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Compact Submanifolds in a Euclidean Space

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Abstract: In this paper we use bounds on the Ricci curvature of an n-dimensional compact submanifold M of the Euclidean space R^{n+p} to obtain a characterization of a sphere (cf. Theorem 1). Also we obtain a lower bound on the integral of the square of the length of mean curvature vector field H of this submanifold and show that the lower bound is attended if and only if the submanifold is a sphere giving another characterization of a sphere (cf. Theorem 2).

1. INTRODUCTION

Let M be an n-dimensional compact submanifold of a Euclidean space R^{n+p}. One of the interesting questions in the geometry of submanifolds in a Euclidean space is to obtain conditions under which the submanifold is a sphere. This question has been studied by many mathematicians using conditions on scalar curvature, Ricci curvature and mean curvature of the submanifold (cf. [1-12]). In this paper using constant vector fields on the Euclidean space R^{n+p}, we obtain a characterization of a sphere in R^{n+p}. Let H the mean curvature vector field of M in the Euclidean space R^{n+p}. Our main results in this paper are the following:

Theorem 1. Let M be an n-dimensional compact and connected submanifold of the Euclidean space R^{n+p} with mean curvature vector field H. If there exists a constant $\lambda > 0$ such that Ricci curvature Ric of M satisfies

$$n^{-1}(n-1)\left((n-1)\lambda^{-2} + \|H\|^2\right) \le Ric \le (n-1)\lambda^{-2}$$

then $M = S^n(\lambda^{-2})$.

Theorem 2. Let M be an n-dimensional compact and connected submanifold of the Euclidean space R^{n+p} with mean curvature vector field H. If $\inf \frac{1}{n-1} Ric = k$ and $\sup \frac{1}{n-1} Ric = K$, then

$$\int_M \|H\|^2 \ge \left\{(n+p)k - (n+p-1)K\right\} Vol(M)$$

and the equality holds if and only if $M = S^n(c)$ for a constant $c > 0$.

2. PRILIMINARIES

Let $\psi : M \to R^{n+p}$ be an n-dimensional immersed submanifold in the Euclidean space R^{n+p}. We denote by g and $\bar{\nabla}$ the Euclidean metric and the Euclidean connection on R^{n+p}. Also we denote by the same letter g and by ∇ the induced metric and the Riemannian connection on the submanifold M. Then we have the following fundamental equations for the submanifold (cf. [3])

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y),\ \bar{\nabla}_X N = -A_N(X) + \nabla^{\perp}_X N, \qquad (2.1)$$

$X, Y \in \mathfrak{X}(M)$, $N \in \Gamma(\upsilon)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M, $\Gamma(\upsilon)$ is the space of smooth sections of the normal bundle υ of M, h is the second fundamental form, A_N is the Weingarten map with respect to the normal vector field N and $\nabla^{\perp}$ is the connection in the normal bundle υ. We also have the following equations of Gauss and Codazzi for the submanifold

$$R(X,Y;Z,W) = g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)) \qquad (2.2)$$

$$(Dh)(X,Y,Z) = (Dh)(Y,Z,X) = (Dh)(Z,X,Y) \qquad (2.3)$$

where R is the curvature tensor field of the submanifold M and $(Dh)(X,Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$ for $X,Y,Z \in \mathfrak{X}(M)$. The Ricci tensor Ric of the submanifold is given by

$$Ric(X,Y) = ng(H, h(X,Y)) - \sum_{i=1}^{n} g(h(X,e_i), h(Y,e_i)) \qquad (2.4)$$

where $\{e_1,..e_n\}$ is a local orthonormal frame and $H = \frac{1}{n}\sum h(e_i,e_i)$ is the mean curvature vector field. The Ricci operator Q is a symmetric (1,1) tensor field defined by $Ric(X,Y) = g(Q(X),Y)$, $X,Y \in \mathfrak{X}(M)$. The scalar curvature S of the submanifold M is given by

$$S = n^2\|H\|^2 - \|h\|^2 \qquad (2.5)$$

which together with Schwarz's inequality gives

$$S \le n(n-1)\|H\|^2 \qquad (2.6)$$

with equality holds if and only if $h(X,Y) = g(X,Y)H$, $X,Y \in \mathfrak{X}(M)$ that is if and only if M is a totally umbilical submanifold.

