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The Analytical Evaluation of the Half-Order Fermi-Dirac Integrals

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Abstract: This paper presents a derivation for analytically evaluating the half-order Fermi-Dirac integrals. A complete analytical derivation of the Fermi-Dirac integral of order $\frac{1}{2}$ is developed and then generalized to yield each half-order Femi-Dirac function. The most important step in evaluating the Fermi-Dirac integral is to rewrite the integral in terms of two convergent real convolution integrals. Once this done, the Fermi-Dirac integral can put into a form in which a proper contour of integration can be chosen in the complex plane. The application of the theorem of residues reduces the Fermi-Dirac integral into one which becomes analytically tractable. The final solution is written in terms of the complementary and imaginary Error functions.

Keywords: Real convolution, fermi-dirac integral, theorem of residues.

I. INTRODUCTION

Fermi-Dirac integrals seem to be omnipresent in the scientific and mathematical literature. They are most often associated with the study of transport phenomena of conductors and semi-conductors [1-4]. However they also find applications in the seemingly unrelated field of multivariate quality control [5]. The scientific literature, spanning about a century, is replete with papers devoted the study of Fermi-Dirac integrals. Many attempts have been made to find more accurate analytical solutions, [6-11] and there have been numerous papers dedicated to their accurate numerical calculation [12-15]. The number of articles written on this subject could probably fill a large tome. In many respects, the techniques which have been developed and employed to evaluate these integrals have been quite successful mainly because researchers have devoted so much time and effort to the study of Fermi-Dirac integrals. So what does this paper bring to table? Stated simply-this paper presents a method for analytically evaluating the Fermi-Dirac integral of order $\frac{1}{2}$ given by

$$F_{\frac{1}{2}}(\xi) = \int_0^\infty \frac{x^{\frac{1}{2}}}{1+e^{x-\xi}}dx \qquad \forall \xi \in \mathbb{R} \quad , \tag{1}$$

where $F_{\frac{1}{2}}(\xi)$ will be defined as the Fermi-Dirac function of order $\frac{1}{2}$. Once the analytical solution is found, one is in position to immediately find $F_{m-\frac{1}{2}}(\xi)\,\forall m \in \mathbb{Z}^{\geq}$ (Appendix). Nothing will be said about what the physical quantities x or ξ may represent. Each will have a specific meaning depending upon how the Fermi-Dirac integral is developed and in what field of study it is used. However, the main goal of this paper is to present a cogent method of attack for analytically evaluating Eq. (1).

It must be stressed that although the Fermi-Dirac integral may have important properties from a purely mathematical point of view, the authors' interests are mostly concerned with the application of the Fermi-Dirac integral to real physical problems. This being said, the authors will consider x and ξ as real quantities which have a physical meaning defined by its particular application.

II. ANALYTICAL EVALUATION OF $F_{\frac{1}{2}}(\xi)$

Let's first restrict our analysis to $\xi > 0$, since this is the most interesting case, and derive an analytical expression for $F_{\frac{1}{2}}(\xi)$ valid $\forall \xi > 0$. Consider rewriting Eq. (1) as follows:

$$F_{\frac{1}{2}}(\xi) = \int_0^\xi \frac{x^{\frac{1}{2}}}{1+e^{-(\xi-x)}}dx + \int_\xi^\infty \frac{x^{\frac{1}{2}}e^{-(x-\xi)}}{1+e^{-(x-\xi)}}dx \qquad \forall \xi > 0 \, . \tag{2}$$

At first glance, Eq. (2) is nothing more than a way to split the integral of Eq. (1) into two convergent integrals no easier to deal with than Eq. (1). What have we really gained from this step? Well, it turns out that the right-hand side of Eq. (2) represents two convergent real convolution integrals. This is the critical step in developing an analytical solution to the Fermi-Dirac integral. We can now rewrite Eq. (2) and put it into a form which makes it more obvious to see that indeed the right-hand side of Eq. (2) represents two convergent convolution integrals. Equation (2) can be written as

$$F_{\frac{1}{2}}(\xi) = \sum_{p=1}^\infty (-1)^{p-1} \int_0^\xi x^{\frac{1}{2}} e^{-(p-1)(\xi-x)}dx + \sum_{p=1}^\infty (-1)^{p-1} \int_\xi^\infty x^{\frac{1}{2}} e^{-p(x-\xi)}dx$$

