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Two-Weight Orlicz Type Integral Inequalities for the Maximal Operator[1]

C.J. Neugebauer*

Department of Mathematics, Purdue University, West Lafayette, in 47907-1395, USA

Abstract: We present a two-weight Orlicz-type integral inequality for the maximal operator which characterizes $(u,v)\in A_p$.

Keywords: Maximal operator, two-weights.

1. INTRODUCTION

In this paper we will study integral inequalities of the type

$$\int_{R^n}\Phi(Mf(x)^p)u(x)dx \le c_1\int_{R^n}\Psi(c_2\,|f(x)|^p)v(x)dx, \qquad (1)$$

where $Mf(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(t)|dt$ is the Hardy-Littlewood maximal operator, and we ask for condtions on $\Phi,\Psi: R_+\to R_+$ such that (1) holds if and only if $(u,v)\in A_p$.

We say that $(u,v)\in A_p$ if $\frac{1}{|Q|}\int_Q u\left(\frac{1}{|Q|}\int_Q v^{1-p'}\right)^{p-1}\le c<\infty, 1<p<\infty$, and $Mu(x)\le cv(x)$, if $p=1$. These weight classes were introduced by Muckenhoupt [4] and Muckenhoupt and Wheeden [5] to study (1) when $\Phi(t)=\Psi(t)=t$. If $1<p<\infty$ and $u=v\in A_p$, (1) holds for $\Phi(t)=\Psi(t)=t$, but not if $p=1$. Also for each $1\le p<\infty$ there exists a pair $(u,v)\in A_p$ so that (1) fails in the special case $\Phi(t)=\Psi(t)=t$ [3, p. 395]. In these exceptional cases we have a weak type inequality. An excellent reference is the book by J.Garcia-Cuerva and J.L.Rubio de Francia [3]. We refer the reader interested in the current state of the two-weight theory to the recent book [1] by Cruz-Uribe, Martell, and Pérez.

The restrictions on Φ,Ψ are: $\Phi(t)=\int_0^t a(s)ds, \Psi(t)=\int_0^t b(s)ds$ with $a,b: R_+\to R_+$ satisfying

$$\int_0^s \frac{a(t)}{t}dt \le c'b(c''s), 0<s<\infty. \qquad (*)$$

Note that this excludes the classical case $\Phi(t)=\Psi(t)=t$. If $(*)$ holds, we say that Φ,Ψ are (c',c'')-related.

We are now ready to state our main result whose proof will be given in section 3.

Theorem 1 The following statements are equivalent for $1\le p<\infty$.

(2) For each Φ and Ψ which are (c',c'')-related, we have

$$\int_{R^n}\Phi(Mf^p)u \le c_1\int_{R^n}\Psi(c_2\,|f|^p)v,$$

for all $f: R^n\to R$, where the constamts c_1, c_2 depend only on c', c'' and p.

(3) We have $(u,v)\in A_p$.

Remark: In the Lebesgue measure case - $u=v=1$ - integral inequalities related to (2) can be found in [6]. It should be noted that $p=1$ is not excluded.

In section 4 we will examine in what sense the condition $\int_0^t \frac{a(s)}{s}ds \le c'b(c''t)$ is also necessary for Theorem 1, and in section 5 we will examine the extrapolation problem: when is it possible to replace p by $p-\varepsilon$ in (2). In sections 6 and 7 we will study the iterated maximal operator and its relation to extrapolation. In section 8 we will collect some unusual and surprising integral inequalities for Mf obtained by choosing Φ,Ψ and applying Theoren 1.

A final comment is in order. I have dedicated this paper to the memory of Richard A. Hunt who made significant contributions to the theory of A_p-weights and to whom I am indebted for introducing me to this subject some 40 years ago.

2. A TWO-WEIGHT DISTRIBUTIONAL INEQUALITY

For convenience all our functions will be non-negative: $f: R^n\to R_+$.

The distributional inequality below for $u=v=1$ - the Lebesgue measure case - and a sublinear operator T instead of M is equivalent with saying that T is both weak-type (p,p) and of type (∞,∞) [11, p. 103].

*Address correspondence to this author at the Department of Mathematics, Purdue University, West Lafayette, in 47907-1395, USA;
E-mail: neug@math.purdue.edu

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Theorem 2 The following statements are equivalent for $1 \le p < \infty$.

(4) There exists $0 < c_0 < \infty$ such that for every $f : R_n \to R_+$ we have for $0 < t < \infty$

$$u\{x : Mf(x) > t\} \le \frac{c_0}{t^p} \int_{t/c_0}^{\infty} v\{x : f(x) > s\} s^{p-1} ds.$$

(5) We have $(u, v) \in A_p$.

Proof. Apart from a minor detail, the proof follows the standard covering argument and we include it for the benefit of the reader.

$(5) \to (4)$. We may assume that M is the centered maximal operator

$$Mf(x) = \sup \frac{1}{|Q|} \int_Q f(\tau) d\tau,$$

where the $\sup$ is extended over all cubes Q centered at x. We consider the case $1 < p < \infty$ first. Fix $f : R_n \to R_+$, and for $0 < t < \infty$ let $f = f^t + f_t$, where

$$f^t(x) = \begin{cases} 0, & f(x) \le t/2 \\ f(x), & f(x) > t/2. \end{cases}$$

Then $Mf(x) \le Mf^t(x) + Mf_t(x)$ so that $\{Mf > t\} \subset \{Mf^t > t/2\} \equiv E_t$. Let $E_{tN} = E_t \cap \{x : |x| \le N\}$. We can now apply the Besicovitch covering Theorem and obtain cubes $\{Q_j\}$ satisfying

$$E_{tN} \subset \cup Q_j, |Q_j| \le \frac{2}{t} \int_{Q_j} f^t, \sum \chi_{Q_j} \le c < \infty.$$

Then

$$u(E_{tN}) \le \sum u(Q_j) \le \frac{c}{t^p} \sum \frac{u(Q_j)}{|Q_j|^p} \left(\int_{Q_j} f^t v^{1/p} v^{-1/p} \right)^p$$

$$\le \frac{c}{t^p} \sum \frac{u(Q_j)}{|Q_j|^p} \int_{Q_j} (f^t)^p v \cdot \left(\int_{Q_j} v^{1-p'} \right)^{p-1}$$

$$\le \frac{c}{t^p} \int_{\{f \ge t/2\}} f^p v.$$

If $A_t = \{x : f(x) \ge t/2\}$, then

$$u(E_{tN}) \le \frac{c}{t^p} \int_{R^n} (f\chi_{A_t})^p v = \frac{c}{t^p} \int_0^{\infty} v\{f\chi_{A_t} > s\} s^{p-1} ds$$

$$= \frac{c}{t^p} \left(\int_{t/2}^{\infty} v\{f > s\} s^{p-1} ds + v(A_t) \int_0^{t/2} s^{p-1} ds \right).$$

It is clear that for some constant c

$$c \int_{t/4}^{t/2} v\{f > s\} s^{p-1} ds \ge v(A_t) \int_0^{t/2} s^{p-1} ds,$$

and hence for some constant c_0

$$u(E_{tN}) \le \frac{c_0}{t^p} \int_{t/c_0}^{\infty} v\{f > s\} s^{p-1} ds.$$

Let now $N \to \infty$. We use the same notation for the case $p = 1$ as above. Since now $u(Q_j)/|Q_j| \le \inf_{Q_j} v$ we get

$$u(E_{tN}) \le \frac{c}{t} \sum \frac{u(Q_j)}{|Q_j|} \int_{Q_j} f^t$$

$$\le \frac{c}{t} \sum \int_{Q_j} f^t v \le \frac{c}{t} \int_{R^n} f\chi_{A_t} v.$$

Proceed now as in the case $1 < p < \infty$.