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Since the immersion ψ can be treated as position vector field on M, we have $\psi \in \mathfrak{X}(R^{n+p})$ and thus we can write $\psi = t + v$, where $t \in \mathfrak{X}(M)$ is the tangential component and $v \in \Gamma(\upsilon)$ is the normal component of ψ. If we denote by $B = A_v$ the Weingarten map with respect to the normal vector field v, then using equations (2.1) we immediately have

$$\nabla_X t = X + B(X), \nabla^{\perp}_X v = -h(X,t), \quad X \in \mathfrak{X}(M) \tag{2.7}$$

Define a smooth function $\phi : M \to R$ by $\phi = \langle H, \psi \rangle$, then using equation (2.7) and the definition of the Laplacian operator $\Delta f = div(\nabla f)$, where ∇f is the gradient of the smooth function f, we immediately have the following (cf. [2]):

Lemma 2.1. For a submanifold M of R^{n+p} and the smooth function $f = \frac{1}{2}\|\psi\|^2$ the Laplacian Δf of the function f is given by

$$\Delta f = n(1+\varphi)$$

Let $x^1,..,x^{n+p}$ be the Euclidean coordinates on the Euclidean space R^{n+p} and $\xi_a \in \mathfrak{X}(R^{n+p})$, $a = 1,..,n+p$ be the coordinate vector fields. Restricting ξ_a to M we can express ξ_a, as $\xi_a = u_a + N_a$, where $u_a \in \mathfrak{X}(M)$ and $N_a \in \Gamma(\upsilon)$. Then using equations (2.1) it is easy to check that

$$\nabla_X u_a = B_a(X), \nabla^{\perp}_X N_a = -h(u_a, X), \quad X \in \mathfrak{X}(M) \tag{2.8}$$

where $B_a = A_{N_a}$ is the Weingarten map with respect to the normal vector field N_a. If we let $f^a = x^a \mid M$ the restriction of a^{th} Euclidean coordinate function to M, then it is straight forward to see that

$$\nabla f^a = u_a, \psi = \sum_{a=1}^{n+p} f^a \xi_a, t = \sum_{a=1}^{n+p} f^a u_a, v = \sum_{a=1}^{n+p} f^a N_a \tag{2.9}$$

Since each $X \in \mathfrak{X}(M)$ and $N \in \Gamma(\upsilon)$ can be express as $X = \sum_a g(X,\xi_a)\xi_a$, $N = \sum_a g(N,\xi_a)\xi_a$, we immediately have the following

Lemma 2.2. For an n-dimensional submanifold M of R^{n+p} and $X \in \mathfrak{X}(M)$ and $N \in \Gamma(\upsilon)$ we have

(i) $X = \sum_a g(X,u_a)u_a$ (ii) $N = \sum_a g(N,N_a)N_a$

(iii) $\sum_a g(X,u_a)N_a = 0$, (iv) $\sum_a g(N,N_a)u_a = 0$.

Next, we define smooth functions $\varphi_a = g(H, N_a)$. Then using equation (2.9) we immediately get the following:

Lemma 2.3. Let M be an n-dimensional compact submanifold of R^{n+p}. Then for the Weingarten maps B_a we have

(i) $trB_a = n\varphi_a$

(ii) $(\nabla B_a)(X,Y) - (\nabla B_a)(Y,X) = R(X,Y)u_a$

(iii) $\sum_a (\nabla B_a)(e_i,e_i) = n\nabla\varphi_a + Q(u_a)$, $\quad X,Y \in \mathfrak{X}(M)$

where $\{e_1,..,e_n\}$ is a local orthonormal frame on M and $(\nabla B_a)(X,Y) = \nabla_X B_a Y - B_a(\nabla_X Y)$.

