Application of Homotopy Perturbation Method in Nonlinear Heat Diffusion-Convection-Reaction Equations

N. Tolou, D.D. Ganji*, M.J. Hosseini and Z.Z. Ganji

Department of Mechanical Engineering, Mazandaran University, P.O. Box 484, Babol, Iran

Abstract: In this Paper, Homotopy- Perturbation method (HPM) which is one of the most recent approximate analytical solutions is implemented to solve diffusion-convection-reaction equations (DCRE). The calculations are carried out for two different types of DCRE's such as the Black-Scholes equation used in financial market option pricing and Fokker-Planck equation from plasma physics. The behavior of the approximate solutions of the distribution functions are shown graphically and are compared with those obtained by other theories (e.g. the Adomian decomposition method, ADM). It shows that the numerical results of these methods are the same; while HPM is much easier, more convenient and more efficient than ADM.

1. INTRODUCTION

Most scientific phenomena occur nonlinearly. We know that except a limited number, most of them do not have precise analytical solutions. Therefore, these nonlinear equations are to be solved using other methods. In recent decades, numerical methods were good means of analyzing the equations related to DCRE; but as numerical calculation methods were improving, semi-exact analytical methods were, too. Most scientists believe that the combination of numerical and semi-exact analytical methods can also end with remarkable results.

Perturbation method is one of the most well-known methods to solve nonlinear equations studied by a large number of researchers such as Bellman [1], Cole [2] and O'Malley [3]. Actually, these scientists had paid more attention to the mathematical aspects of the subject which included a loss of physical verification. This loss in the physical verification of the subject was recovered by Nayfeh [4] and Van Dyke [5].

Since there are some limitations with the common perturbation method, and also because the basis of the common perturbation method was upon the existence of a small parameter, developing the method for different applications is very difficult. Therefore, many different methods have recently introduced some ways to eliminate the small parameter, such as artificial parameter method introduced by Liu [6], the homotopy perturbation method by Ganji [7, 8] and the variational iteration method by He [9-11].

In this paper, a brief review of the method is firstly presented, it is applied to two types of DCR equations and then the behavior of the approximate solutions of the distribution functions are shown graphically and are compared with those obtained by other techniques, like ADM.

2. BASIC IDEA OF HOMOTOPY PERTURBATION METHOD

In this letter, we apply the homotopy perturbation method [26-33] to the discussed problem. To illustrate the basic ideas of the new method, we consider the following nonlinear differential equation.

$$A(u) - f(r) = 0 , \quad r \in \Omega \tag{1}$$

with boundary conditions

$$B(u \, \partial u / \partial n) = 0 , r \in \Gamma \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω.

The operator A can be general divided into two parts L and N, where L is linear, whereas N is nonlinear. Therefore, Eq. (1) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \tag{3}$$

In case the nonlinear equation (1) has no "small parameter", we can construct the following homotopy,

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{4}$$

where p is called homotopy parameter.

According to the homotopy perturbation method, the approximation solution of Eq. (4) Can be expressed as a series of the power of p, i, e.

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ..., \tag{5}$$

When Eq. (4) correspond to Eq. (1), (5) becomes the approximate solution of Eq. (1) some interesting results have been attained using this method [12-18].

*Address correspondence to this author at the Department of Mechanical Engineering, Mazandaran University, P.O. Box 484, Babol, Iran; E-mail: ddg_davood@yahoo.com

3. APPLICATIONS

3.1. The Black-Scholes Equation (BSE)

DRCE and stochastic analysis have interesting applications in mathematical modeling of financial market option pricing [19-22]. The most well- known stochastic model for the equilibrium condition among the expected return on the option, the expected return on the stock and the risk-less interest rate is the Black-Scholes equation [21].