$(4) \to (5)$. For the case $p = 1$ we fix a cube Q_0 and let $f = \chi_Q$, where Q is an arbitrary subcube of Q_0. Then

$$Q_0 \subset \left\{ Mf \ge \frac{1}{|Q_0|} \int_Q f = \frac{|Q|}{|Q_0|} \equiv t \right\}.$$

Thus $u(Q_0) \le c_0(|Q_0|/|Q|)v(Q)$, and thus $u(Q_0)/|Q_0)| \le c_0 \inf_{Q_0} v$.

If $1 < p < \infty$ we take the usual test function $f = \chi_Q v^{1-p'}$ with $t = \frac{1}{|Q|} \int_Q f$. Then

$$u(Q) \le c_0 \frac{|Q|^p}{\left(\int_Q f \right)^p} \int_{t/c_0}^{\infty} v\{f > s\} s^{p-1} ds$$

$$\le c_0 \frac{|Q|^p}{\left(\int_Q f \right)^p} \int_Q f^p v$$

$$= c_0 |Q|^p \left(\int_Q v^{1-p'} \right)^{1-p},$$

and the A_p-condition follows.

3. PROOF OF THEOREM 1. (3) $\to$ (2).

$$\int_{R^n} \Phi[Mf(x)^p] u(x) dx = \int_0^{\infty} u\{Mf^p > t\} a(t) dt =$$

$$\int_0^{\infty} u\{Mf > t^{1/p}\} a(t) dt \le c_0 \int_0^{\infty} \frac{1}{t} \int_{t^{1/p}/c_0}^{\infty} v\{f > s\} s^{p-1} ds a(t) dt =$$

$$c_0 \int_0^{\infty} \int_0^{(c_0 s)^p} \frac{a(t)}{t} v\{f > s\} s^{p-1} dt ds \le$$

$$c_0 c' \int_0^{\infty} b(c''(c_0 s)^p) v\{f > s\} s^{p-1} ds =$$

$$\frac{c_0 c'}{p} \int_0^{\infty} b(c_* t) v\{f^p > t\} dt \le c_1 \int_{R^n} \Psi[c_2 f(x)^p] v(x) dx.$$

It is clear that the constants c_1 and c_2 have the desired properties.

(2) $\to$ (3). We assume that

$$L \equiv \int_0^\infty u\{Mf^p > t\}a(t)dt \le c_1 \int_0^\infty v\{c_2 f^p > t\}b(t)dt \equiv R.$$

Fix $0 < \lambda_0 < \infty$ and let

$$a(t) = 1h\chi_{[\lambda_0, \lambda_0 + h]}(t).$$

Set

$$b(t) = \int_0^t a(s)sds = 0, 0 \le t \le \lambda_0 1h\log(t/\lambda_0), \lambda_0 < t \le \lambda_0 + h1h$$
$$\log\lambda_0 + h\lambda_0, t > \lambda_0 + h.$$

With this choice Φ and Ψ are $(1,1)$-related independent of h and λ_0 and hence c_1 and c_2 do not depend on h or λ_0. Then

$$L = 1h\int_{\lambda_0}^{\lambda_0+h} u\{Mf^p > t\}dt \to u\{Mf^p > \lambda_0\},$$

as $h \to 0$. The right side R is

$$R = c_1 h\int_{\lambda_0}^{\lambda_0+h} v\{c^2 f^p > t\}\log(t/\lambda_0)dt + c_1 h\log\lambda_0 + h\lambda_0$$

$$\int_{\lambda_0+h}^\infty v\{c_2 f^p > t\}dt = I_1(h) + I_2(h).$$

We see that $I_1(h) \to 0$ as $h \to 0$ and

$$I_2(h) \to c_1\lambda_0 \int_{\lambda_0}^\infty v\{c_2 f^p > t\}dt = c_1 c_2 \lambda_0 \int_{\lambda_0/c_2}^\infty v\{f^p > t\}dt.$$

Since λ_0 was arbitrary we get for some constant $c_0 > 1$

$$u\{Mf^p > \lambda\} \le c_0\lambda \int_{\lambda/c_0}^\infty v\{f^p > t\}dt.$$

We now make the substitution $\lambda = s^p$ and then $t \to t^p$ to get

$$u\{Mf > s\} \le c_0 s^p \int_{s/c_0}^\infty v\{f > t\}t^{p-1}dt.$$

By Theorem 2 this is the same as saying $(u,v) \in A_p$.

Remark. Theorem 1 is not true with M replaced by a singular integral operator T. If it were true, then the argument as on the previous page shows that

$$u\{|Tf| > s\} \le c_0 s^p \int_{s/c_0}^\infty v\{f > t\}t^{p-1}dt,$$

and hence for $s > c_0\|f\|_{\infty,v}, u\{|Tf| > s\} = 0$ and $\|Tf\|_{\infty,u} < \infty$. But T is not of type (∞,∞) [10].

4. A CONVERSE

For a given $a, b : R_+ \to R_+$ and $\Phi(t) = \int_0^t a(s)ds$, $\Psi(t) = \int_0^t b(s)ds$ we wish to examine when (2) of Theorem 1 implies that

$$\int_0^s a(t)tdt \le c'b(c''s), 0 < s < \infty.$$

Since this condition is independent of $(u,v) \in A_p$, we are allowed to take any $(u,v) \in A_p$, in particular $u = v = 1$, the Lebesgue measure case, or $u = v$ in RH_∞. We prefer the second alternative since it is based on an extension of the reverse weak type inequality. We say that $u \in RH_\infty$ if for every cube Q, $\sup_Q u(x) \le c1|Q|\int_Q u$. The inf of all such c's is called the RH_∞-constant of u. This class was studied in [2] and plays roughly the same role among the reverse Hölder classes $RH_r, r \to \infty$, as A_1 does among A_p, $p \searrow 1$. Typical examples of RH_∞-weights in R_+ are $u(x) = x^\alpha$, $\alpha > 0$.

Theorem 3 Let $u \in RH_\infty$. Then there are constants $0 < c_1, c' < \infty$ such that for all $f : R^n \to R_+$ and $0 < t < \infty$

$$1t\int_{\{f>t\}} f(x)u(x)dx \le c_1 u\{Mf > c't\},\cdot$$

where $1/c' = c_*$ is the RH_∞-constant of u.

Proof. Since $u(x)dx$ is a doubling measure [3], we have available the Calderon-Zygmund decomposition at height t and this gives us disjoint cubes $\{Q_k\}$ such that

$$t \le 1u(Q_k)\int_{Q_k} fu \le ct$$

$$f(x) \le t, \text{ on } R^n \setminus \cup Q_k.$$

Then

$$1t\int_{\{f>t\}} fu \le 1t\sum\int_{Q_k} fu \le c\sum u(Q_k) = cu(\cup Q_k) \le cu\{M_u f > t\},$$

where $M_u f(x) = \sup_{x\in Q} 1u(Q)\int_Q fu$. Since $u \in RH_\infty$

$$1u(Q)\int_Q fu \le \sup_Q u u(Q)/|Q| 1|Q|\int_Q f \le c_* Mf(x),$$

if $x \in Q$. Hence $m_u f(x) \le c_* Mf(x)$ and the proof is complete.

