2$ admits a self-adjoint extension.*

*Proof.* The formal self-adjointness of $\hat{p}$ implies according to Theorem 7 that its symbol satisfies

$$p - Re(p) \in S_\rho^{m(x,\xi)-(2\rho-1)},$$

hence $|Re(p(x,\xi))| \geq c.|p(x,\xi)|$, with $c > 0$, for large enough $|\xi|$. We can thus suppose that $Re(p(x,\xi)) > 0$ for large enough $|\xi|$ (if $Re(p)$ has the opposite sign, then the following argument applies to $-\hat{p}$). We can therefore apply Lemma 13 of the next section to show that

$$\hat{p} = \hat{b}^*\hat{b} - \hat{K}$$

with $\hat{b} \in \Psi_\rho^{\frac{1}{2}m(x,\xi)}$ and $\hat{K} \in \Psi^{-\infty}$. Since $\hat{K}$ is bounded in $L^2$, it follows that $\hat{p}$ is bounded from below in $L^2$:

$$(\hat{p}u, u) \geq \left|\hat{b}u\right|^2 - \left|\hat{K}\right| |u|^2.$$

This allows us to construct the Friedrichs extension of $\hat{p}$, which is self-adjoint on a domain in $L^2$ (see e.g. [43, p. 317]).

□

**Lemma 12.** *Let $\hat{p} \in \Psi_\rho^{m(x,\xi)}$ be an elliptic and formally self-adjoint operator. Then there exists a formally self-adjoint and smoothing operator $\hat{r} \in \Psi_\rho^{-\infty}$ such that $\hat{p} + \hat{r}$ is invertible $C^\infty(M) \to C^\infty(M)$ with inverse in $\Psi_\rho^{-m(x,\xi)}$.*

*Proof.* This is also a standard construction. First, Lemma 10 tells us that $\hat{p} : H^{m^+} \to L^2$ with $m^+ = \limsup m(x,\xi)$ has a finite dimensional kernel contained in $C^\infty(M)$. But this implies that viewed as an unbounded operator on $L^2$ with domain $H^{m^+}$, the operator $\hat{p}$ has also a finite dimensional kernel contained in $C^\infty(M)$. On the other hand, $\hat{p}$ has a self-adjoint extension on $L^2$ thanks to Lemma 11. Denote by $\mathcal{D}$ the domain of this extension. This leads to the orthogonal decomposition $L^2 = \overline{\mathrm{im}(\hat{p})} \overset{\perp}{\oplus} \ker(\hat{p})$ and the restriction $\hat{p} : \overline{\mathrm{im}(\hat{p})} \cap \mathcal{D} \to \mathrm{im}(\hat{p})$ is thus invertible. Therefore, the operator $\hat{P}$ given in matrix form by

$$\hat{P} := \begin{pmatrix} \hat{p}|_{\overline{\mathrm{im}(\hat{p})}\cap\mathcal{D}} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

is invertible. We first remark that $\hat{P}$ is related to $\hat{p}$ by

$$\hat{P} = \hat{p}(1-\pi) + \pi$$

with $\pi : L^2 \to \ker\hat{p}$ the orthogonal $L^2$-projection. Since $\ker p$ is a finite dimensional subspace of $C^\infty(M)$, the projection $\pi$ is a smoothing operator. It follows that $\hat{r} := \hat{P} - \hat{p} = (1-\hat{p})\pi$ is self-adjoint and smoothing. In particular $\hat{P} \in \Psi_\rho^{m(x,\xi)}$ is a PDO and defines an injective map $C^\infty(M) \to C^\infty(M)$. On the other hand, the existence of a parametrix $\hat{Q} \in \Psi_\rho^{-m(x,\xi)}$ for $\hat{p}$, and thus for $\hat{P}$, implies that $\hat{P}$ is also surjective $C^\infty(M) \to C^\infty(M)$. Finally, denote by $\hat{P}^{-1} : C^\infty(M) \to C^\infty(M)$ the inverse of $\hat{P}$ which is continuous by the open mapping Theorem. One has

$$\hat{Q} = \hat{Q}\hat{P}\hat{P}^{-1} = \hat{P}^{-1} + \hat{r}\hat{P}^{-1}$$

with $\hat{r}$ smoothing. The last term is also smoothing since $\hat{P}^{-1}$ is continuous and $\hat{r}$ smoothing. This implies that $\hat{P}^{-1}$ is itself a PDO in $\Psi_\rho^{-m(x,\xi)}$.

