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On the Irreducibility of Wada's Representation of the Pure Braid Group, P_4

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Abstract: We consider the reduced Wada's representation of the pure braid group, namely $P_4 \to GL_3(\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}, t_4^{\pm1}])$. We then specialize the parameters t_1, t_2, t_3, t_4 to nonzero complex numbers z_1, z_2, z_3, z_4 . Our main theorem asserts that the reduced Wada's representation, $\varphi_4 : P_4 \to GL_3(\mathbb{C})$, is reducible if and only if $z_1^2 = z_2^2 = z_3^2 = z_4^2$.

Keywords: Pure braid group, Wada's representation,irreducible.

1. INTRODUCTION

Let B_n be the braid group on n strings. We consider a normal subgroup, namely the pure braid group, denoted by P_n . In section 2, we define Wada's representation of pure braid group on four strings. Under that representation, the automorphism corresponding to σ_i , takes $x_i \to x_i x_{i+1}^{-1} x_i$, $x_{i+1} \to x_i$; and fixes all other free generators. We then specialize the indeterminates used in defining the representation $P_4 \to GL_4(\mathbb{Z}[t_1^{\pm1},...,t_4^{\pm1}])$ to nonzero complex numbers a, b, c and d . In [1], it was shown that the reduced Wada's representation $B_n \to GL_{n-1}(\mathbb{C})$ is irreducible if and only if n is an odd integer. In section 3, we consider the question of the irreducibility after we restrict the representation to the normal subgroup of B_4, namely the pure braid group P_4 . In other words, we determine necessary and sufficent conditions under which $\varphi_4(a,b,c,d) : P_4 \to GL_3(\mathbb{C})$ is reducible.

2. DEFINITIONS

Definition 1

The braid group on n strings, B_n , is the abstract group with presentation

$$B_n = \{\sigma_1, ..., \sigma_{n-1} / \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1,...,n-2, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1\}.$$

The generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ are called the standard generators of B_n (See [3]).

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Definition 2

The pure braid group, denoted by P_n, is defined as the kernel of the homomorphism $B_n \to S_n$ defined by $\sigma_i \to (i, i+1)$, $1 \le i \le n-1$ (See [2]). It is finitely generated by the elements

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}...\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}...\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, 1 \le i < j \le n.$$

Let F_n be the free group of rank n, with free basis $x_1,...,x_n$. According to Wada's representation, the action of braid generators σ_i on the basis $\{x_1,...,x_n\}$ is defined as follows:

$$\sigma_i : \begin{cases} x_i \to x_i x_{i+1}^{-1} x_i \\ x_{i+1} \to x_i \\ x_j \to x_j \text{ for } j \notin \{i, i+1\} \end{cases}$$

By applying the Magnus representation to the image of the pure braid group under Wada' s representation, we determine the linear representation $P_4 \to GL_4(\mathbb{Z}[t_1^{\pm1},...,t_4^{\pm1}])$. Now we specialize the indeterminantes $t_1,...,t_4$ in Wada's representation to nonzero complex numbers a , b , c and d respectively. We then conjugate this representation by a matrix T defined by

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Having done this, we observe that entries $(2,1)$, $(3,1)$ and $(4,1)$ of the images of all the generators of P_4 under Wada's representation are zeros. Therefore, we may delete the first row and the first column to obtain a representation

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of degree 3, and we denote the representation by φ_4. For simplicity, we still call $T^{-1}A_{ij}T$ by A_{ij} for $1 \le i < j \le 4$.

Definition 3

For $(a,b,c,d) \in (\mathbb{C}^*)^4$, the reduced Wada's representation $\varphi_4(a,b,c,d): P_4 \to GL_3(\mathbb{C})$ is given by

$$A_{12} = \begin{pmatrix} \frac{a^2}{b^2} & 0 & 0 \\ -\frac{a+b}{b} & 1 & 0 \\ -\frac{a+b}{b} & 0 & 1 \end{pmatrix}, A_{13} = \begin{pmatrix} 1+\frac{ac(b^2+ac)}{b^4} & \frac{a(b^2+ac)}{b^3} & 0 \\ -\frac{b(b^2+ac)}{a^2c} & \frac{-b^2}{ac} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{23} = \begin{pmatrix} 1 & \frac{b(b+c)}{c^2} & 0 \\ 0 & \frac{b^2}{c^2} & 0 \\ 0 & \frac{b(b+c)}{c^2} & 1 \end{pmatrix}, A_{34} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{c^2+cd+d^2}{d^2} & \frac{-c(c+d)}{d^2} \\ 0 & \frac{c+d}{d} & -\frac{c}{d} \end{pmatrix},$$