Defintion. (1) $b : R_+ \to R_+$ is quasi-increasing (qi) if there is a constant $0 < c_0 < \infty$ such that $t' \le t''$ implies $b(t') \le c_0 b(c_0 t'')$.

(2) A measure μ on R_+ is weakly doubling if there is a constant $0 < c < \infty$ such that $\mu([0,2d]) \le c\mu([d,2d]), 0 < d < \infty$.

If a measure is doubling, it is also weakly doubling. The converse is not true as the measure $d\mu = e^x dx$ shows. In fact if

$f: R_+ \to R_+$ is nondecreasing, then $d\mu = f(x)dx$ is weakly doubling. The measure $d\mu = dx/(1+x)$ is not weakly doubling.

Theorem 4 Assume that $b(t)$ is qi and assume that for some n and $u_0 \in RH_\infty(R^n)$ we have

$$\int_{R^n} \Phi(Mf^p)u_0 \le c_1 \int_{R^n} \Psi(c_2 f^p)u_0.$$

Then

$$\int_0^s a(t)tdt \le c' b(c''s), 0 < s < \infty$$

holds if $p=1$, and if $1<p<\infty$ it holds under the additional assumption that the measure $d\mu = a(t)tdt$ is weakly doubling.

Proof. In distributional form the integral inequality is

$$L \equiv \int_0^\infty u_0\{Mf^p > t\}a(t)dt \le c_1 \int_0^\infty u_0\{c_2 f^p > t\}b(t)dt \equiv R.$$

The constants $c_1, c_2, \ldots$ appearing below only depend upon the constants in the overall hypothesis. By Lemma 3

$$L \ge c_3 \int_0^\infty a(t)t^{1/p} \int_{\{f > c_4 t^{1/p}\}} f(x)u_0(x)dxdt.$$

We apply this to the test functions $f(x) = r\chi_Q(x)$, $0 < r < \infty$, $Q = [0,1]^n$ and get

$$L \ge c_3 \int_0^{c_5 r^p} a(t)t^{1/p} r u_1 dt, u_1 = \int_Q u_0(x)dx.$$

The right side $R = \int_0^{c_6 r^p} u_1 b(t)dt$. Hence

$$c_3 r \int_0^{c_5 r^p} a(t)t^{1/p}dt \le c_1 \int_0^{c_6 r^p} b(t)dt.$$

With $s = c_5 r^p$ this becomes

$$c_7 s^{1/p} \int_0^s a(t)t^{1/p}dt \le c_1 \int_0^{c_8 s} b(t)dt \le c_0 s b(c_{10}s),$$

since b is quasi-increasing. The left side is

$$\ge c_7 s^{1/p} \int_{s/2}^s t^{1/p'} a(t)dt \ge c_{11} s \int_0^s a(t)tdt,$$

by the weak type doubling condition, which clearly is not needed when $p=1$.

Remark: 1. The special case $p=1$ and $u_0 \sim 1$ - the Lebesgue measure case - is Theorem 7 in [6].

2. The weak doubling hypothesis of the measure $d\mu = a(t)tdt$ cannot be omitted if $1<p<\infty$. The classical norm inequality for $u \in A_p$ is

$$\int_{R^n} Mf^p u \le c \int_{R^n} f^p u.$$

This is the $\Phi(t) = \Psi(t) = t$ case, and $a(t) = 1$.

5. EXTRAPOLATION

As before $a, b: R_+ \to R_+$ and $\Phi(s) = \int_0^s a(t)dt$, $\Psi(s) = \int_0^s b(t)dt$. We wish to examine the relationship between the following statements.

I. There exists $0 < \varepsilon < p, 1 \le p < \infty$, such that for $(u,v) \in A_p$ we have

$$\int_{R^n} \Phi(Mf^{p-\varepsilon})u \le c_1 \int_{R^n} \Psi(c_2 f^{p-\varepsilon})v.$$

II. There exists $\eta > 0$ such that

$$\int_0^s a(t)t^{1+\eta}dt \le c' b(c''s)s^\eta, 0 < s < \infty.$$

The constants ε, η, and p are related by $\varepsilon = \eta p/(1+\eta)$ or $\eta = \varepsilon/(p-\varepsilon)$.

Theorem 5 $II \Rightarrow I$, and, if b is quasi-increasing and $u = v = 1$, the converse $I \Rightarrow II$ holds if $p=1$, and if $1<p<\infty$ it holds if the measure $d\mu = a(t)/t^{1+\eta}dt$ is weakly doubling.

Proof. $II \Rightarrow I$. Fix $1 \le p < \infty$ and let $\varepsilon = \eta p/(1+\eta)$. If $q = p - \varepsilon$, then $p/q = 1+\eta$. By Theorem 2

$$\int_{R^n} \Phi(Mf^q)u \le c_0 \int_0^\infty a(t)t^{1+\eta} \int_{t^{1/q}/c_0}^\infty v\{f > s\}s^{p-1}dsdt = c_0 \int_0^\infty \int_0^{(c_0 s)^q}$$
$$a(t)t^{1+\eta}dt v\{f > s\}s^{p-1}ds \le c \int_0^\infty b[(c_0' s)^q](c_0 s)^{q\eta} v\{f > s\}s^{p-1}ds = c \int_0^\infty$$
$$b(\sigma)\sigma^\eta v\{f > \sigma^{1/q}\}\sigma^{(p-1)/q}\sigma^{1/q-1}d\sigma = c \int_0^\infty v\{f > \sigma^{1/q}\}b(\sigma)d\sigma = c_1$$
$$\int_{R^n} \Psi(c_2 f^q)v.$$

$I \Rightarrow II$. First let $p=1$ and $u=v=1$. If $q = 1-\varepsilon$, then the statement I in distributional form is

$$L = \int_0^\infty |\{Mf > t^{1/q}\}| a(t)dt \le c_1 \int_0^\infty |\{f > c_3 t^{1/q}\}| b(t)dt = R.$$

By Lemma 3,

$$L \ge c_4 \int_0^\infty a(t)t^{1/q} \int_{\{f > c_5 t^{1/q}\}} f(x)dxdt.$$

We apply this to the test functions $f(x) = r\chi_{[0,1]}(x)$, $0 < r < \infty$. Then

$$L \ge c_4 \int_0^{c_6 r^q} ra(t)t^{1/q}dt, R = \int_0^{c_7 r^q} b(t)dt \le c_8 r^q b(c_9 r^q),$$

because b is quasi-increasing. Hence

$$\int_0^{c_6 r^q} a(t)t^{1/q}dt \le c_{10} r^{q-1} b(c_9 r^q).$$

Let $s = c_6 r^q$ and $1/q = 1+\eta$. Then $\eta = \varepsilon/(1-\varepsilon)$ and

$$\int_0^s a(t)t^{1+\eta}dt \le c_{11} s^{(q-1)/q} b(c_{12}s),$$

and $(q-1)/q=-\eta$.