□

Collecting the results of this section, we obtain the following corollary.

**Corollary 4.** *For any real elliptic symbol $q \in S_\rho^{m(x,\xi)}$, there is an operator $\hat{p} \in \Psi_\rho^{m(x,\xi)}$ satisfying $\hat{p} - \hat{q} \in \Psi_\rho^{m(x,\xi)-(2\rho-1)}$, which is formally self-adjoint and invertible $C^\infty(M) \to C^\infty(M)$.*

*Proof.* Let $\hat{q} \in \Psi_\rho^{m(x,\xi)}$ be the quantized of the symbol $q$ and take the real part $\hat{a} := \frac{1}{2}(\hat{q} + \hat{q}^*)$. It is self-adjoint and according to Theorem 7, its symbol satisfies $a(x,\xi) = q(x,\xi) \bmod S_\rho^{m(x,\xi)-(2\rho-1)}$. Then, Lemma 9 implies that the ellipticity of $q$ is not destroyed by a modification of order $m(x,\xi) - (2\rho - 1)$. Therefore $a \in S_\rho^{m(x,\xi)}$ is elliptic as-well. Finally, Theorem 10 shows that this implies the existence of a parametrix for $\hat{a}$. Consequently, one can find a self-adjoint PDO $\hat{p} = \hat{a} \bmod \Psi^{-\infty}$ which is invertible $C^\infty(M) \to C^\infty(M)$ and with inverse in $\Psi_\rho^{-m(x,\xi)}$ (see Lemma 12).

□

### A.4.3 $L^2$-continuity and quasi-compacity

The next result, due originally to Hörmander, is very useful. It tells us that one can take the "square root" of a positive elliptic operator.

**Lemma 13.** *Let $p \in S_\rho^{m(x,\xi)}$ be an elliptic symbol satisfying $p - Re\,(p) \in S_\rho^{m(x,\xi)-\varepsilon}$ for some $\varepsilon > 0$ and $Re\,(p\,(x,\xi)) > 0$ for $\langle\xi\rangle \geq c$. Then, there exists $\hat{b} \in \Psi_\rho^{\frac{1}{2}m(x,\xi)}$ such that*

$$\hat{p} - \hat{b}^*\hat{b} \in \Psi^{-\infty}.$$

*Proof.* Thanks to Lemma 9, the real part $\mathrm{Re}\,(p)$ is elliptic and satisfies therefore $\mathrm{Re}\,(p) \geq C\,\langle\xi\rangle^{m(x,\xi)}$ for large $\langle\xi\rangle$. This implies that we can certainly find a smooth $b_0\,(x,\xi)$ which coincides with $\sqrt{\mathrm{Re}\,(p)}$ for large $\langle\xi\rangle$, i.e., outside from a compact set. Straightforward computations show that $b_0 \in S_\rho^{\frac{1}{2}m(x,\xi)}$. On the other hand we have $|b_0|^2 = \mathrm{Re}\,(p) \mod S^{-\infty}$. Then, the symbolic calculus gives

$$\hat{b}_0^*\hat{b}_0 = \widehat{\mathrm{Re}\,(p)} \mod \Psi_\rho^{m(x,\xi)-(2\rho-1)} = \widehat{p} \mod \Psi_\rho^{m(x,\xi)-\varepsilon},$$

where we have assumed without loss of generality that $\varepsilon \leq 2\rho - 1$. The rest of the procedure is a standard iterative construction which shows that for any $N$, there are $\hat{b}_j \in \Psi_\rho^{m(x,\xi)-j\varepsilon}$, $j : 1..N$, satisfying

$$\left(\hat{b}_0^* + ... + \hat{b}_N^*\right)\left(\hat{b}_0 + ... + \hat{b}_N\right) = \widehat{p} \mod \Psi_\rho^{m(x,\xi)-(N+1)\varepsilon}.$$

Then, a Borel resummation (Theorem 6) yields the result.

□

The last but not the least result of this appendix is standard, since it concerns symbols of (constant) order 0. It is usually a way to prove $L^2$-continuity of PDO's, but it yields also a way to show that a PDO's with a "small" symbol is quasi-compact, which is the property we use in the context of Ruelle-Pollicott resonances.