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$$=\int_0^{\xi} x^{\frac{1}{2}}\,dx-\sum_{p=1}^{\infty}(-1)^{p+1}\int_0^{\xi} x^{\frac{1}{2}}e^{-p(\xi-x)}dx+\sum_{p=1}^{\infty}(-1)^{p+1}\int_{\xi}^{\infty} x^{\frac{1}{2}}e^{-p(x-\xi)}dx$$

$$=\frac{2}{3}\xi^{\frac{3}{2}}-\sum_{p=1}^{\infty}\left(-1\right)^{p+1}\int_0^{\xi} x^{\frac{1}{2}}e^{-p(\xi-x)}dx+\sum_{p=1}^{\infty}(-1)^{p+1}\int_{\xi}^{\infty} x^{\frac{1}{2}}e^{-p(x-\xi)}dx \quad (3)$$

Firstly, the binomial expansion of the denominator of the first integral in Eq. (2) is mathematically justified since $e^{-(\xi-x)}<1\ \forall x<\xi$ within the limits of integration. Also, the binomial expansion of the denominator of the second integral in Eq. (2) is mathematically justified since $e^{-(x-\xi)}<1\ \forall\xi<x$ within the limits of integration. Notice that each of the two integrals on the right-hand side of Eq. (3) has exactly the mathematical structure of a real convolution integral. Armed with this observation allows one to rewrite the two integrals in Eq. (3) as follows:

$$\int_0^{\xi} x^{\frac{1}{2}}e^{-p(\xi-x)}dx=\frac{\sqrt{\pi}}{2}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\xi s}}{s^{\frac{3}{2}}\left(s+p\right)}ds\ , \quad (4)$$

$$\int_{\xi}^{\infty} x^{\frac{1}{2}}e^{-p(x-\xi)}dx=\frac{\sqrt{\pi}}{2}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\xi s}}{s^{\frac{3}{2}}\left(-s+p\right)}ds \quad (5)$$

The right-hand side of Eqs. (4) and (5) give the *s-domain* representation of the corresponding integrals on the left-hand side of Eqs. (4) and (5). This is nothing more than an application of the Faltung theorem for the Laplace Transform found on pages 30 and 31 of Sneddon [16]. Substituting Eqs. (4) and (5) into Eq. (3) and simplifying the algebra yields the following expression for $F_{\frac{1}{2}}\left(\xi\right)$ given by

$$F_{\frac{1}{2}}\left(\xi\right)=\frac{2}{3}\xi^{\frac{3}{2}}-\sqrt{\pi}\sum_{p=1}^{\infty}\left(-1\right)^{p+1}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\xi s}}{s^{\frac{1}{2}}\left(s^2-p^2\right)}ds \quad \forall\xi>0 \quad (6)$$

Equation (6) represents a mathematically exact expression for $F_{\frac{1}{2}}\left(\xi\right)$. The next step in the simplification process is to evaluate the integral in Eq. (6). To this end, the contour illustrated in Fig. (**1**) is chosen. Notice that the contour is closed in the left-hand plane in order to ensure convergence.

One can now employ the residue theorem of complex variable theory to the Fermi-Dirac contour of Fig. (**1**) and write the following:

$$\begin{aligned}\frac{1}{2\pi i}\oint\frac{e^{\xi s}}{s^{\frac{1}{2}}\left(s^2-p^2\right)}ds&=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\xi s}}{s^{\frac{1}{2}}\left(s^2-p^2\right)}ds\\&+\frac{1}{2\pi i}\int_{\infty}^{0}\frac{e^{i\xi y}}{\left(ye^{\frac{\pi i}{2}}\right)^{\frac{1}{2}}\left[\left(ye^{\frac{\pi i}{2}}\right)^2-p^2\right]}idy\\&+\frac{1}{2\pi i}\int_{0}^{\infty}\frac{e^{i\xi y}}{\left(ye^{-\frac{3\pi i}{2}}\right)^{\frac{1}{2}}\left[\left(ye^{-\frac{3\pi i}{2}}\right)^2-p^2\right]}idy\\&=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\xi s}}{s^{\frac{1}{2}}\left(s^2-p^2\right)}ds+\frac{e^{-\frac{\pi i}{4}}}{2\pi}\int_0^{\infty}\frac{e^{i\xi y}}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy\\&-\frac{e^{\frac{3\pi i}{4}}}{2\pi}\int_0^{\infty}\frac{e^{i\xi y}}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy\end{aligned} \quad (7)$$