The case $1<p<\infty$ with $q=p-\varepsilon$, and $u=v=1$ follows the same steps as above and we get

$$\int_0^s \frac{a(t)}{t^{1/q}}dt \le c_{11}s^{(q-1)/q}b(c_{12}s).$$

We use now the weak doubling condition and get

$$\int_0^s \frac{a(t)}{t^{1/q}}dt = \int_0^s \frac{a(t)t^{1+\eta-1/q}}{t^{1+\eta}} \ge c_{13}s^{1+\eta-1/q}\int_{s/2}^s \frac{a(t)}{t^{1+\eta}}dt \ge c_{14}s^{1+\eta-1/q}\int_0^s \frac{a(t)}{t^{1+\eta}}dt.$$

Hence

$$\int_0^s \frac{a(t)}{t^{1+\eta}}dt \le c_{15}b(c_{12}s)/s^{\eta}.$$

The result that we discuss now essentially says that, in the presence of condition II, extrapolation for (u,v) is the same as $(u,v)\in A_p$.

Theorem 6 Let $1\le p<\infty, \eta\ge 0, \varepsilon=\eta p/(1+\eta)$, and

$$\int_0^s \frac{a(t)}{t^{1+\eta}}dt \le \frac{c'b(c''s)}{s^{\eta}}, 0<s<\infty.$$

Then the following statements are equivalent.

$$\int_{R^n}\Phi(Mf^{p-\varepsilon})u \le c_1\int_{R^n}\Psi(c_2f^{p-\varepsilon})v, \qquad (2)$$

where c_1, c_2 depend only upon c', c'' and p.

(2) We have $(u,v)\in A_p$.

Remark: Theorem 1 is the special case $\eta=0$.

Proof. $(2)\Rightarrow(1)$. This is $\text{II}\Rightarrow\text{I}$ of Theorem 5. $(1)\Rightarrow(2)$. We proceed as in the proof of Theorem 1 and let

$$a(t)=\frac{1}{h}\chi_{[\lambda,\lambda+h]}(t), \lambda>0, h>0.$$

We let $b(s)=s^{\eta}\int_0^s \frac{a(t)}{t^{1+\eta}}dt$. We may assume that $\eta>0$ since the case $\eta=0$ is Theorem 1. Then

$$b(s)=0, 0\le s\le\lambda\frac{(s/\lambda)^{\eta}-1}{h\eta}, \lambda\le s\le\lambda+h\frac{(s/\lambda)^{\eta}-(s/(\lambda+h))^{\eta}}{h\eta}, s\ge\lambda+h.$$

Our hypothesis in distributional form is

$$L_h \equiv \int_0^{\infty} u\{Mf^{p-\varepsilon}>t\}a(t)dt \le c_1\int_{\lambda}^{\infty} v\{c_2f^{p-\varepsilon}>t\}b(t)dt \equiv R_h.$$

First

$$L_h=\frac{1}{h}\int_{\lambda}^{\lambda+h} u\{Mf^{p-\varepsilon}>t\}dt \to u\{Mf^{p-\varepsilon}>\lambda\},$$

as $h\to 0$. The right side R_h splits into two integrals

$$R_h=c_1\left(\int_{\lambda}^{\lambda+h}+\int_{\lambda+h}^{\infty}\right)=I_1+I_2.$$

I_1 is easily disposed of

$$I_1=c_1\int_{\lambda}^{\lambda+h}\frac{(t/\lambda)^{\eta}-1}{h\eta}v\{c_2f^{p-\varepsilon}>t\}dt \to 0,$$

as $h\to 0$. Next

$$I_2=c_1\frac{\lambda^{-\eta}-(\lambda+h)^{-\eta}}{h\eta}\int_{\lambda+h}^{\infty} v\{c_2f^{p-\varepsilon}>t\}t^{\eta}dt \to \frac{c_1}{\lambda^{\eta+1}}\int_{\lambda}^{\infty} v\{c_2f^{p-\varepsilon}>t\}t^{\eta}dt,$$

as $h\to 0$. The substitution $\tau=t^{\eta+1}$ gives

$$I_2 \to \frac{c_3}{\lambda^{\eta+1}}\int_{\lambda^{\eta+1}}^{\infty} v\{c_2f^{p-\varepsilon}>\tau^{1/(\eta+1)}\}d\tau,$$

and since $(p-\varepsilon)(\eta+1)=p(\eta+1)-p\eta=p$,

$$I_2 \to \frac{c_3}{\lambda^{\eta+1}}\int_{\lambda^{\eta+1}}^{\infty} v\{c_4f^{p}>\tau\}d\tau.$$

Hence for some constant $c_0>1$

$$u\{Mf^{p-\varepsilon}>\lambda\} \le \frac{c_0}{\lambda^{\eta+1}}\int_{\lambda^{\eta+1}/c_0}^{\infty} v\{f^p>t\}dt.$$

With $\lambda=\sigma^{p-\varepsilon}$ we get

$$u\{Mf>\sigma\} \le \frac{c_0}{\sigma^p}\int_{\sigma^p/c_0}^{\infty} v\{f^p>t\}dt=\frac{c_{0'}}{\sigma^p}\int_{\sigma/c_{0'}}^{\infty} v\{f>t\}t^{p-1}dt.$$

This shows that $(u,v)\in A_p$ by Theorem 2.

Remark: The following observation may be of interest in connection with condition II: if $\int_0^s \frac{a(t)}{t}dt \le c_0a(s)$, then there exists $\eta>0$ such that

$$\int_0^s \frac{a(t)}{t^{1+\eta}}dt \le c\frac{a(s)}{s^{\eta}},$$

and hence Theorem 5 about extrapolation applies.

Proof. By hypothesis

$$L \equiv \int_0^{s_1}\frac{1}{s}\int_0^s \frac{a(t)}{t}dtds \le c_0\int_0^{s_1} \le c_0^2a(s_1).$$

Also

$$L=\int_0^{s_1}\int_t^{s_1}\frac{a(t)}{ts}dsdt=\int_0^{s_1}\frac{a(t)}{t}\log\frac{s_1}{t}dt \le c_0^2a(s_1).$$

We repeat this argument and finally get

$$\int_0^s \frac{a(t)}{t}\frac{1}{j!}\log^j\frac{s}{t}dt \le c_0^{j+1}a(s).$$

Let $c_1>c_0$. Then

$$\int_0^s \frac{a(t)}{t}\sum\frac{1}{j!}\frac{1}{c_1^j}\log^j\frac{s}{t}dt \le ca(s),$$

and the sum $=(s/t)^{\eta}$ with $\eta=1/c_1$.

6. ITERATED MAXIMAL OPERATOR. LET

$$M_j f(x) = \underbrace{M \circ M \circ \cdots \circ M}_{j-\text{times}} f(x).$$

The purpose of this section is to present some weighted integral inequalities involving $M_j f$.

