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On First-Hitting Time of a Linear Boundary by Perturbed Brownian Motion

Mario Abundo*

Dipartimento di Matematica, Università Tor Vergata Via della Ricerca Scientifica, 00133 Roma, Italy

Abstract: We consider the first-hitting time, τ_Y, of the linear boundary $S(t)=a+bt$ by the process $X_Y(t)=x+B_t+Y$, with $a\ge x, b\ge 0$, where B_t is Brownian motion and Y is a random variable independent of B_t and such that $P(x+Y\ge a)=1$. For a given distribution function F, we find the distribution of Y in such a way $P(\tau_Y\le t)=F(t)$.

Keywords: Brownian motion, diffusion, first-hitting time.

1. INTRODUCTION

For $a,b\ge 0$ and $x\in \mathsf{R}$, let us consider the linear boundary $S(t)=a+bt$, and the process $X_Y(t):=x+B_t+Y$ obtained by superimposing to Brownian motion B_t an additive random perturbation Y which is independent of B_t and such that $P(x+Y\ge a)=1; X_Y(t)$ can be thought as Brownian motion starting from the perturbed position $x+Y\ge a$. Then, we consider the first-hitting time of the boundary $S(t)$ by $X_Y(t)$, that is $\tau_Y:=\inf\{t>0: X_Y(t)\le S(t)\}$, and we denote by $\tau_Y(y)$ the first-hitting time, conditional to $Y=y$, that is the first-passage time of Brownian motion starting from $x+y\ge a$, below the boundary $S(t)$; thanking to the conditions $x+y\ge a$ and $b\ge 0$, it results $P(\tau_Y(y)<+\infty)=1$ and $\tau_Y(y)$ has the inverse Gaussian distribution, namely its density is (see e.g. [1]):

$$f(t\mid y)=\frac{x+y-a}{t^{3/2}}\,\varphi\left(\frac{x+y-a-bt}{\sqrt{t}}\right),$$

where $\varphi(u)=e^{-u^2/2}/\sqrt{2\pi}$. By conditioning on $Y=y$ we obtain that also τ_Y is finite with probability one. For a given distribution function F, our aim is to find the density of Y, if it exists, in such a way $P(\tau_Y\le t)=F(t)$. This problem has interesting applications in Mathematical Finance, in particular in credit risk modeling, where the first-hitting time of $a+bt$ represents a default event of an obligor.

2. MAIN RESULTS

By using the arguments of [2], with a replaced with $a-x$, we are able to obtain the following results.

2.1. Proposition

For $x\in \mathsf{R}$, and $a\ge x, b\ge 0$, let us consider the boundary $S(t)=a+bt$ and the process $X_Y(t)=x+B_t+Y$, where $Y\ge a-x$ is a random variable, independent of B_t, whose density g has to be found; suppose that the first-hitting time, τ_Y, of S by $X_Y(t)$, has an assigned probability density $f=F'$ and denote by $Lf(\theta)=\int_0^\infty e^{-\theta t}f(t)dt, \theta\ge 0$, the Laplace transform of f (see e.g. [3]). Then, if there exists the density g of Y such that $P(\tau_Y\le t)=F(t)$, its Laplace transform $Lg(\theta)$ must satisfy the equation:

$$Lg(\theta)=e^{-(a-x)\theta}Lf\left(\frac{\theta(\theta+2b)}{2}\right) \tag{1}$$

If $Lf(\theta)$ is analytic in a neighbor of $\theta=0$, then the k-th order moments of τ_Y exist finite and they are obtained in terms of $Lf(\theta)$ by $E(\tau_Y^k)=(-1)^k\frac{\partial^k}{\partial\theta^k}Lf(\theta)|_{\theta=0}$. The same thing holds for the moments of Y, since by (1) also $Lf(\theta)$ turns out to be analytic. Then, by (1) one easily obtains that

$$\begin{aligned} &E(Y)=a-x+bE(\tau_Y)\ \ and\\ &Var(Y)=b^2Var(\tau_Y)-E(\tau_Y)\end{aligned} \tag{2}$$

Since it must be $Var(Y)\ge 0$, we get the compatibility condition:

$$b^2Var(\tau_Y)-E(\tau_Y)\ge 0 \tag{3}$$

which is necessary so that there exist a random variable $Y\ge a-x$ which solves our problem (i.e. $P(\tau_Y\le t)=F(t)$), in the case of analytic Laplace transforms Lf and Lg Notice that, if e.g. $S(t)=a$ (i.e. $b=0$), the moments of τ_Y are infinite and (3) loses meaning.