Using $\nabla_X u_a = B_a X$, from equation (2.8) and Lemma 2.3, we get $divu_a = n\varphi_a$. Thus we have

Lemma 2.4. Let M be an n-dimensional compact submanifold M of R^{n+p}. Then the functions φ_a satisfy

$$\int_M \varphi_a dV = 0$$

Lemma 2.5. Let M be an n-dimensional compact submanifold M of R^{n+p}. Then

$$\int_M \left\{ Ric(u_a,u_a) + \|B_a\|^2 - n^2\phi_a^2 \right\} dV = 0$$

Proof. Let $\{e_1,..,e_n\}$ be a pointwise constant local orthonormal frame. We compute the divergence of the vector field $B_a(u_a)$ as

$$divB_a u_a = \sum_i e_i g(u_a, B_a e_i) = \sum_i g(B_a e_i, B_a e_i) + \sum_i g(u_a, (\nabla B_a)(e_i,e_i))$$

$$= \|B_a\|^2 + ng(u_a, \nabla\phi_a) + Ric(u_a,u_a) \tag{2.10}$$

where we have used (iii) of Lemma 2.3. Also using Lemma 2.3 we arrive at

$$g(u_a, \nabla\varphi_a) = div(\varphi_a u_a) - n\varphi_a^2 \tag{2.11}$$

Using equation (2.11) in (2.10) and integrating the resulting equation we get the Lemma.

Lemma 2.6. For a submanifold M of R^{n+p} we have

$$\sum_{a=1}^{n+p} \|u_a\|^2 = n \text{ and } \sum_{a=1}^{n+p} \|N_a\|^2 = p$$

Proof. Using equations (2.8), (2.9) and Lemma 2.3, we compute $\Delta f^a = div(\nabla f^a) = n\varphi_a$. Also note that $f = \frac{1}{2}\sum_a \left(f^a\right)^2$ holds and consequently by Lemma 2.1, we have

$$n(1+\phi) = \sum_{a=1}^{n+p} \Delta \frac{1}{2}(f^a)^2 = \sum_{a=1}^{n+p} (f^a \Delta f^a + \|\nabla f^a\|^2)$$

$$= n\sum_{a=1}^{n+p} f^a \phi_a + \sum_{a=1}^{n+p} \|\nabla f^a\|^2 \tag{2.12}$$

Since,

$$\varphi = g(H,\psi) = \sum_{a=1}^{n+p} g(H, f^a \xi_a) = \sum_{a=1}^{n+p} f^a g(H,N_a) = \sum_{a=1}^{n+p} f^a \varphi_a \tag{2.13}$$

Thus using equations (2.13) and (2.9) in (2.12), we get the first part of the Lemma. Then using $\|N_a\|^2 = 1 - \|u_a\|^2$ in the first result of the Lemma we get the second result in the Lemma.

3. PROOF OF THE THEOREMS

Proof of Theorem 1

Let M be an n-dimensional compact submanifold of the Euclidean space R^{n+p} with mean curvature vector field H. Then the condition on the Ricci curvature in the statement gives

$$\|H\|^2 \le \lambda^{-2} \tag{3.1}$$

Let U_a be the open subset of M where $u_a \neq 0$. Then by Lemma 2.6, not all U_a's are empty. Choose a local orthonormal frame $\{e_1,..,e_n\}$ on U_a such that $u_a = \|u_a\| e_1$. Then we have

$$Ric(u_a,u_a) = \|u_a\|^2 Ric(e_1,e_1) = \|u_a\|^2 \left(S - \sum_{i=2}^{n} Ric(e_i,e_i) \right)$$

and consequently Lemma 2.5 gives

$$\int_M \left\{ \|u_a\|^2 S + \|u_a\|^2 \sum_{i=2}^{n} \left((n-1)\lambda^{-2} - Ric(e_i,e_i) \right) \right. \tag{3.2}$$

$$\left. -(n-1)\lambda^{-2} \|u_a\|^2 + \left(\|B_a\|^2 - n\phi_a^2 \right) - n(n-1)\phi_a^2 \right\} dV = 0$$

Thus using the condition in the Theorem and the Schwarz's inequality $\|B_a\|^2 \ge n\varphi_a^2$, we arrive at

$$\int_M \left\{ \|u_a\|^2 S - (n-1)^2 \lambda^{-2} \|u_a\|^2 - n(n-1)\phi_a^2) \right\} dv \le 0 \tag{3.3}$$