Consider The Black - Scholes equation. Risky

$$\frac{\partial}{\partial t}c(s,t) = \frac{v^2}{2}s^2\frac{\partial^2 c(s,t)}{\partial s^2} + rs\frac{\partial c(s,t)}{\partial s} - rc(s,t), \tag{6}$$

where s is the asset price, which undergoes geometric Brownian motion, $c(s,t)$ is the call price, v is the velocity and r is the risk-less interest rate.

Using the following transformation [20]

$$x = Ln(s) \tag{7}$$

and

$$P(x,t) = exp(rt)c(s,t) \tag{8}$$

we apply homotopy-perturbation to Eq. (6). We construct a homotopy in the from:

$$(1-p)\left[\left(\frac{\partial}{\partial t}c(s,t)\right) + rc(s,t) - rs - \frac{r}{s^{(7/5)}}\right] \tag{9}$$

$$+p\left[\left(\frac{\partial}{\partial t}c(s,t)\right) - \frac{1}{2}v^2s^2\left(\frac{\partial^2}{\partial s^2}c(s,t)\right) - rs\left(\frac{\partial}{\partial s}c(s,t)\right) + rc(s,t)\right] = 0$$

It is clear that when $p = 0$, Eq. (9) becomes a linear equation and when $p = 1$, it becomes the original nonlinear one. So the changing process of p Form zero to unity is just that of the function of a linear oscillator to the original nonlinear one.

The initial condition is:

$$C(s,0) = s + \frac{1}{s^{7/5}} \tag{10}$$

According to homotopy perturbation theory, we assume that the solution of Eq. (6) can be expressed in a series of p :

$$c(s,t) = c_0(s,t) + pc_1(s,t) + p^2c_2(s,t) + p^3c_3(s,t) \tag{11}$$

and so on. Substituting Esq. (17)-(19) and other calculations into Eq. (11) gives the solution:

Substituting $c(s,t)$ from Eq. (11) into Eq. (9) and after some simplifications and substitutions, we have:

$$\left[-\frac{1}{2}v^2s^2p^3\left(\frac{\partial^2}{\partial s^2}c_2(s,t)\right) - \frac{1}{2}v^2s^2p^2\left(\frac{\partial^2}{\partial s^2}c_1(s,t)\right) + p^2\left(\frac{\partial}{\partial t}c_2(s,t)\right) + prc_1(s,t)\right.$$

$$+rp^2c_2(s,t) + rp^3c_3(s,t) - \frac{42v^2p}{25s^{(7/5)}} + \frac{12rp}{5s^{(7/5)}} + p^3\left(\frac{\partial}{\partial t}c_3(s,t)\right) - \frac{1}{2}v^2s^2p^4\left(\frac{\partial^2}{\partial s^2}c_3(s,t)\right) \tag{12}$$

$$\left.-rsp^2\left(\frac{\partial}{\partial s}c_1(s,t)\right) - rsp^3\left(\frac{\partial}{\partial s}c_2(s,t)\right) - rsp^4\left(\frac{\partial}{\partial s}c_3(s,t)\right) + \left(\frac{\partial}{\partial t}c_1(s,t)\right)\right] = 0$$

Rearranging based on powers of p - terms, we have:

$$p^0 : 0 \tag{13}$$

$$p^1 : \frac{50\left(\frac{\partial}{\partial t}c_1(s,t)\right)s^{(7/5)} - 84v^2 + 50rc_1(s,t)s^{(7/5)} + 120r}{50s^{(7/5)}} = 0 \tag{14}$$

$$p^2 : -\frac{1}{50s^{(7/5)}}\left(25v^2s^{(17/5)}\left(\frac{\partial^2}{\partial s^2}c_1(s,t)\right) - 50rc_2(s,t)s^{(7/5)} - 50\left(\frac{\partial}{\partial t}c_2(s,t)\right)s^{(7/5)} + 50rs^{(12/5)}\left(\frac{\partial}{\partial s}c_1(s,t)\right)\right) = 0 \tag{15}$$

$$p^3 : -\frac{1}{50s^{(7/5)}}\left(-50rc_3(s,t)s^{(7/5)} - 50\left(\frac{\partial}{\partial t}c_3(s,t)\right)s^{(7/5)} + 25v^2s^{(17/5)}\left(\frac{\partial^2}{\partial s^2}c_2(s,t)\right) + 50rs^{(12/5)}\left(\frac{\partial}{\partial s}c_2(s,t)\right)\right) = 0 \tag{16}$$