$$A_{14} = \begin{pmatrix} 1+\frac{ac^2(ac^2+b^2d)}{b^4d^2} & \frac{a(c+d)(ac^2+b^2d)}{b^3d^2} & \frac{-ac(ac^2+b^2d)}{b^3d^2} \\ 0 & 1 & \frac{-c(c+d)}{d^2} \\ \frac{c(b^2d+ac^2)}{b^3d} & \frac{(b^2+ac)(ac^2+b^2d)}{b^4d} & -\frac{c}{d} \end{pmatrix}$$

and

$$A_{24} = \begin{pmatrix} 1 & \frac{b(c+d)(c^2+bd)}{c^4} & -\frac{b(c^2+bd)}{c^3} \\ 0 & \frac{c^4+b(c^2+bd)(c+d)}{c^4} & \frac{-b(c^2+bd)}{c^3} \\ 0 & \frac{(c^2+bd)(bc^4+c^5+b^3dc+b^3d^2)}{b^2c^4d} & -\frac{(c^5+b^2c^2d+b^3d^2)}{bc^3d} \end{pmatrix}.$$

3. IRREDUCIBILITY OF ϕ_4

We determine necessary and sufficient conditions under which the complex specialization $\phi_4(a,b,c,d)$ is irreducible.

Lemma 4

For $(a,b,c,d) \in (\mathbb{C}^*)^4$, the reduced Wada's represntation $\phi_4(a,b,c,d): P_4 \to GL_3(\mathbb{C})$ is irreducible if $a^2 \ne b^2$ or $b^2 \ne c^2$ or $c^2 \ne d^2$.

Proof. Let $a^2 \ne b^2$. We diagonalize the matrix that corresponds to the pure braid A_{12} by an invertible matrix M defined by

$$M = \begin{pmatrix} 0 & 0 & 1-ab^{-1} \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Direct computations show that

$$M^{-1}A_{12}M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^2b^{-2} \end{pmatrix}.$$

Now we conjugate the reduced Wada's representation, φ_4, by M to get an equivalent representation of degree 3. For simplicity, we still denote $M^{-1}A_{ij}M$ by A_{ij} for $1 \le i < j \le 4$. The matrices are given by

$$A_{24} = \begin{pmatrix} \frac{-c^2}{bd} - \frac{abn}{(a-b)c^3} & \frac{pn}{(a-b)b^2c^4d} & \frac{nc}{b^2d} + \frac{abdn}{(a-b)c^4} \\ \frac{abn}{(-a+b)c^3} & \frac{q}{(a-b)c^4} & \frac{abdn}{(a-b)c^4} \\ \frac{b^2n}{(a-b)c^3} & \frac{-b^2n(c+d)}{(a-b)c^4} & 1-\frac{b^2dn}{(a-b)c^4} \end{pmatrix},$$

$$A_{13} = \begin{pmatrix} 1 & \frac{a(b^2+ac)}{(a-b)b^2} & \frac{-a(b^2+ac)(-b^2+ac-bc)}{(a-b)b^4} \\ 0 & \frac{-b^2}{ac} + \frac{a(b^2+ac)}{(a-b)b^2} & \frac{-b(b^2+ac)}{a^2c} + \frac{a(b^2+ac)(b(b+c)+ac)}{(a-b)b^4} \\ 0 & \frac{-a(b^2+ac)}{(a-b)b^2} & \frac{-b^5-ab^3c+a^3c^2-a^2bc^2}{(a-b)b^4} \end{pmatrix},$$