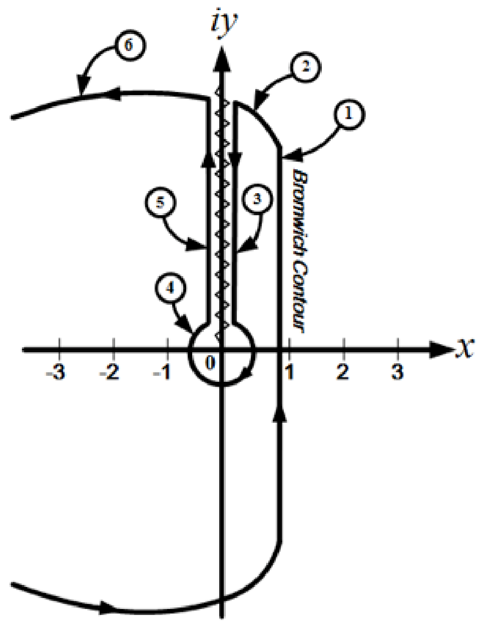

Fig. (1). Fermi-Dirac contour.

Only the line integrals along the contours 1, 3, and 5 contribute a nonzero quantity to the closed line integral on the left-hand side of Eq. (7). The integral on the left-hand side of Eq. (7) encloses only simple poles since the origin has been excluded from the contour of integration. Evaluating the integral on the left-hand side of Eq. (7) by applying the residue theorem yields the following:

$$\frac{1}{2\pi i}\oint\frac{e^{\xi s}}{s^{\frac{1}{2}}\left(s^2-p^2\right)}ds=-\frac{ie^{-p\xi}}{2p^{\frac{3}{2}}} \quad (8)$$

Substituting Eq. (8) into Eq. (7) allows one to write the following expression.

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\xi s}}{s^{\frac{1}{2}}\left(s^2-p^2\right)}ds=-\frac{ie^{-p\xi}}{2p^{\frac{3}{2}}}-\frac{e^{-\frac{\pi i}{4}}}{\pi}\int_0^{\infty}\frac{e^{i\xi y}}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy \quad (9)$$

One can now rewrite Eq. (9) as follows:

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\xi s}}{s^{\frac{1}{2}}\left(s^2-p^2\right)}ds=-\frac{1}{\pi\sqrt{2}}\int_0^{\infty}\frac{\cos\left(\xi y\right)+\sin\left(\xi y\right)}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy$$

$$+i\left[-\frac{e^{-p\xi}}{2p^{\frac{3}{2}}}+\frac{1}{\pi\sqrt{2}}\int_0^\infty\frac{\cos(\xi y)-\sin(\xi y)}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy\right]. \tag{10}$$

Now, since $F_{\frac{1}{2}}(\xi)$ is a real quantity, one immediately concludes that

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\xi s}}{s^{\frac{1}{2}}\left(s^2-p^2\right)}ds=-\frac{1}{\pi\sqrt{2}}\int_0^\infty\frac{\cos(\xi y)+\sin(\xi y)}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy, \tag{11}$$

since

$$0=-\frac{e^{-p\xi}}{2p^{\frac{3}{2}}}+\frac{1}{\pi\sqrt{2}}\int_0^\infty\frac{\cos(\xi y)-\sin(\xi y)}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy. \tag{12}$$

Equation (12) expresses the fact that the imaginary term in Eq. (10) must be zero. Therefore, from Eq. (12), one now knows that

$$\int_0^\infty\frac{\cos(\xi y)-\sin(\xi y)}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy=\frac{\pi e^{-p\xi}}{\sqrt{2}p^{\frac{3}{2}}}, \tag{13}$$

and rewriting Eq. (13) yields:

$$\int_0^\infty\frac{\cos(\xi y)}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy=\frac{\pi e^{-p\xi}}{\sqrt{2}p^{\frac{3}{2}}}+\int_0^\infty\frac{\sin(\xi y)}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy. \tag{14}$$

Substituting Eq. (14) into Eq. (11) yields:

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\xi s}}{s^{\frac{1}{2}}\left(s^2-p^2\right)}ds=-\frac{e^{-p\xi}}{2p^{\frac{3}{2}}}-\frac{\sqrt{2}}{\pi}\int_0^\infty\frac{\sin(\xi y)}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy. \tag{15}$$