Theorem 7 Let $u \in A_p, 1 \le p < \infty,$ and assume that $a, b : R_+ \to R_+$ satisfy

$$\int_0^s \frac{a(t)}{t} \log^{j-1}(s/t)dt \le c' b(c'' s).$$

Then, if $\Phi(t) = \int_0^t a(s)ds, \Psi(t) = \int_0^t b(s)ds$,

$$\int_{R^n} \Phi(M_j f^p)u \le c_{j'} \int_{R^n} \Psi(c_{j''} f^p)u.$$

Proof. By Theorem 2,

$$u\{M_j f > t\} \le \frac{c_o}{t^p} \int_{t/c_o}^\infty u\{M_{j-1} f > s_1\} s_1^{p-1} ds_1 \le \frac{c_0^2}{t^p} \int_{t/c_o}^\infty \frac{s_1^{p-1}}{s_1^p} \int_{s_1/c_o}^\infty$$

$$u\{M_{j-2} f > s_2\} s_2^{p-1} ds_2 ds_1 = \frac{c_o^2}{t^p} \int_{t/c_o^2}^\infty \int_{t/c_o}^{c_0 s_2} \frac{ds_1}{s_1} u\{M_{j-2} f > s_2\} s_2^{p-1} ds_2$$

$$= \frac{c_o^2}{t^p} \int_{t/c_o^2}^\infty \log \frac{c_o^2 s_2}{t} u\{M_{j-2} > s_2\} s_2^{p-1} ds_2 \le \cdots \le \frac{c_o^j}{(j-1)! t^p} \int_{t/c_0^j}^\infty$$

$$\log^{j-1} \frac{c_o^j s}{t} u\{f > s\} s^{p-1} ds.$$

The left side of the conclusion is

$$\int_0^\infty u\{M_j f > t^{1/p}\} a(t)dt \le \frac{c_o^j}{(j-1)!} \int_0^\infty \frac{a(t)}{t} \int_{t^{1/p}/c_o^j}^\infty \log^{j-1} \frac{c_o^j s}{t^{1/p}} u\{f > s\} s^{p-1} ds dt = \frac{c_o^j}{(j-1)!} \int_0^\infty \frac{a(t)}{t} \int_{t/c_o^{jp}}^\infty \log^{l-1}(c_o^j(\sigma/t)^{1/p}) u\{f > \sigma^{1/p}\} d\sigma dt =$$

$$\frac{cc_0^j}{(j-1)!} \int_0^\infty \int_0^{c_o^{jp}\sigma} \frac{a(t)}{t} \log^{j-1}(c_o^{jp}\sigma/t) dt u\{f > \sigma^{1/p}\} d\sigma = \frac{cc_o^j}{(j-1)!} \int_0^\infty$$

$$b(c'' c_o^{jp}\sigma) u\{f > \sigma^{1/p}\} d\sigma \le c_{j'} \int_{R^n} \Psi(c_{j''} f^p)u.$$

Remark: (1) The $\log$ term in the hypothesis of Theorem 7 can be omitted if $u \sim 1$, the Lebesgue measure case and $1 < p < \infty$. The operator $M_j f$ is weak (p,p) and (∞,∞) and hence by [11, p. 103]

$$|\{M_j f > t\}| \le \frac{c_j}{t^p} \int_{t/c_j}^\infty |\{f > s\}| s^{p-1} ds.$$

>From this we get

$$\int_{R^n} \Phi(M_j f(x)^p)dx = \int_0^\infty |\{M_j f > t^{1/p}\}| a(t)dt \le c_j \int_0^\infty \frac{a(t)}{t} \int_{t^{1/p}/c_j}^\infty$$

$$|\{f > s\}| s^{p-1} ds dt = c_j \int_0^\infty \int_0^{(c_j s)^p} \frac{a(t)}{t} |\{f > s\}| s^{p-1} dt ds = c_1 \int_0^\infty b(c_2 s^p)$$

$$|\{f > s\}| s^{p-1} ds = c_3 \int_0^\infty b(c_2 t) |\{f^p > t\}| dt = c_{j'} \int_{R^n} \Psi(c_{j''} f(x)^p)dx.$$

(2) There is a converse to the above. If b is qi, the integral inequality

$$\int_{R^n} \Phi(M_j f(x)^p)dx \le c_{j'} \int_{R^n} \Psi(c_{j''} f(x)^p)dx$$

implies

$$\int_0^s \frac{a(t)}{t} dt \le c' b(c'' s), 0 < s < \infty,$$

if $p = 1$, and if $p > 1$ this holds if the measure $d\mu = \frac{a(t)}{t} dt$ is weakly doubling. This follows from

$$\int_{R^n} \Phi(Mf(x)^p)dx \le \int_{R^n} \Phi(M_j f(x)^p)dx \le c_{j'} \int_{R^n} \Psi(c_{j''} f(x)^p)dx,$$

and Theorem 4 applies.

7. THE ITERATED MAX OPERATOR AND EXTRAPOLATION

There is a connection between the behavior of $M_j f$ and extrapolation [7-9]. The next two Theorems will explore this connection in our setting. Again let $a, b : R_+ \to R_+$ and let $\Phi(s) = \int_0^s a(t)dt, \Psi(s) = \int_0^s b(t)dt$ with b quasi-increasing..

Theorem 8 Let $1 \le p < \infty$ and assume that for $j \in N$

$$\int_R \Phi(M_j f(x)^p)dx \le A^j \int_R \Psi(c_2 f(x)^p)dx,$$

with c_2 independent of j . Let $A < c_* < \infty$ and let $\eta = 1/(c_* p)$. If in the case $1 < p$ the measure $d\mu = \frac{a(t)}{t^{1+\eta}} dt$ is weakly doubling, then for $(u, v) \in A_p(R^n)$

$$\int_{R^n} \Phi(Mf^{p-\varepsilon})u \le c_{1'} \int_{R^n} \Psi(c_{1''} f^{p-\varepsilon})v,$$

where $\varepsilon = \eta p/(1+\eta)$.

Proof. Our goal is to prove

$$\int_0^s \frac{a(t)}{t^{1+\eta}} dt \le \frac{c' b(c'' s)}{s^\eta},$$

and then Theorem 5 gives us our conclusion.

In distributional form our hypothesis is

$$L \equiv \int_0^\infty |\{M_j f > t^{1/p}\}| a(t)dt \le A^j \int_0^\infty |\{f > (t/c_2)^{1/p}\}| b(t)dt \equiv R.$$

By Lemma 3

$$L \ge c_1 \int_0^\infty \frac{a(t)}{t^{1/p}} \int_{\{M_{j-1} f > c_3 t^{1/p}\}} M_{j-1} f(x) dx dt$$

with c_1, c_3 independent of j. We apply this to the test functions $f(x) = r\chi_{[0,1]}(x), 0 < r < \infty$. Then

$$M_i f(x) = r\{\chi_{[0,1]}(x) + \frac{1}{x} \phi_{i-1}(x)\chi_{[1,\infty)}(x)\}, \phi_k(x) = \sum_0^k \frac{\log^i x}{i!}.$$

Therefore the inner integral is

$$\int_{\{M_{j-1}f>c_3t^{1/p}\}} M_{j-1}f(x)dx \ge \int_{\{(r/x)\phi_{j-2}(x)>c_3t^{1/p}\}} M_{j-1}f(x)dx.$$

For $0<t<(r/c_3)^p$, the set $\{(r/x)\phi_{j-2}(x)>c_3t^{1/p}\}\supset[1,\sigma(t))$, where $\sigma(t)$ is defined by

$$(r/\sigma(t))\phi_{j-2}(\sigma(t))=c_3t^{1/p}.$$

Since $\phi_{j-2}(x)\ge 1$, we get $\sigma(t)\ge r/(c_3t^{1/p})$. Hence

$$\int_{\{M_{j-1}>c_3t^{1/p}\}} M_{j-1}f(x)dx \ge \int_1^{r/(c_3t^{1/p})} \frac{r}{x(j-2)!}\log^{j-2}x dx = \frac{r}{(j-1)!}\log^{j-1}\frac{r}{c_3t^{1/p}} = \frac{r}{p^{j-1}(j-1)!}\log^{j-1}\frac{r^p}{c_3^p t}.$$

Thus

$$L\ge \frac{c_1 r}{p^{j-1}(j-1)!}\int_0^{(r/c_3)^p}\frac{a(t)}{t^{1/p}}\log^{j-1}\frac{r^p}{c_3^p t}dt.$$