$$A_{14} = \begin{pmatrix} \frac{acm}{(b-a)b^2d^2} - \frac{ac^2}{b^2d} & \frac{m(b^2+ac)}{b^4d} + \frac{ma(c+d)}{b^2d^2(a-b)} & \frac{mc}{b^3d} + \frac{am(bn-ac^2)}{b^4d^2(a-b)} \\ \frac{-acm}{(a-b)b^2d^2} & 1+\frac{am(c+d)}{(a-b)b^2d^2} & \frac{-amc^2}{b^4d^2} + \frac{am}{(a-b)b^2d} \\ \frac{acm}{(a-b)b^2d^2} & \frac{-am(c+d)}{(a-b)b^2d^2} & \frac{a^2c^4}{b^4d^2} - \frac{m}{(a-b)bd} \end{pmatrix},$$

$$A_{34} = \begin{pmatrix} \frac{-c}{d} & \frac{c+d}{d} & 0 \\ \frac{-c(c+d)}{d^2} & \frac{c^2+cd+d^2}{d^2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{23} = \begin{pmatrix} 1 & \frac{-ab(b+c)}{(-a+b)c^2} & \frac{ab(b+c)}{(a-b)c^2} \\ 0 & \frac{-b^2(a+c)}{(-a+b)c^2} & \frac{(b+c)(ab-ac+bc)}{(a-b)c^2} \\ 0 & \frac{-b^2(b+c)}{(a-b)c^2} & 1+\frac{-b^2(b+c)}{(a-b)c^2} \end{pmatrix},$$

where

$m = ac^2 + b^2d,$

$n = c^2 + bd,$

$p = abc^4 - b^2c^4 + ac^5 - bc^5 + ab^3cd + ab^3d^2,$

$q = abc^3 + ac^4 - bc^4 + ab^2cd + abc^2d + ab^2d^2.$

Suppose to get contradiction that φ_4 is reducible. Then there exists a proper nonzero invariant subspace S, where the dimension of S is either 1 or 2. We will show that a contradiction is obtained in each of the following cases.

1. Assume that the dimension of S is one. From the diagonal matrix, A_{12}, we see that the subspace S has to be generated by e_1+ue_2, e_2 or e_3, where u is a complex number.

Case 1.

$S=<e_1+ue_2>$. Since $e_1+ue_2\in S$, it follows that $A_{23}(e_1+ue_2)\in S$, which implies that $(b+c)u=0$. We have to consider the case $b+c=0$ and the case $u=0$.

(a) *Let* $b+c=0$.

- If $u\neq 0$, then $A_{13}(e_1+ue_2)\notin S$, a contradiction.
- If $u=0$, then $A_{34}e_1\in S$ and $A_{14}e_1\in S$. This implies that $a+b=0$, a contradiction.

(b) Let $u=0$.

- If $c+d\neq 0$, then $A_{34}e_1\notin S$, a contradiction.
- If $c+d=0$ and $b-c\neq 0$, then $A_{24}e_1\notin S$, a contradiction.
- If $c+d=0$ and $b-c=0$, then $A_{14}e_1\notin S$, a contradiction.

Case 2.

$S=<e_2>$. Since $e_2\in S$, it follows that $A_{23}e_2\in S$ which implies that $b+c=0$. If $b+c=0$, then $A_{13}e_2\notin S$, a contradiction.

Case 3.

$S=<e_3>$. Since $e_3\in S$, it follows that $A_{23}e_3\in S$ which implies that $b+c=0$. If $b+c=0$, then $A_{13}e_3\notin S$, a contradiction.

2. Assume that the dimension of S is two. We consider the cases $<e_2,e_3>$ and $<e_1+ue_2,e_3>$.

Case 4.

$S=<e_2,e_3>$. The proof goes along exactly same lines as Case 2.

Case 5.

$S=<e_1+ue_2,e_3>$. Since $e_1+ue_2\in S$, it follows that $A_{23}(e_1+ue_2)\in S$. This implies that $(c+b)(u-\frac{ab-ac+bc}{ab})u=0$.

(a) Let $c+b=0$.

- If $u\neq 0$ and $u\neq \frac{a^2+ab-b^2}{a^2}$, then $A_{13}(e_1+ue_2)\notin S$, a contradiction.
- If $u=0$ and $b-d\neq 0$, then $A_{34}e_1\notin S$, a contradiction.
- If $u=0$ and $b-d=0$, then $A_{24}e_1\notin S$, a contradiction.
- If $u=\frac{a^2+ab-b^2}{a^2}$, then $A_{13}e_3\notin S$, a contradiction.