So we now have an analytical expression for the integral given in Eq. (6). Employing Eq. (15) allows Eq. (6) to be written as

$$\begin{aligned}F_{\frac{1}{2}}(\xi)=&\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\sqrt{\pi}}{2}\sum_{p=1}^{\infty}(-1)^{p+1}\frac{e^{-p\xi}}{p^{\frac{3}{2}}}\\&+\sqrt{\frac{2}{\pi}}\sum_{p=1}^{\infty}(-1)^{p+1}\int_0^\infty\frac{\sin(\xi y)}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy\quad\forall\xi>0\end{aligned}. \tag{16}$$

However, the integral in Eq. (16) is readily evaluated as follows:

$$\begin{aligned}\int_0^\infty\frac{\sin(\xi y)}{y^{\frac{1}{2}}\left(y^2+p^2\right)}dy=&\frac{\pi}{2\sqrt{2}p^{\frac{3}{2}}}\\&\left[e^{p\xi}Erfc\left(\sqrt{p\xi}\right)+e^{-p\xi}Erfi\left(\sqrt{p\xi}\right)-e^{-p\xi}\right]\end{aligned}, \tag{17}$$

where $Erfc(\bullet)$ and $Erfi(\bullet)$ are defined as the complementary Error function and imaginary Error function respectively. In fact, if one uses Mathematica [17] to evaluate the integral in Eq. (17) then the solution one obtains is in terms of $Erfc(\sqrt{p\xi})$ and $Erfi(\sqrt{p\xi})$. Substituting Eq. (17) into Eq. (16) yields the analytical solution to the Fermi-Dirac integral $F_{\frac{1}{2}}(\xi)$, where we can now relax our initial restriction and allow $\xi\geq 0$.

One can now write the total solution valid $\forall\xi\in\mathbb{R}$ as follows:

$$F_{\frac{1}{2}}(\xi)=F_{\frac{1}{2}}(\xi)_{\xi<0}+F_{\frac{1}{2}}(\xi)_{\xi\geq 0}, \tag{18}$$

or

$$F_{\frac{1}{2}}(\xi)=F_{\frac{1}{2}}(\xi)_{\xi\leq 0}+F_{\frac{1}{2}}(\xi)_{\xi>0}. \tag{19}$$

We will choose to use Eq. (18) and define the following:

$$F_{\frac{1}{2}}(\xi)_{\xi<0}=\frac{\sqrt{\pi}}{2}e^{-|\xi|}\sum_{p=0}^{\infty}\frac{(-1)^p e^{-p|\xi|}}{(p+1)^{\frac{3}{2}}}\quad\forall\xi\in\mathbb{R}<0. \tag{20}$$

Equation (20) is easily derived by direct integration of Eq. (1) after letting $\xi\to-\xi$.

$$\begin{aligned}F_{\frac{1}{2}}(\xi)_{\xi\geq 0}=&\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\sqrt{\pi}}{2}\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p^{\frac{3}{2}}}\\&\left[e^{p\xi}Erfc\left(\sqrt{p\xi}\right)+e^{-p\xi}Erfi\left(\sqrt{p\xi}\right)\right]\quad\forall\xi\in\mathbb{R}\geq 0\end{aligned} \tag{21}$$

Equation (21) shows that the Fermi-Dirac integral of order $\frac{1}{2}$, which has been stated in the literature to be analytically intractable [18-21], is indeed analytically tractable. However, the summation given in Eq. (21) does not yield a closed-form expression in terms of elementary functions.

The Error functions in Eq. (21) are easily computed numerically by using Mathematica. However, one may find alternative functional representations instead of Error functions. For example, Ulrich *et al.,* [18] uses Polylogarithms to accurately compute the numerical values of Fermi-Dirac integrals. Also, Mathematica employs a Polylogarithm function algorithm to compute Fermi-Dirac type integrals. However, the method developed in this paper is a direct attack on the Fermi-Dirac integral itself. In other words, the authors have developed a complete analytical solution from first principles. The summation term in Eq. (21) cannot be summed in closed form in terms of elementary functions, however, this in no way negates the authors' claim that the solution is a completely analytical solution. No mathematical approximations were made in deriving Eq. (17) and this, of course, lead directly to the Fermi-Dirac integral given by Eq. (21).