Also

$$R\le A^j\int_0^{c_4r^p} b(t)dt\le A^jc'r^pb(c''r^p),$$

since b is quasi-increasing. Let $s=(r/c_3)^p$. Then

$$\frac{c_5}{p^{j-1}(j-1)!}s^{1/p}\int_0^s\frac{a(t)}{t^{1/p}}\log^{j-1}(s/t)dt\le A^jc_6sb(c_7s).$$

Then

$$\frac{c_5}{c_*}s^{1/p}\int_0^s\frac{a(t)}{t^{1/p}}\sum\frac{1}{(j-1)!p^{j-1}c_*^{j-1}}\log^{j-1}(s/t)dt\le c_8sb(c_7s),$$

since $c_*>A$. Since the sum inside the integral $=(s/t)^\eta$, we get

$$L_1\equiv c_9s^{1/p}\int_0^s\frac{a(t)}{t^{1/p}}(s/t)^\eta dt\le c_8sb(c_7s).$$

If $1=p$ we stop, and if $p>1$ we note that

$$L_1\ge c_{10}s\int_{s/2}^s\frac{a(t)}{t}(s/t)^\eta dt.$$

Finally, the weak doubling condition gives us

$$\int_0^s\frac{a(t)}{t^{1+\eta}}dt\le\frac{c_{11}b(c_7s)}{s^\eta}.$$

There is a converse to Theorem 8 which reads as follows.

Theorem 9 Let $1\le p<\infty$ and assume that for some $\varepsilon>0$

$$\int_{R^n}\Phi(Mf^{p-\varepsilon}(x))dx\le c_1\int_{R^n}\Psi(c_2f^{p-\varepsilon}(x))dx.$$

If in case $p>1$ the measure $d\mu=\frac{a(t)}{t^{1+\eta}}dt$, $\eta=\varepsilon/(p-\varepsilon)$, is weakly doubling, then for $j\in N$ and $u\in A_p$

$$\int_{R^n}\Phi(M_jf^p)u\le c_{j'}\int_{R^n}\Psi(c_{j''}f^p)u.$$

Proof. By Theorem 5, $\int_0^s\frac{a(t)}{t^{1+\eta}}dt\le\frac{c'b(c''s)}{s^\eta}$. Then

$$\int_0^s\frac{a(t)}{t}\sum\frac{\eta^{j-1}}{(j-1)!}\log^{j-1}(s/t)dt\le c'b(c''s).$$

Thus for each $j\in N$

$$\int_0^s\frac{a(t)}{t}\log^{j-1}(s/t)dt\le c_{j'}b(c''s).$$

Theorem 7 completes the proof.

8. APPLICATIONS

We give some examples of Φ and Ψ which are (c',c'')-related and investigate the inplications of Theorem 1. We will get some unusual and surprising integral inequalities.

I. If $(u,v)\in A_p$ for some $1\le p<\infty$, then

$$\int_{R^n}Mf^ru\le c\int_{R^n}f^rv,$$

for $p<r<\infty$.

Proof. This is well-known [4]. It also follows from Theorem 1 by taking $\Phi(t)=t^\alpha,\alpha>1$. An easy calculation shows that we can take $\Psi(t)=t^\alpha$.

II. If $(u,v)\in A_p$ for some $1\le p<\infty$, then for $\alpha>1$

$$\int_{R^n}\log^\alpha(1+Mf^p)u\le c\int_{R^n}f^p\log^{\alpha-1}(1+f^p)v.$$

Proof. Let $\Phi(t)=\log^\alpha(1+t)$. Then $a(t)=\alpha\frac{\log^{\alpha-1}(1+t)}{1+t}$ and

$$\int_0^s\frac{a(t)}{t}dt=\alpha\int_0^s\frac{\log^{\alpha-1}(1+t)}{t(1+t)}dt\le\alpha\int_0^s\frac{\log^{\alpha-2}(1+t)}{1+t}dt=\frac{\alpha}{\alpha-1}\log^{\alpha-1}(1+s)=b(s).$$

Also

$$\int_0^t b(s)ds\le\frac{\alpha}{\alpha-1}t\log^{\alpha-1}(1+t)\equiv\Psi(t).$$

The desired integral inequality follows from Theorem 1, since $\log(1+cx)\le c\log(1+x)$ if $c\ge1$.

Remark: We cannot replace the right side by the more symmetric $\int_{R^n}\log^\alpha(1+f^p)v$. As an example let $u=v=1$ and $n=1$. If $f(x)=r\chi_{[0,1]}(x), 0<r<\infty$, then $\int_R\log^\alpha(1+f^p)=\log^\alpha(1+r^p)$. Since $Mf(x)\ge r/x, x\ge1$, we get

$$\int_R\log^\alpha(1+Mf^p)dx\ge\int_1^\infty\log^\alpha(1+(r/x)^p)dx.$$

The integrand

$$\log^\alpha(1+(r/x)^p)=(\log(x^p+r^p)-\log x^p)^\alpha\ge\left(\frac{r^p}{x^p+r^p}\right)^\alpha\ge 1/2^\alpha,$$

if $x\le r$. Hence

$$\int_R \log^\alpha(1+Mf^p)dx \ge \int_1^r \frac{dx}{2^\alpha} = \frac{r-1}{2^\alpha}.$$

Our assertion follows since $\frac{\log^\alpha(1+r^p)}{r-1} \to 0$ as $r\to\infty$.

III. If $(u,v)\in A_p$ for some $1\le p<\infty$, then

$$\int_{\{Mf>1\}} Mf^p u \le c_1 \int_{\{f>c_2\}} f^p \log(1+f^p) v.$$

Proof. Let $\Phi(t)=(t-1)\chi^1(t)$, where $\chi^1(t)=\chi_{[1,\infty)}(t)$. Then $a(t)=\chi^1(t)$. We let

$$b(s)=\int_0^s \frac{a(t)}{t}dt=(\log s)\chi^1(s).$$

Then $\Psi(t)=\int_0^t b(s)ds \le (t\log t)\chi^1(t)$. By Theorem 1 we get

$$\int_{\{Mf\ge1\}}(Mf^p-1)u \le c_1\int_{\{f\ge 1/c'\}} f^p\log(c_2 f^p)v,$$

where $c'=c_2^{1/p}$. By Theorem 2

$$u\{Mf\ge1\}\le c_0\int_{1/c_0}^\infty v\{f\ge s\}s^{p-1}ds=\frac{c_0}{p}\int_{c''}^\infty v\{f^p>t\}dt\le\frac{c_0}{p}\int_{\{f\ge c''\}} f^p v,$$

where $c''=1/(c_0^p)<1$. Thus we get

$$\int_{\{Mf\ge1\}} Mf^p u\le c_1\int_{\{f\ge c_*\}} f^p(1+\log(c_2f^p))v\le c_{1'}\int_{\{f\ge c_*\}} f^p\log(1+f^p)v,$$

since $1+\log(cx)\le ec\log(1+x)$ if $ec>1$.

Remark: As a special case, if $(u,v)\in A_1$ and $K\subset R^n$ is compact, then $\int_{R^n} f\log(1+f)v<\infty$ implies $Mf\chi_K\in L^1(u)$. This is a two-weight version of the well-known fact that $Mf\chi_K\in L^1$, if $f\in L\log L$ [10].