(b) Let $u=\frac{ab-ac+bc}{ab}$.

- If $d\neq\frac{-ac^2}{b^2}$ and $d\neq\frac{ac}{b}$, then $A_{14}e_3\notin S$, a contradiction.
- If $d=\frac{-ac^2}{b^2}$ and $a^2+ab+bc\neq 0$, then $A_{24}e_3\notin S$, a contradiction.
- If $d=\frac{-ac^2}{b^2}$ and $a^2+ab+bc=0$, then $A_{24}(e_1+ue_2)\notin S$, a contradiction.
- If $d=\frac{ac}{b}$ and $c\neq b$, then $A_{34}(e_1+ue_2)\notin S$, a contradiction.
- If $d=\frac{ac}{b}$ and $c=b$ and $b\neq\frac{-a}{2}$, then $A_{24}(e_1+ue_2)\notin S$, a contradiction.
- If $d=\frac{ac}{b}$ and $c=b=\frac{-a}{2}$, then $A_{13}(e_1+ue_2)\notin S$, a contradiction.

(c) Let $u=0$.

If $c+d\neq 0$ then $A_{34}e_1\notin S$, a contradiction.

If $c+d=0$ and $b-c\neq 0$, then $A_{24}e_1\notin S$, a contradiction.

If $c+d=0$ and $b-c=0$, then $A_{14}e_1\notin S$, a contradiction.

Almost the same proof, as in the case $a^2\neq b^2$, is applied to each of the cases $b^2\neq c^2$ and $c^2\neq d^2$. In each of these cases, we conjugate the corresponding representation by the invertible matrices

$$N=\begin{pmatrix}0&1&1\\0&0&1-b^{-1}c\\1&0&1\end{pmatrix}\text{and}P=\begin{pmatrix}0&1&0\\1&0&cd^{-1}\\1&0&1\end{pmatrix}$$

respectively.

Lemma 5.

For $(a,b,c,d)\in(\mathbb{C}^*)^4$, the reduced Wada's represntation $\varphi_4(a,b,c,d):P_4\to GL_3(\mathbb{C})$ is reducible if $a^2=b^2=c^2=d^2$.

Proof. It is clear that we have 2^3 cases. They are

$(1) -a = +b = +c = +d$	$(5) +a = +b = -c = -d$
$(2) +a = -b = +c = +d$	$(6) +a = -b = +c = -d$
$(3) +a = +b = -c = +d$	$(7) +a = -b = -c = +d$
$(4) +a = +b = +c = -d$	$(8) +a = +b = +c = +d.$

Under each condition, we find a proper nonzero invariant subspace of the complex specialization of the reduced Wada's representation φ_4. The subspaces for (1), (2), (3), (4), (5),(6), (7) and (8) are $< e_1 >$, $< e_1 + e_2 + e_3 >$, $< e_2 >$, $< e_3 >$, $< e_2 + e_3 >$, $< e_1 + e_2 >$, $< e_1 + e_3 >$ and $< e_1 - e_2, e_1 + e_3 >$, respectively.

Combining Lemma 4 and Lemma 5, we get our main theorem:

Theorem 6.

For $(a,b,c,d) \in (\mathbb{C}^*)^4$, the reduced Wada's represntation $\varphi_4(a,b,c,d) : P_4 \to GL_3(\mathbb{C})$ is reducible if and only if $a^2 = b^2 = c^2 = d^2$.

ACKNOWLEDGEMENTS

None declared.

CONFLICT OF INTERESTS

None declared.

REFERENCES

[1] Abdulrahim MN. Wada's Representation is of Burau Type. J Algebra Comput 2001; 17(3):1-9.

[2] Birman JS. Braids, links and mapping class groups. annals of mathematical studies. vol. 82. New Jersey: Princeton University Press 1975.

[3] Hansen VL. Braids and Coverings. London Math. Soc., Cambridge: Cambridge University Press 1989.

Received: February 18, 2012 Revised: April 22, 2012 Accepted: April 30, 2012