III. ILLUSTRATIVE EXAMPLE

Let's consider the following numerical example. For $0 \le \xi \le 5$, one can make a few representative plots comparing a numerical evaluation of the Fermi-Dirac integral given by Eq. (1) and its analytical evaluation given in Eq. (21). Figs. (**2-5**) below illustrate the results. Also, $p_{\max}$ is the maximum number of terms retained in the summation of Eq. (21). All numerical computations were done using Mathematica.

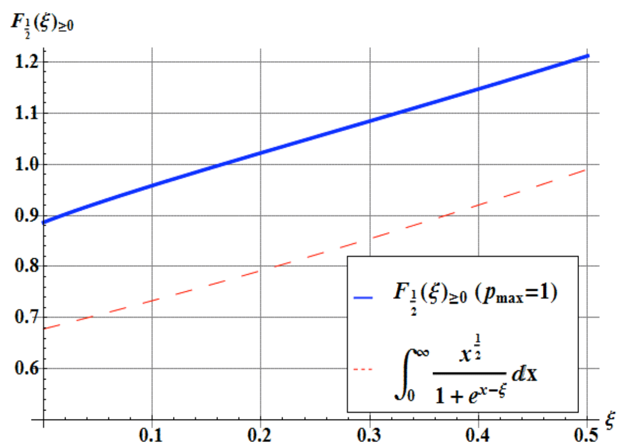

Fig. (2). $p_{\max} = 1$.

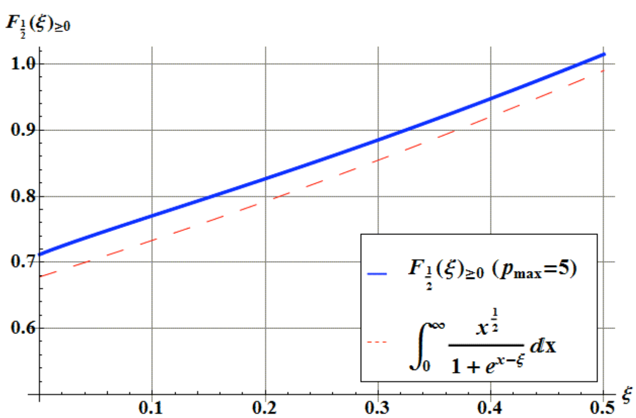

Fig. (3). $p_{\max} = 5$.

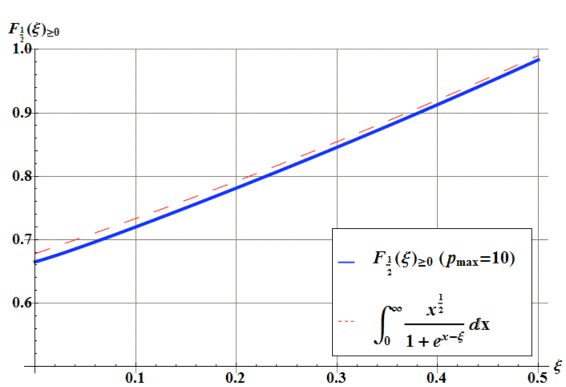

Fig. (4). $p_{\max} = 10$.

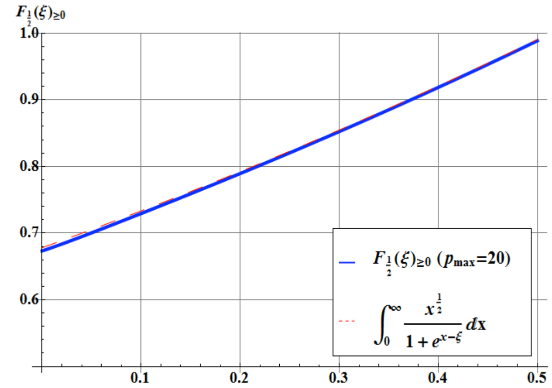

Fig. (5). $p_{\max} = 20$.