Solving Eq. (13-16) we have:

$$c_1(s,t) = -\frac{6e^{(-rt)}(7v^2 - 10r)}{25rs^{(7/5)}} + \frac{6(7v^2 - 10r)}{25rs^{(7/5)}} \tag{17}$$

$$c_2(s,t) = \left(\frac{42(7v^2 - 10r)(-5r + 6v^2)\left(-t + \frac{e^{(rt)}}{r}\right)}{625rs^{(7/5)}} - \frac{42(7v^2 - 10r)(-5r + 6v^2)}{625r^2s^{(7/5)}}\right)e^{(-rt)} \tag{18}$$

$$c_3(s,t) = \left[-\frac{294(-5r+6v^2)^2(7v^2-10r)\left(t+\frac{1}{2}rt^2-\frac{e^{(rt)}}{r}\right)}{15625r^2s^{(7/5)}} - \frac{294(-5r+6v^2)(7v^2-10r)}{15625\,r^3s^{(7/5)}} \right]e^{(-rt)} \tag{19}$$

$$c(s,t) = s + \frac{1}{s^{(7/5)}} + p\left(-\frac{6e^{(-rt)}(7v^2-10r)}{25rs^{(7/5)}} + \frac{6(7v^2-10r)}{25rs^{(7/5)}} \right)$$

$$+p^2\left[\frac{42(7v^2-10r)(-5r+6v^2)\left(-t+\frac{e^{(rt)}}{r}\right)}{625rs^{(7/5)}} - \frac{42(7v^2-10r)(-5r+6v^2)}{625\,r^2s^{(7/5)}} \right]e^{(-rt)}$$

$$+p^3\left[-\frac{294(-5r+6v^2)^2\,(7v^2-10r)\left(t+\frac{1}{2}rt^2-\frac{e^{(rt)}}{r}\right)}{15625\,r^2\,s^{(7/5)}} - \frac{294(-5r+6v^2)^2+(7v^2-10r)}{15625\,r^3\,s^{(7/5)}} \right]e^{(-rt)} \tag{20}$$

When p reaches $p \to 1$, $r \to 1$, $v \to 1$ then

$$c(s,t)_{HPM} = s + \frac{7}{25s^{(7/5)}} + \frac{18e^{(-t)}}{25s^{(7/5)}} + \left(-\frac{126(-t+e^t)}{625s^{(7/5)}} + \frac{126}{625s^{(7/5)}} \right)e^{(-t)}$$

$$+\left(\frac{882\left(t+\frac{1}{2}t^2-e^t\right)}{15625\,s^{(7/5)}} + \frac{882}{15625\,s^{(7/5)}} \right)e^{(-t)} \tag{21}$$

Inserting Eq. (21) into Eq. (8) gives the solution of Eq. (6).

$$P_{HPM}(x,t) = e^t\left(e^x + \frac{7}{25(e^x)^{(7/5)}} + \frac{18e^{(-t)}}{25(e^x)^{(7/5)}} + \left(-\frac{126(-t+e^t)}{625(e^x)^{(7/5)}} + \frac{126}{625(e^x)^{(7/5)}} \right)e^{(-t)} \right.$$

$$\left. +\left(\frac{882\left(t+\frac{1}{2}t^2-e^t\right)}{15625(e^x)^{(7/5)}} + \frac{882}{15625(e^x)^{(7/5)}} \right)e^{(-t)} \right) \tag{22-a}$$