2.2. Proposition

Suppose that the first-hitting time density f is the Gamma density with parameters (γ,λ). Then, there exists an absolutely continuous random variable $Y\ge a-x$ such that $P(\tau_Y\le t)=F(t)$, provided that $b\ge\sqrt{2\lambda}$, and the Laplace transform of the density g of Y is given by:

$$Lg(\theta)=\left[\frac{\left(b-\sqrt{b^2-2\lambda}\right)^\gamma}{\left(\theta+b-\sqrt{b^2-2\lambda}\right)^\gamma}\right]\cdot\left[\frac{\left(b+\sqrt{b^2-2\lambda}\right)^\gamma}{\left(\theta+b+\sqrt{b^2-2\lambda}\right)^\gamma}\right]e^{-(a-x)\theta} \tag{4}$$

*Address correspondence to this author at the Dipartimento di Matematica, Università Tor Vergata Via della Ricerca Scientifica, 00133 Roma, Italy; Tel: +39 06 72594627; Fax: +39 06 72594699;
E-mail: abundo@mat.uniroma2.it

which is the Laplace transform of the random variable $Z = a - x + Z_1 + Z_2$, where Z_i are independent random variables with Gamma distribution of parameters γ and λ_i $(i = 1,2)$, with $\lambda_1 = b - \sqrt{b^2 - 2\lambda}$ and $\lambda_2 = b + \sqrt{b^2 - 2\lambda}$.

2.3. Remark

If f is the Gamma density, the compatibility condition (3) writes $b \geq \sqrt{\lambda}$, which is satisfied under the assumption $b \geq \sqrt{2\lambda}$ required by Proposition 2.2. In the special case when f is the exponential density with parameter λ, then $Y = a - x + Z_1 + Z_2$, where Z_i are independent and exponential with parameter $\lambda_i, i = 1,2$.

2.4. Proposition

Suppose that the Laplace transform of the first-hitting time density f has the form:

$$Lf(\theta) = \sum_{k=1}^{N} \frac{A_k}{(\theta + B_k)^{c_k}} \tag{5}$$

for some $A_k, B_k, c_k > 0, k = 1,\ldots,N$. Then, a value $b^* > 0$ exists such that, if $b \geq b^*$ there exists an absolutely continuous random variable $Y \geq a - x$ for which τ_Y has density f.

The following Proposition deals with the case when $b = 0$.

2.5. Proposition

Let be $b = 0$ and suppose that the Laplace transform of the first hitting time density f has the form:

$$Lf(\theta) = \sum_{k=1}^{N} \frac{A_k}{(\sqrt{2\theta} + B_k)^{c_k}} \tag{6}$$

for some $A_k, B_k, c_k > 0, k = 1,\ldots,N$. Then, there exists an absolutely continuous random variable $Y \geq a - x$ for which τ_Y has density f.

2.6. Remark

The function $Lf(\theta)$ given by (6) is not analytic in a neighbor of $\theta = 0$, so the moments of τ_Y are indeed infinite.

We consider now the piecewise-continuous process $\overline{X}_Y(t)$, obtained by superimposing to $X_Y(t)$ a jump process, namely we set $\overline{X}_Y(t) = X_Y(t)$ for $t < T$, where T is an exponential distributed time with parameter $\mu > 0$; we suppose that for $t = T$ the process $\overline{X}_Y(t)$ makes a downward jump and it crosses the linear boundary $S(t) = a + bt$, irrespective of its state before the occurrence of the jump. This kind of behavior is observed e.g. in the presence of a so called *catastrophes*. Then, for $Y \geq a - x$, the first-hitting time of S by $\overline{X}_Y(t)$ is $\overline{\tau}_Y = \inf\{t > 0 : \overline{X}_Y(t) \leq a + bt\}$. By proceeding in analogous manner as in [2], with a replaced by $a - x$, and by correcting a typographical error, there present (see [4]), we obtain:

2.7. Proposition

If there exists an absolutely continuous random variable $Y \geq a - x$ such that $P(\overline{\tau}_Y \leq t) = F(t)$, then its Laplace transform is given by

$$\overline{Lg}(\theta) = e^{-(a-x)\theta}\left(1 - \frac{2\mu}{\theta(\theta + 2b)}\right)^{-1} [\overline{Lf}\left(\frac{\theta(\theta+2b)}{2} - \mu\right) - \frac{2\mu}{\theta(\theta+2b) - 2\mu}] \tag{7}$$

where $\overline{Lf}$ denotes the Laplace transform of $\overline{\tau}_Y$.