One can see from Figs. (**2**) through (**5**) that for $p_{\max} = 20$ in Eq. (21) it becomes difficult to distinguish between the plot of Eq. (1) and Eq. (21). If one plots the ratio of these two equations, then one gets a better feel for how Eq. (21) is behaving. Fig. (**6**) illustrates the ratio of Eq. (21), with $p_{\max} = 50$, to Eq. (1), evaluated numerically. Computing the sum for large values of $p_{\max}$ using Mathematica is easily done. As more terms are retained in Eq. (21), the ratio of Eq. (21) to Eq. (1) as shown in Fig. (**6**), approaches unity especially at points near $\xi = 0$. This is illustrated in Fig. (**7**) where 100 terms in Eq. (21) were retained. As stated earlier, the goal of this article was not to delve into the numerical properties of Eq. (21). However, from the standpoint of computer numerics, it appears that Eq. (21) could be used for numerical purposes. As ξ moves away from the origin, fewer and fewer terms in Eq. (21) need to be retained. This is, of course, exactly what is to be expected. For example, Fig. (**8**) is a plot of the ratio of Eq. (21) to Eq. (1) for $0.5 \le \xi \le 5$ using $p_{\max} = 30$. One can see that as ξ moves away from the origin, $F_{\frac{1}{2}}(\xi)_{\xi \ge 0}$ needs fewer and fewer terms to compute the Fermi-Dirac integral of order $\frac{1}{2}$. Fig. (**9**) represents a plot of the full solution given by Eq. (18) for $-2 \le \xi \le 2$.

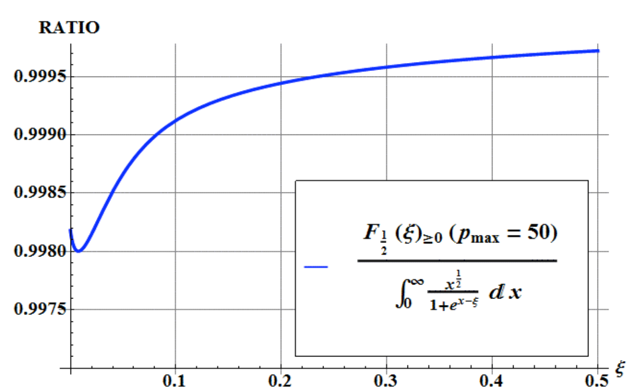

Fig. (6). $p_{\max} = 50$.

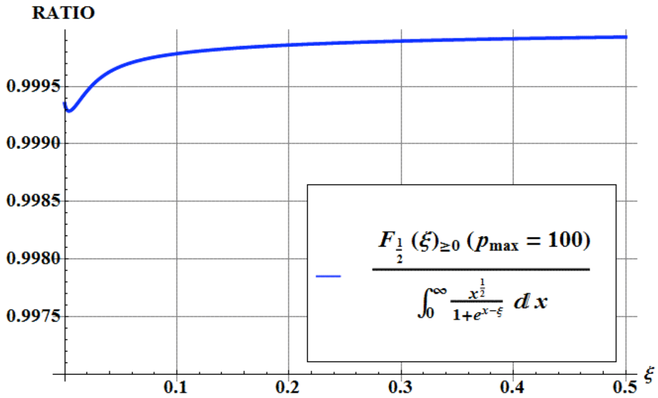

Fig. (7). $p_{\max} = 100$.

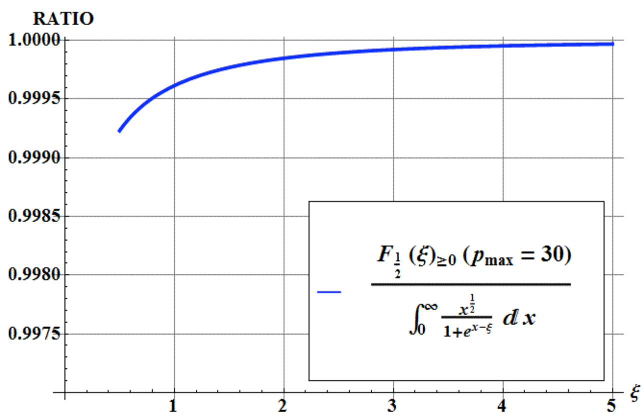

Fig. (8). $p_{\max} = 30$.

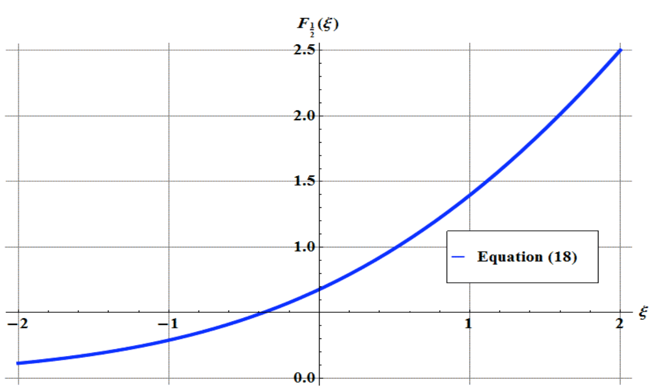

Fig. (9). $F_{\frac{1}{2}}(\xi)\ \forall \xi \in \mathbb{R}$.