We have the ADM solution [36] of Eq. (6) in the from:

$$P(x,t)_{ADM} = \frac{1}{(e^x)^{(7/5)}}\left(10^{-4}e^t\left(-13907(e^x)^{(7/5)} + 10^4.t + 10^4.t^2 + 10^4.t^3 \right. \right.$$

$$\left. \left. +10^4.t^4 + 10^4.(e^x)^{(12/5)} + 10^4 \right) \right) \tag{22-b}$$

3.2. Fokker-Planck Equation in Plasma Physics

Consider the Fokker-Planck equation (FPE) from plasma physics in terms of quite complicated integrals [23], the governing equation is taken as [24].

$$\frac{\partial P}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{2}x^{1-2\epsilon}P \right) + -\frac{\partial}{\partial x}\left(\frac{1}{4}x^{-2\epsilon}P \right) \tag{23}$$

With $\epsilon \neq -1/2$, Eq. (24) can be written in the fractional form as follows:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{1}{4}x^{1-2\epsilon}P(x,t) \right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{2}x^{1-2\epsilon}P(x,t) \right) \tag{24}$$

With the following replacements [25]:

$$P(x,t) = \sqrt{2}\,x^{(2\epsilon-1)/2}\omega(\tau,y),\ \tau = t,\ y = \frac{2\sqrt{2}}{2\epsilon+1}x^{(2\epsilon+1)/2} \tag{25}$$

Eq. (24) transforms FPE for the linear Brownian motion as:

$$\frac{\partial \omega(\tau, y)}{\partial \tau} = -\frac{\partial}{\partial y}\left(\frac{\omega}{2y}\right) + \frac{1}{2}\frac{\partial^2 \omega}{\partial y^2}, \qquad (26)$$

With the initial condition of :

$$\omega(o, y) = erf\left[\frac{1}{2}\sqrt{2}y\right]e^{y^2/2} + e^{y^2/2} \qquad (27)$$

Again, to find the solution of this equation by HPM, we simply take the equation in an operator form as

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0 \qquad (28)$$

Then we construct a homotopy in the from

$$\left(\frac{\partial}{\partial \tau}\omega(\tau, y)\right) + p\left(\frac{\frac{\partial}{\partial y}\omega(\tau, y)}{2y} - \frac{\omega(\tau, y)}{2y^2}\right) - 0.5p\left(\frac{\partial^2}{\partial y^2}\omega(\tau, y)\right) = 0 \qquad (29)$$

Assuming:

$$\omega(\tau, y) := \omega_0(\tau, y) + p\omega_1(\tau, y) + p^2\omega_2(\tau, y) \qquad (30)$$

Similar to example 1, by substituting Eq. (27), (30) in Eq. (29) and after organizing based on powers, we have:

$$p^0 : 0 \qquad (31)$$

$$p^1 : -\frac{1}{y2}\left(2.82\times10^{-10}\left(-3.54\times10^9\left(\frac{\partial}{\partial \tau}\omega_1(\tau, y)\right)y^2 + 1.77\times10^9\ erf(0.707y)e^{(0.5\ y^2)} - 1.41\times10^9\ ye^{(3\times10^{-10}\ y^2)}\right.\right.$$
$$\left.\left.+1.77\times10^9\ y^4e^{(0.5\ y^2)} + 1.41\times10^9\ y^3\ e^{(3\times10^{-10}\ y^2)} + 1.77\times10^9\ e^{(0.5\ y^2)} + 1.77\times10^9\ erf(0.707y)y^4e^{(0.5\ y^2)}\right)\right) = 0 \qquad (32)$$

$$p^2 : -\frac{1}{y2}\left(2.82\times10^{-10}\left(1.77\times10^9\left(\frac{\partial^2}{\partial y^2}\omega_1(\tau, y)\right)y^2 - 1.77\times10^9\left(\frac{\partial}{\partial y}\omega_1(\tau, y)\right)y - 3.54\times10^9\left(\frac{\partial}{\partial \tau}\omega_2(\tau, y)\right)y^2 + 1.77\times10^9\ \omega_1(\tau, y)\right)\right) = 0 \qquad (33)$$

$$p^3 : -\frac{1}{y^2}\left(2.82\times10^{-10}\left(-1.77\times10^9\left(\frac{\partial}{\partial y}\omega_2(\tau, y)\right)y + 1.77\times10^9\ \omega_2(\tau, y) + 1.77\times10^9\left(\frac{\partial^2}{\partial y^2}\omega_2(\tau, y)\right)y^2\right)\right) = 0 \qquad (34)$$