IV. Let $(u,v)\in A_p$ for some $1\le p<\infty$, and let $0<\alpha<1$. Then

$$\int_{\{Mf>1\}}(Mf^{\alpha p}-1)u\le\frac{c_1}{1-\alpha}\int_{\{c_2f^p>1\}}(c_2f^p-c_2^\alpha f^{\alpha p})v.$$

Proof. Let $\Phi(t)=(t^\alpha-1)\chi^1(t)$. Then $a(t)=\alpha t^{\alpha-1}\chi^1(t)$. We set

$$b(t)=\alpha\int_1^t s^{\alpha-2}ds\chi^1(t)=\frac{\alpha}{1-\alpha}(1-t^{\alpha-1})\chi^1(t).$$

Hence

$$\Psi(t)=\frac{\alpha}{1-\alpha}\int_1^t(1-s^{\alpha-1})ds\chi^1(t)=\left(\frac{\alpha}{1-\alpha}(t-t^\alpha/\alpha)+1\right)\chi^1(t).$$

>From Theorem 1, using $v\{c_2f^p>1\}\le\int_{\{c_2f^p>1\}} c_2f^p v$, we get the desired inequality.

V. Let $(u,v)\in A_p$ for some $1\le p<\infty$, and let $0<k<\infty$. Then

$$\int_{\{Mf>1\}}\left(1-\frac{1}{Mf^p}\right)^k u\le c\int_{\{f>1/c'\}} f^p(1-1/(c_2f^p))^{k+1}v,\ c'=c_2^{1/p}.$$

Proof. Let $\Phi(t)=(1-1/t)^k\chi^1(t)$. Then $a(t)=k(1-1/t)^{k-1}1/(t^2)\chi^1(t)$. We set

$$b(t)=k\int_1^t(1-1/s)^{k-1}\frac{1}{s^3}ds\chi^1(t)\le(1-1/t)^k\chi^1(t).$$

>From this we see that

$$\Psi(t)=\int_1^t(1-1/s)^k ds\chi^1(t)\le(1-1/t)^k(t-1)\chi^1(t)=t(1-1/t)^{k+1}\chi^1(t),$$

and the inequality follows.

VI. Let $(u,v)\in A_p$ for some $1\le p<\infty$. Then

$$\int_{R^n} e^{-1/(Mf^p)}u\le c_1\int_{R^n} f^p e^{-1/(c_2f^p)}v.$$

Proof. Let $\Phi(t)=e^{-1/t}, t>0$ and $\Phi(0)=0$. Then $a(t)=e^{-1/t}1/t^2$ and

$$b(t)=\int_0^t\frac{e^{-1/s}}{s^3}ds=e^{-1/t}(\frac{1}{t}+1).$$

>From this

$$\Psi(t)=\int_0^t e^{-1/s}(\frac{1}{s}+1)ds=te^{-1/t}.$$

Theorem 1 gives the desired integral inequality.

Remark: The factor f^p in the above inequality cannot be omitted as examples of the type $f_N=N\chi_{[0,1]}$ show.

VII. Suppose $a(t)=\Phi'(t)$ is convex with $a(0)=0$. If $(u,v)\in A_p$ for some $1\le p<\infty$, then

$$\int_{R^n}\Phi(Mf^p)u\le c_1\int_{R^n}\Phi(c_2f^p)v.$$

Proof. This follows from

$$\int_0^t\frac{a(s)}{s}ds\le\int_0^t a'(s)ds=a(t).$$

Remark: Examples illustrating (VII) are $\Phi(t)=t^2e^t, e^t-t-1, \sum_{n\ge2}a_nt^n, a_n\ge0$. As an application we will present an inequality involving e^{Mf^p}.

VIII. If $(u,v)\in A_p$ for some $1\le p<\infty$, then there exist constants $0<c_1,c_2<\infty$ such that for every $f:R^n\to R_+$

$$\int_{\{Mf>1\}} e^{Mf^p}u\le c_1\int_{\{c_2f^p>1\}} e^{c_2f^p}v.$$

Proof. Let $\Phi(t)=(e^t-te)\chi^1(t)$. Then $a(t)=(e^t-e)\chi^1(t)$ and thus from VII

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$$\int_{\{Mf>1\}}(e^{Mf^p}-Mf^pe)u\le c'\int_{\{c''f^p>1\}}(e^{c''f^p}-c''f^pe)v\le c'\int_{\{c''f^p>1\}}e^{c''f^p}v.$$

We only need to verify now that

$$\int_{\{Mf>1\}}Mf^pu\le c_1\int_{\{c_2f^p>1\}}e^{c_2f^p}v.$$

This is easy by letting $\Phi(t)=(t-1)\chi^1(t)$. Then $a(t)=\chi^1(t)$ and thus $b(t)=\log t\chi^1(t)\le e^t\chi^1(t)$.

IX. If $(u,v)\in A_p$ for some $1\le p<\infty$, then

$$\int_{\{Mf>1\}}\sqrt{Mf^p-1}u\le c_1\int_{\{c_2^{1/p}f>1\}}(c_2f^p\tan^{-1}\sqrt{c_2f^p-1}-\sqrt{c_2f^p-1})v.$$

Proof. Let $\chi^1(t)=\chi_{[1,\infty)}(t)$, and take $\Phi(t)=\sqrt{t-1}\chi^1(t)$. Then $a(t)=1/(2\sqrt{t-1})\chi^1(t)$ and

$$b(t)=\int_1^t\frac{1}{2s(s-1)^{1/2}}ds\chi^1(t)=\tan^{-1}\sqrt{t-1}\chi^1(t).$$

Also

$$\Psi(t)=\int_1^t\tan^{-1}\sqrt{s-1}ds\chi^1(t)=(t\tan^{-1}\sqrt{t-1}-\sqrt{t-1})\chi^1(t).$$

Theorem 1 gives us the desired integral inequality.

Remark: It is tempting to replace the right side of IX by the more symmetric

$$c_1\int_{\{c_2f^p>1\}}\sqrt{c_2f^p-1}v.$$

Examples of the form $f_N=N\chi_1$ as $N\to\infty$ show that this is not possible.

X. If $(u,v)\in A_p$ for some $1\le p<\infty$, then

$$\int_{\{Mf>y\}}\log\left(e\frac{Mf}{y}\right)^pu\le\frac{c_1}{y^p}\int_{\{f>c_2y\}}f^pv,$$

with c_1,c_2 independent of f.

Proof. Fix $y>0$ and let $e^p\tau=y^p$. If $a(t)=(1/t)\chi^\tau(t)$ and

$$b(t)=\int_0^t\frac{a(s)}{s}ds=\int_\tau^t\frac{ds}{s^2}\chi^\tau(t)\le(1/\tau)\chi^\tau(t),$$

then $\Phi(t)=\log(t/\tau)\chi^\tau(t)$ and $\Psi(t)\le(t/\tau)\chi^\tau(t)$. >From Theorem 1 we get

$$\int_{\{Mf^p>\tau\}}\log\left(\frac{Mf^p}{\tau}\right)u\le\frac{c'}{\tau}\int_{\{f^p>c''\tau\}}f^pv.$$

Finally

$$\int_{\{Mf>y\}}\log\left(e^p\frac{Mf^p}{y^p}\right)u\le\int_{\{Mf^p>\tau\}}\log\left(\frac{Mf^p}{\tau}\right)u\le\frac{c_1}{y^p}\int_{\{f>c_2y\}}f^pv.$$

Remark: The above inequality is a generalization of the weak-type inequality $u\{Mf>y\}\le\frac{c}{y^p}\int_{R^n}f^pv$.