IV. DISCUSSION

The Fermi-Dirac integral has received much attention throughout the twentieth century and for good reason. It is well known for its application in areas such as solid-state physics, statistical mechanics, quantum statistical mechanics, condensed matter physics, and solid-state electronics. However, it also sees a renewed interest in other areas such as the flow of traffic in communication networks [22]. Its wide-range of applicability coupled with the belief that no analytical solution existed coaxed the authors to make a concerted effort to analytically solve this seemingly intractable integral.

In the authors' opinion, the most important aspect of this article does not solely lie in the mathematical steps used to analytically evaluate the Fermi-Dirac integral, but in the observation that real convolution could be used. Once this is recognized, the mathematical machinery is straightforward. The calculus of residues, used in this article, is well known. No new mathematical formulations needed to be introduced. No complicated mathematical transformations were employed. There are hundreds of papers throughout the literature devoted to the analytical approximation and the numerical computation of Fermi-Dirac integrals. Some of the techniques require a fairly sophisticated knowledge of various special functions, asymptotic expansions, and computational mathematics. The method developed in this paper required only a basic knowledge of the theory of residues and real convolution to produce a complete analytical solution.

Although the main focus of this article was to develop, from first principles, a complete analytical solution for the half-order Fermi-Dirac functions, a critical aspect not discussed in this article is the numerical efficacy of Eq. (21). Whether Eq. (21) has any numerical advantages over other existing methods certainly needs to be addressed. There are numerous articles which tackle the Fermi-Dirac integral from a numerical point of view. For example, A. D. Kozhukhovskii *et al.,* [23] employs shifted Chebyshev polynomials of the first kind for aiding in the evaluation of Fermi-Dirac integrals. W. J. Cody *et al.,* [24] employs rational Chebyshev approximations to tackle the same problem as A. D. Kozhukhovskii *et al.,* J. Macleod [25] employs Chebyshev polynomials and introduces a very useful algorithm for accurately computing Fermi-Dirac integrals. M.Goano [26] develops an algorithm based upon a hypergeometric series expansion for computing the complete and incomplete Fermi-Dirac integral. In fact, the authors have employed, within the framework of Mathematica [17], the algorithms developed by both J. Macleod and M. Goano in order to verify the plot illustrated in Fig. (**9**) of this article. N. Mohankumar [11] has derived a very interesting equation which makes use of the complementary Error function. The results of his article may help to further develop the numerical properties of Eq. (21). Also, N. Mohankumar *et al.* [27, 28] provides an algorithm for computing the half-order complete and incomplete Fermi-Dirac integrals. Interestingly, their method employs a modified trapezoidal rule. This approach corrects for any loss in the accuracy which is due to the poles in the integrand of the Fermi-Dirac integral. In fact, this technique may be directly applied to the integral given on the right-hand side of Eq. (16). This avoids having to evaluate the complementary Error function and the imaginary Error function given in Eq. (21).

It is clear that a much more in-depth analysis of the numerical efficacy of Eq. (21) needs to be performed. It is hoped that this article will spawn future articles which will shine light on whether the mathematical expression given by Eq. (21) is indeed useful from a computational standpoint.

Appendix

HALF-ORDER FERMI-DIRAC FUNCTIONS

The derivation of $F_{\frac{1}{2}}(\xi)_{\xi\geq 0}$ can easily be extended to include all $F_{m-\frac{1}{2}}(\xi)\forall m\in\mathbb{Z}^{\geq}$. In order to accomplish this, let's define the Fermi-Dirac function introduced by Dingle [29], as follows:

$$\Im_{m-\frac{1}{2}}(\xi)=\frac{1}{\Gamma\left(m+\frac{1}{2}\right)}\int_0^\infty\frac{x^{m-\frac{1}{2}}}{1+e^{x-\xi}}dx\,, \tag{A.1}$$

$$=\frac{1}{\Gamma\left(m+\frac{1}{2}\right)}F_{m-\frac{1}{2}}(\xi)\qquad\forall\xi\geq 0\text{ and }\forall m\in\mathbb{Z}^{\geq}\,. \tag{A.2}$$