Solving Eqs. (32), we have:

$$\omega_1(\tau, y) = \frac{1}{10^{10}\ y^2}\left(3\left(1.6\times10^9 erf\left(\frac{7.07\times10^8}{10^{10}}y\right)e^{\left(\frac{1}{2}y^2\right)} - 1.3\times10^9\ ye^{\left(\frac{3}{10^{10}}y^2\right)} + 1.6\times10^9\ y^4\ e^{\left(\frac{1}{2}y^2\right)}\right.\right.$$
$$\left.\left.+1.3\times10^9\ y^3e^{\left(\frac{3}{10^{10}}y^2\right)} + 1.6\times10^9\ e^{\left(\frac{1}{2}y^2\right)} + 1.6\times10^9\ erf\left(\frac{7.07\times10^8}{10^9}y\right)y^4\ e^{\left(\frac{1}{2}y^2\right)}\right)\tau\right) \qquad (35)$$

Since, $\omega_2(\tau, y), \omega_3(\tau, y)$ are to long too be written in here, they are illustrated in diagrams. Substituting the above results in Eq. (30), having them simplified and finally putting $p=1$, we have:

$$\omega(\tau, y) = erf(0.707y)e^{\left(\frac{1}{2}y^2\right)} + e^{\left(\frac{1}{2}y^2\right)} + \frac{1}{10^{10}\ y^2}\left(3\left(1.6\times10^9\ erf\left(\frac{7.07\times10^8}{10^9}y\right)e^{\left(\frac{1}{2}y^2\right)} - 1.3\times10^9\ ye^{\left(\frac{3}{10^{10}}y^2\right)} + 1.6\times10^9\ y\ e^{\left(\frac{1}{2}y^2\right)}\right.\right.$$

$$\left.\left.+1.3\times10^9\ y^3\ e^{\left(\frac{3}{10^{10}}y^2\right)} + 1.6\times10^9\ e^{\left(\frac{1}{2}y^2\right)} + 1.6\times10^9\ erf\left(\frac{7.07\times10^8}{10^9}y\right)y^4\ e^{\left(\frac{1}{2}y^2\right)}\right)\tau\right) - \frac{\tau^2\ erf\left(0.707\right)e^{\left(\frac{1}{2}y^2\right)}}{2y^2}$$

$$+0.25\tau^2\ erf\left(0.707y\right)e^{\left(\frac{1}{2}y^2\right)} + \frac{1}{2}\tau^2\ y^2\ erf\left(0.707y\right)e^{\left(\frac{1}{2}y^2\right)} + \frac{225\ \tau^2\ erf\left(0.707y\right)e^{\left(\frac{1}{2}y^2\right)}}{200\ y^4} - \frac{\tau^2\ e^{\left(\frac{1}{2}y^2\right)}}{2y^2} + 0.25\tau^2\ e^{\left(\frac{1}{2}y^2\right)}$$

$$+\frac{1}{2}\tau^2\ y^2\ e^{\left(\frac{1}{2}y^2\right)} - \frac{0.4\ \tau^2e^{\left(3\times10^{-10}\ y^2\right)}}{y^3} + 0.125\tau^2\ y^4\ e^{\left(\frac{1}{2}y^2\right)} + 3.6\times10^{-20}\ \tau^2\ y^3\ e^{\left(\frac{3}{10^{10}}y^2\right)} + 1.12\times10^{-10}\ \tau^2\ e^{\left(\frac{3}{10^{10}}y^2\right)} + 1.19\times10^{-10}\ \tau^2\ y\ e^{\left(\frac{3}{10^{10}}y^2\right)}$$