XI. If $(u,v)\in A_p$ and $p<s<r<\infty$, then

$$\int_{\{Mf>1\}}(Mf^r-Mf^s)u\le\frac{c_1}{\beta-1}\int_{\{c_2f^p>1\}}(c_2^\alpha f^r-c_2^\beta f^s)v,$$

where $\alpha=r/p,\beta=s/p$.

Proof. Let $\Phi(t)=(t^\alpha-t^\beta)\chi^1(t)$. Then $a(t)=(\alpha t^{\alpha-1}-\beta t^{\beta-1})\chi^1(t)$ and

$$b(t)=\left(\frac{\alpha}{\alpha-1}t^{\alpha-1}-\frac{\beta}{\beta-1}t^{\beta-1}+c_{\alpha\beta}\right)\chi^1(t),$$

where $c_{\alpha\beta}=\beta/(\beta-1)-\alpha/(\alpha-1)$. Consequently

$$\Psi(t)=\left(\frac{t^\alpha}{\alpha-1}-\frac{t^\beta}{\beta-1}+c_{\alpha\beta}t\right)\chi^1(t)\le\left((\frac{1}{\alpha-1}+c_{\alpha\beta})t^\alpha-\frac{1}{\beta-1}t^\beta\right)$$
$$\chi^1(t)=\frac{1}{\beta-1}(t^\alpha-t^\beta)\chi^1(t).$$

Theorem 1 gives the desired inequality.

Remark: If $s=p$ above, then using the same type of argument with $\Phi(t)=(t^\alpha-t)\chi^1(t)$, etc, we get for $(u,v)\in A_p$

$$\int_{\{Mf>1\}}(Mf^r-Mf^p)u\le\frac{c_1\alpha}{\alpha-1}\int_{\{c_2f^p>1\}}(c_2^\alpha f^r-c_2f^p)v.$$

XII. The fact that $Mf\notin L^1(R^n)$ unless $f=0$ gives rise to the question for which $\Phi:R_+\to R_+$ is $\Phi(Mf)\in L^1(R^n)$. Let $a:R_+\to R_+$ be in $L^1_{loc}((0,\infty))$ and let $\Phi(t)=\int_0^ta(s)ds$.

Theorem 10 The following statements are equivalent for $f\in L^\infty\cap L^1(R^n)$:

$$\Phi(Mf(x))\in L^1(R^n),\tag{3}$$

$$\int_0^s\frac{a(t)}{t}dt<\infty,0<s<\infty.\tag{4}$$

Proof. (2) $\to$ (1). Since $|\{Mf>t\}|\le c_0/t\,\|f\|_1$ and Mf is (∞,∞), we get

$$\int_{R^n}\Phi(Mf(x))dx=\int_0^{\|f\|_\infty}|\{Mf>t\}|a(t)dt\le c\|f\|_1\int_0^{\|f\|_\infty}\frac{a(t)}{t}dt.$$

(1) $\to$ (2). We may assume that $a(t)\ne0$ on any interval $(0,\varepsilon))$ and $f\ne0$. By Lemma 3, $\frac{1}{t}\int_{\{f>t\}}f(x)dx\le c|\{Mf>t\}|$, and thus for $f\in L^\infty\cap L^1(R^n)$

$$\infty>\int_{R^n}\Phi(Mf(x))dx=\int_0^{\|f\|_\infty}|\{Mf>t\}|a(t)dt\ge c\int_0^{\|f\|_\infty}\frac{a(t)}{t}$$
$$\int_{\{f>t\}}f(x)dxdt=c\int_{R^n}\int_0^{f(x)}\frac{a(t)}{t}f(x)dtdx=c\int_{R^n}\Psi(f(x))f(x)dx,$$

where $\Psi(r)=\int_0^r \frac{a(t)}{t}dt$. Therefore, $\Psi(f(x))f(x)<\infty$, a.e x, and hence $\Psi(f(x))<\infty$, a.e. x.

Incidentally, we have established the following inequality:

$$c_1\int_{R^n}\Psi(f(x))f(x)dx \le \int_{R^n}\Phi(Mf(x))dx \le c_2\Psi(\|f\|_\infty)\|f\|_1.$$

XIII. Let $a:R_+\to R_+, b(s)=\int_0^s \frac{a(t)}{t}dt$ and $\Phi(t)=\int_0^t a(s)ds$. If $1\le p,q<\infty$ and $(u,v)\in A_p$, then

$$\int_{R^n}\Phi(Mf^p)u \le c_1\int_{R^n}\Psi_{p,q}(c_2 f^q)v,$$

where $\Psi_{p,q}(t)=\int_0^{t^{p/q}} b(s)ds$.

Proof. This follows from Theorem 1 since $\Psi_{p,q}(t)=\Psi(t^{p/q})$.

Remark: Theorem 1 deals with functions Φ,Ψ non-decreasing. It is sometimes convenient to have a version of Theorem 1 with Φ,Ψ non-increasing.

Let $a:R_+\to R_+$ and let $\Phi(t)=\int_t^\infty a(s)ds$. The function $b:R_+\to R_+$ is related to a by

$$\int_s^\infty ta(t)dt \le c'b(c''s), 0<s<\infty.$$

Finally, let

$$\Psi(t)=\int_t^\infty \frac{b(s)}{s^2}ds.$$

Theorem 11 The following statements are equivalent for $1\le p<\infty$.

(6) Whenever Φ and Ψ are related as above, then for every $f:R^n\to R_+$

$$\int_{R^n}\Phi\left(\frac{1}{Mf^p}\right)u \le c_1\int_{R^n}\Psi\left(\frac{c_2}{f^p}\right)v,$$

where the constants c_1, c_2 depend only on c', c'' and p.

(7) $(u,v)\in A_p$.

Proof. The change of variables $s\to 1/s$ shows that condition (2) of Theorem 1 is equivalent with condition (6): $\Phi(t),\Psi(t)$ satisfy (6) if and only if $\Phi_*(t)=\Phi(1/t),\Psi_*(t)=\Psi(1/t)$ satisfy (2) of Theorem 1.

As an example let $\Phi(t)=\int_t^\infty e^{-s}ds$. An easy calculation shows that we get VI. Another interesting example is $\Phi(t)=(1-t^\alpha)\chi_1(t)$, $0<\alpha<\infty$, where $\chi_1(t)=\chi_{[0,1]}(t)$. Then $a(t)=\alpha t^{\alpha-1}\chi_1(t)$ and s

$$b(t)=\frac{\alpha}{\alpha+1}(1-t^{\alpha+1})\chi_1(t).$$

Thus

$$\Psi(t)=\int_t^\infty \frac{b(s)}{s^2}ds\chi_1(t)=\left\{\frac{\alpha}{\alpha+1}(1/t+t^\alpha/\alpha)-1\right\}\chi_1(t).$$

If $(u,v)\in A_p$ for some $1\le p<\infty$, Theorem 7 gives

$$\int_{\{Mf>1\}}\left(1-\frac{1}{Mf^{\alpha p}}\right)u \le c_1\int_{\{c_2^{1/p}<f\}}\left(\frac{f^p}{c_2}+\frac{c_2^\alpha}{\alpha f^{\alpha p}}\right)v - c_1 v\{f>c_2^{1/p}\}.$$

CONFLICT OF INTEREST

The authors confirm that this article content has no conflicts of interest.

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