Using Eq. (A.2) and the known recurrence relationship [29, 30] given by

$$\Im_{m-\frac{1}{2}}(\xi)=\frac{d}{d\xi}\Im_{m+\frac{1}{2}}(\xi)\,, \tag{A.3}$$

one can compute all the half-order Fermi-Dirac functions. A few of these are listed below.

$$\Im_{-\frac{1}{2}}(\xi)=\frac{d}{d\xi}\Im_{\frac{1}{2}}(\xi)=\frac{2}{\sqrt{\pi}}\xi^{\frac{1}{2}}-\sum_{p=1}^{\infty}\frac{(-1)^p}{p^{\frac{1}{2}}}\left[e^{p\xi}Erfc\left(\sqrt{p\xi}\right)-e^{-p\xi}Erfi\left(\sqrt{p\xi}\right)\right], \tag{A.4}$$

$$\Im_{\frac{1}{2}}(\xi)=\int\Im_{-\frac{1}{2}}(\xi)d\xi=\frac{4}{3\sqrt{\pi}}\xi^{\frac{3}{2}}-\sum_{p=1}^{\infty}\frac{(-1)^p}{p^{\frac{3}{2}}}\left[e^{p\xi}Erfc\left(\sqrt{p\xi}\right)+e^{-p\xi}Erfi\left(\sqrt{p\xi}\right)\right], \tag{A.5}$$

$$\Im_{\frac{3}{2}}(\xi)=\int\Im_{\frac{1}{2}}(\xi)d\xi=\frac{8}{15\sqrt{\pi}}\xi^{\frac{5}{2}}-\sum_{p=1}^{\infty}\frac{(-1)^p}{p^{\frac{5}{2}}}\left[\frac{4\sqrt{p\xi}}{\sqrt{\pi}}+e^{p\xi}Erfc\left(\sqrt{p\xi}\right)-e^{-p\xi}Erfi\left(\sqrt{p\xi}\right)\right], \tag{A.6}$$

$$\Im_{\frac{5}{2}}(\xi)=\int\Im_{\frac{3}{2}}(\xi)d\xi=\frac{16}{105\sqrt{\pi}}\xi^{\frac{7}{2}}-\sum_{p=1}^{\infty}\frac{(-1)^p}{p^{\frac{7}{2}}}\left[\frac{8(p\xi)^{\frac{3}{2}}}{3\sqrt{\pi}}+e^{p\xi}Erfc\left(\sqrt{p\xi}\right)+e^{-p\xi}Erfi\left(\sqrt{p\xi}\right)\right], \tag{A.7}$$

$$\Im_{\frac{7}{2}}(\xi)=\int\Im_{\frac{5}{2}}(\xi)d\xi=\frac{32}{945\sqrt{\pi}}\xi^{\frac{9}{2}}-\sum_{p=1}^{\infty}\frac{(-1)^p}{p^{\frac{9}{2}}}\left[\begin{array}{l}\frac{4\sqrt{p\xi}}{\sqrt{\pi}}+\frac{16(p\xi)^{\frac{5}{2}}}{15\sqrt{\pi}}+e^{p\xi}Erfc\left(\sqrt{p\xi}\right)\\ \qquad\qquad -e^{-p\xi}Erfi\left(\sqrt{p\xi}\right)\end{array}\right], \tag{A.8}$$

$$\Im_{m+\frac{1}{2}}(\xi)=\int\Im_{m-\frac{1}{2}}(\xi)d\xi\,. \tag{A.9}$$

The higher half-order Fermi-Dirac Functions are easily derived by employing Eq. (21), Eq. (A.2) and Eq. (A.3). Also, Eq. (A.3) implies that analytical continuation is possible and one may employ this idea to compute $\Im_{m-\frac{1}{2}}(\xi)$ for values of m for which the integral in Eq. (A.1) is divergent. In fact, A. Trellakis *et al.,* [20] and Lether [14] develop methods for computing $\Im_{-\frac{3}{2}}(\xi)$ based upon analytical continuation. The analytical expressions for the half-order Fermi-Dirac functions introduced in this Appendix may prove to be fruitful within that domain for which analytical continuation is valid.

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