$$+\frac{1.125\tau^2\ e^{\left(\frac{1}{2}y^2\right)}}{y^4} + 0.1\tau^2\ y^3\ e^{\left(2.64\times10^{-10}\ y^2\right)} + 0.0125\ \tau^2\ y^4\ erf\left(.707y\right)e^{\left(\frac{1}{2}y^2\right)} + 0.3\tau^2\ y^3\ e^{\left(2.63\times10^{-10}\ y^2\right)} \qquad (36)$$

$$+\frac{0.1\ \tau^2\ e^{\left(2.64\times10^{-10}\ y^2\right)}}{y} - \frac{0.5\tau^2\ e^{\left(2.64\times10^{-10}\ y^2\right)}}{y^3}$$
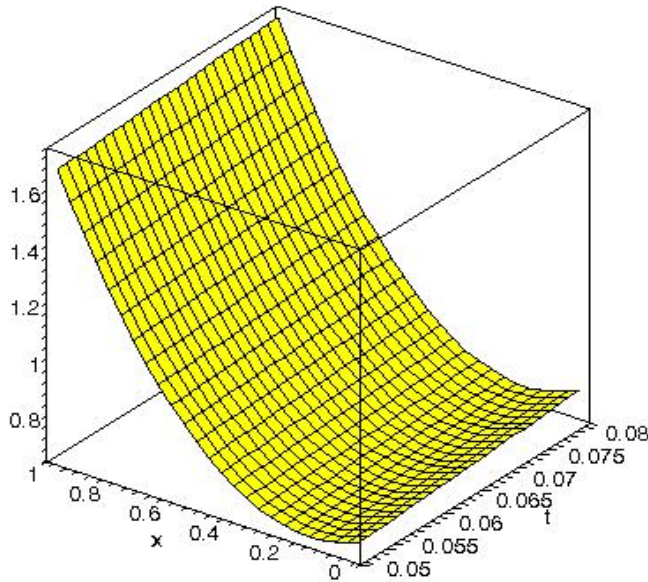
Substituting Eq. (25) in to Eq. (36) gives the solution of probability.
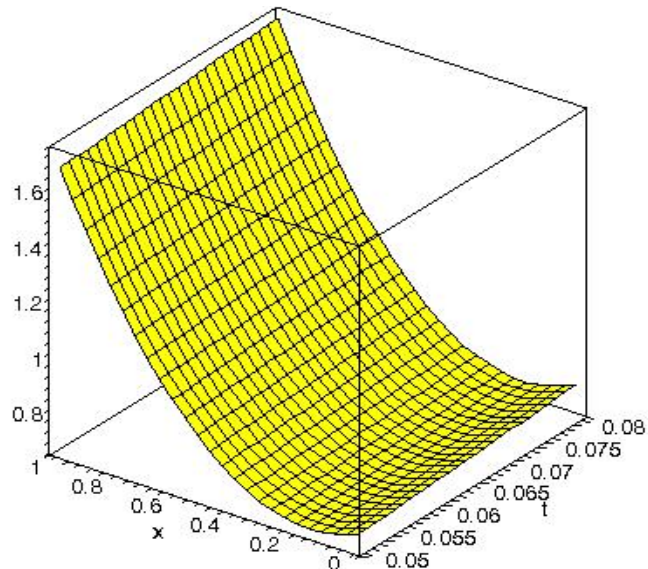
4. RESULTS AND DISCUSSION

To demonstrate the convergence of HPM, the results of the numerical example are shown graphically.