2.8. Remark

For $\mu = 0$, namely when no jump occurs, (7) reduces to (1).

Example

(i) For $a \geq x, \mu > 0$, let be

$$\overline{Lf}(\theta) = \frac{a\mu\sqrt{2(\theta+\mu)} + \theta - \theta e^{-a\sqrt{2(\theta+\mu)}}}{a(\theta+\mu)\sqrt{2(\theta+\mu)}}$$

and take $S(t) = a$. Suppose that the density of the first-hitting time of S is $\overline{f} = \overline{F}'$, i.e. $P(\overline{\tau}_Y \leq t) = \overline{F}(t)$, then Y is uniformly distributed in the interval $[a - x, 2a - x]$. In fact, from (7) with $b = 0$, it follows that $\overline{Lg}(\theta) = \frac{e^{-(a-x)\theta}(1 - e^{-a\theta})}{a\theta}$, which is indeed the Laplace transform of $\overline{g}(y) = 1_{[a-x,2a-x]}(y) \cdot \frac{1}{a}$.

(ii) For $c, \mu > 0$, let be

$$\overline{Lf}(\theta) = \frac{c(\theta+\mu) + \mu\sqrt{2(\theta+\mu)}}{(\theta+\mu)\left(c + \sqrt{2(\theta+\mu)}\right)}$$

and take $S(t) = a$. Suppose that the density of the first-hitting time of S is $\overline{f} = \overline{F}'$, then the density of Y is $\overline{g}(y) = ce^{-c(y+x-a)}, y \geq a - x$, namely $Y = a - x + Z$, where Z is exponentially distributed with parameter c. In fact, from (7) with $b = 0$, it follows that $\overline{Lg}(\theta) = e^{-(a-x)\theta} \cdot \frac{c}{c+\theta}$, which is indeed the Laplace transform of $a - x + Z$, with Z exponential of parameter c.

2.9. Remark

In the case without jump, we have considered the first-passage time of $x + Y + B_t$ below the linear boundary $S(t) = a + bt$, with $Y \geq a - x, a \geq x, b \geq 0$. In analogous way, one could consider the first-passage time of $x + \overline{Y} + B_t$ over the boundary $\overline{S}(t) = a - bt$, with $\overline{Y} \leq a - x, a \geq x, b \geq 0$. In fact, since $-B_t$ has the same distribution as B_t, we get that $\inf\{t > 0 : x + \overline{Y} + B_t \geq a - bt\}$ has the same distribution as $\inf\{t > 0 : B_t \leq x + \overline{Y} - a + bt\} = \inf\{t > 0 : x - \overline{Y} + B_t \leq 2x - a + bt\}$, that is the first-passage time, $\tau(-\overline{Y})$, of $X(-\overline{Y})(t)$ below the linear boundary $2x - a + bt$. Thus, by using the arguments of Proposition 2.1, we obtain that, if the first-passage time of $x + \overline{Y} + B_t$ over $\overline{S}$ has density f, then the Laplace transform, $L\overline{g}(\theta)$ of $\overline{Y}$ must satisfy the equation:

$$L\overline{g}(\theta) = e^{-(a-2x)\theta}\, Lf\left(\frac{\theta(\theta - 2b)}{2}\right) \tag{8}$$

2.10. Remark

The results of the present paper can be extended to processes such as $X_Y(t) := x + Z(t) + Y$, where $Z(t)$ is a one-dimensional, time-homogeneous diffusion, starting from $Z(0) = 0$, with respect to a non-linear boundary $S(t)$ with $S(0) \le x + Y$. Indeed, this is possible when $Z(t)$ can be reduced to Brownian motion by a deterministic transformation and a random time-change (see [2] for a few examples); then, the results concerning $Z(t)$ can be obtained by those for B_t, by using the arguments of [2] with a replaced by $a - x$.

CONFLICT OF INTEREST

The author confirms that this article content has no conflict of interest.

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MATHEMATICS SUBJECT CLASSIFICATION

60J60, 60H05, 60H10.

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