**Lemma 14.** *Let $p \in S_\rho^0$ be a symbol and denote*

$$L = \limsup_{(x,\xi)\in T^*M} |p\,(x,\xi)|\,.$$

*Then, for any $\varepsilon > 0$ there is a decomposition*

$$\hat{p} = \hat{p}_\varepsilon + \hat{K}_\varepsilon$$

*with $\hat{K}_\varepsilon \in \Psi^{-\infty}$ and $\|\hat{p}_\varepsilon\| \leq L + \varepsilon$.*

*Proof.* The first remark is that for any $\varepsilon > 0$ the operator $\left(L^2+\varepsilon\right)\mathbb{I}-\hat{p}^*\hat{p} =: \hat{q}$ is self-adjoint and in the class $S^0_\rho$, which means $q-\mathbb{R}\left(q\right) \in S_\rho^{-(2\rho-1)}$. On the other hand, $q = \left(L^2+\varepsilon\right)-|p|^2$ modulo $\Psi_\rho^{-(2\rho-1)}$, which is positive for large $\xi$. Therefore we can apply Lemma 13 and obtain $\hat{b} \in \Psi^0_\rho$ such that $\hat{q} = \hat{b}^*\hat{b}-\hat{K}$ with $K \in \Psi^{-\infty}$. Then, for any $u \in L^2\left(M\right)$ one has

$$\begin{aligned} \left\|\hat{p}\left(u\right)\right\|^2 &= \left(\hat{p}^*\hat{p}\left(u\right),u\right) \\ &= \left(L^2+\varepsilon\right)\|u\|^2-\left(\hat{b}^*\hat{b}\left(u\right),u\right)+\left(\hat{K}\left(u\right),u\right) \\ &= \left(L^2+\varepsilon\right)\|u\|^2-\left\|\hat{b}\left(u\right)\right\|^2+\left(\hat{K}\left(u\right),u\right). \end{aligned}$$

From this follows the upper bound

$$\left\|\hat{p}\left(u\right)\right\|^2 \leq \left(L^2+\varepsilon\right)\|u\|^2+\left(\hat{K}\left(u\right),u\right). \tag{62}$$

The next step is to introduce the spectral projector $\pi_\lambda$ of the Laplacian $-\Delta$ on $(-\infty,\lambda]$ for large enough $\lambda$, which will be chosen later depending on $\varepsilon$ in a suitable way. Notice that this projection is smoothing. Then, we decompose

$$\hat{p} = \hat{p}_\varepsilon+\hat{r}_\varepsilon := \hat{p}\left(1-\pi_\lambda\right)+\hat{p}\pi_\lambda.$$

It follows first that $\hat{r}_\varepsilon$ is smoothing. On the other hand, the upper bound (62) yields

$$\begin{aligned} \left\|\hat{p}_\varepsilon\left(u\right)\right\|^2 &\leq \left(L^2+\varepsilon\right)\left\|\left(1-\pi_\lambda\right)u\right\|^2+\left(\hat{K}\left(1-\pi_\lambda\right)\left(u\right),\left(1-\pi_\lambda\right)\left(u\right)\right) \\ &\leq \left(L^2+\varepsilon\right)\|u\|^2+\left\|\hat{K}\left(1-\pi_\lambda\right)\right\|\|u\|^2 \end{aligned}$$

where we have used $\|1-\pi_\lambda\| \leq 1$ and the Cauchy-Schwarz inequality. Finally, we show that we can make $\left\|\hat{K}\left(1-\pi_\lambda\right)\right\|$ arbitrarily small. Since $\hat{K}$ is smoothing, it is continuous $H^s \to L^2$ for any $N > 0$. In particular, we can decompose

$$\hat{K}\left(1-\pi_\lambda\right) = \hat{K}\left(1-\Delta\right)^N\left(1-\Delta\right)^{-N}\left(1-\pi_\lambda\right)$$

and $\left\|\hat{K}\left(1-\Delta\right)^N\right\|_{L^2} \leq C_N$. On the other hand, the spectral theorem yields $\left\|\left(1-\Delta\right)^{-N}\left(1-\pi_\lambda\right)\right\| \leq \lambda^{-N}$. Therefore we have showed that $\left\|\hat{K}\left(1-\pi_\lambda\right)\right\| \leq \frac{C_N}{\lambda^N}$ which can be made arbitrarily small, by taking $\lambda = \frac{1}{\varepsilon}$ for example. This proves $\left\|\hat{p}_\varepsilon\left(u\right)\right\|^2 \leq \left(L^2+2\varepsilon\right)\|u\|^2$.

□

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