Adding these $n+p$ inequalities and using Lemma 2.6, we get

$$\int_M \left\{ S - (n-1)((n-1)\lambda^{-2} + \|H\|^2)) \right\} dv \le 0 \tag{3.4}$$

where we have used $\|H\|^2 = \sum_a g(H,\xi_a)^2 = \sum_a \phi_a^2$. Since the condition in the Theorem implies $(n-1)((n-1)\lambda^{-2} + \|H\|^2) \le S$, this together with inequality (3.4) yields

$$S = (n-1)((n-1)\lambda^{-2} + \|H\|^2) \tag{3.5}$$

Using the fact that, $\sum_a b_a = 0$ with $b_a \le 0$ implies $b_a = 0$, in equality (3.4) (in light of (3.5) and that it is a sum) and (3.3), we get equality in (3.3). Consequently this equality in (3.3) together with equation (3.2) and the condition in the Theorem implies

$$Ric(e_i,e_i) = (n-1)\lambda^{-2}, \text{and} \|B_a\|^2 = n\varphi_a, \tag{3.6}$$

for $i = 2,..,n$ and $a = 1,..,n+p$.

Thus using equations (3.5) and (3.6) we get

$$Ric(e_1,e_1) = S - \sum_{i=2}^{n} Ric(e_i,e_i) = (n-1)((n-1)\lambda^{-2} + \|H\|^2) - (n-1)^2 \lambda^{-2}$$

$$= (n-1)\|H\|^2 \tag{3.7}$$

Since, $Ric \ge n^{-1}(n-1)[(n-1)\lambda^{-2} + \|H\|^2]$, the equation (3.7) gives $(n-1)\|H\|^2 \ge n^{-1}(n-1)[(n-1)\lambda^{-2} + \|H\|^2]$, that is,

$$\|H\|^2 \ge \lambda^{-2} \tag{3.8}$$

Consequently equations (3.1) and (3.8) give $\|H\|^2 = \lambda^{-2}$, which combined with equation (3.5) gives $S = n(n-1)\|H\|^2$. Combining the last equation with inequality (2.6) we get that M is totally umbilical, that is, $h(X,Y) = g(X,Y)H$. $X,Y \in \mathfrak{X}(M)$ with $\|H\|^2 = \lambda^{-2}$. This proves $M = S^n(\lambda^{-2})$.

Proof of Theorem 2

Using a local orthonormal frame $\{e_1,..,e_n\}$ such that $u_a = \|u_a\| e_1$ in Lemma 2.5, we arrive at,

$$\int_M \left\{ \|u_a\|^2 \left(S - \sum_{i=2}^{n} Ric(e_i,e_i) \right) + C_a \right\} dV = n(n-1) \int_M \varphi_a^2 dV$$

where $C_a = \|B_a\|^2 - n\varphi_a^2$. Using $\|u_a\|^2 = 1 - \|N_a\|^2$ and bounds on Ricci curvature as given in the statement, in above equation we arrive at,

$$\int_M \left\{ S - \|N_a\|^2 S - (n-1)^2 K \|u_a\|^2 + C_a \right\} dV \le n(n-1) \int_M \phi_a^2 dV$$

Since, $n(n-1)k \le S \le n(n-1)K$, above inequality takes the form

$$\int_M \left\{ nk - \|N_a\|^2 nK - (n-1)K\|u_a\|^2 + C_a \right\} dV \le n \int_M \phi_a^2 dV$$

Adding these $n+p$ inequalities and using $\sum_a \varphi_a^2 = \|H\|^2$ and Lemma 2.6, we arrive at

$$(n+p)k - (n+p-1)K]Vol(M) + \frac{1}{n} \int_M \left(\sum_{a=1}^{n+p} C_a \right) dV \le \int_M \|H\|^2 dV \tag{3.9}$$

Since the numbers C_a are non negative the inequality in Theorem follows from (3.9). If the equality holds then by (3.9), we get $C_a = 0$ for each $a = 1,..,n+p$ and consequently by Schwarz's inequality $\|B_a\|^2 = n\varphi_a^2$ if and only if $B_a = \varphi_a I$. Then by Lemma 2.2, we have

$$h(X,Y)=\sum_{a=1}^{n+p}g(h(X,Y),N_a)N_a=\sum_{a=1}^{n+p}g(B_aX,Y)N_a$$

$$=g(X,Y)\sum_{a=1}^{n+p}\varphi_a N_a=g(X,Y)\sum_{a=1}^{n+p}g(H,N_a)N_a$$

$$=g(X,Y)H$$

Thus M is totally umbilical submanifold and consequently we have $M=S^n(c)$, for constant $c=\|H\|^2$. Conversely, if $M=S^n(c)$, then trivially the equality holds.

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