4.1. The Black-Scholes Equation

Fig. (**1**) compares the function distribution by HPM, Eq. (22-a), and ADM, Eq. (22-b), under the proposed condition, Eq. (10). There is an excellent agreement between the results by HPM and ADM. Fig. (**2**) shows the result for constant t; t=0.05 (Fig. **2a**) and t=0.08 (Fig. **2b**) for better comparison.
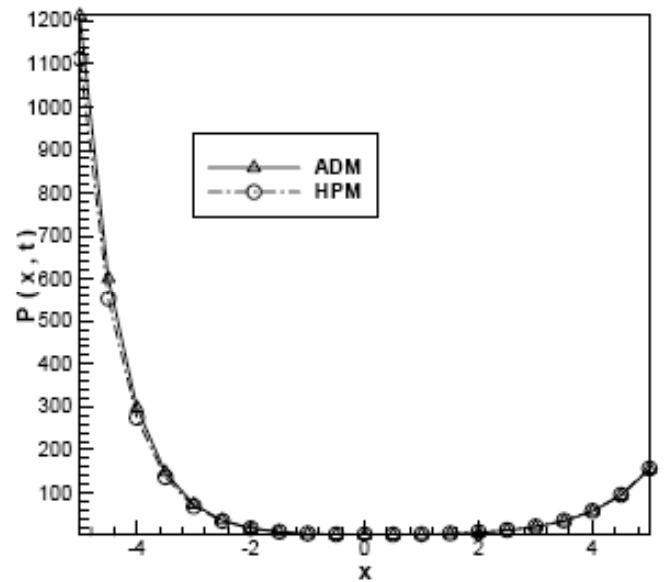

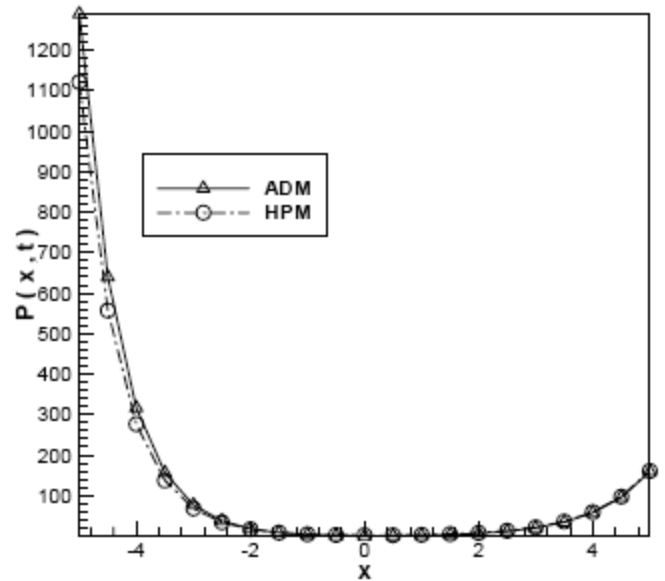**Fig. (1a).** HPM results of $P(x,t)$.

Fig. (1b). ADM results of $P(x,t)$.

4.2. Fokker-Planck Equation

Fig. (**3**) presents the function distribution obtained by the proposed methods, Eq. (36), under the introduced initial condition, Eq. (27). There is an excellent agreement between the results by HPM and ADM.

Fig. (2a). The comparison of the results of $P(x,t)$ for the two methods, at t=0.05.

Fig. (2b). The comparison of the results of $P(x,t)$ the two methods, at t=0.08.

CONCLUSIONS

In this survey, Homotopy- perturbation method (HPM) was used to solve the diffusion-convection reaction equations. The calculations were done for two different types of DCRE's such as the Black-Scholes equation used in financial market option pricing, and Fokker-Planck equation from plasma physics. In this work, the basic idea of the HPM is introduced and we extend the analysis of Homotopy perturbation method [5-15] to solve the DCRE equation.

The obtained solutions were compared with those of Adomian's decomposition Method. All examples show that the results of the present method are in excellent agreement with those obtained by the Adomian's decomposition method. Applying HPM to DCRE's has more advantages than ADM; it overcomes the difficulties arising in the calculation of Adomian polynomials. Also HPM does not require

small parameters in the equation, so that the limitations of the traditional perturbation methods can be eliminated, and thereby the calculations are simple and straightforward.
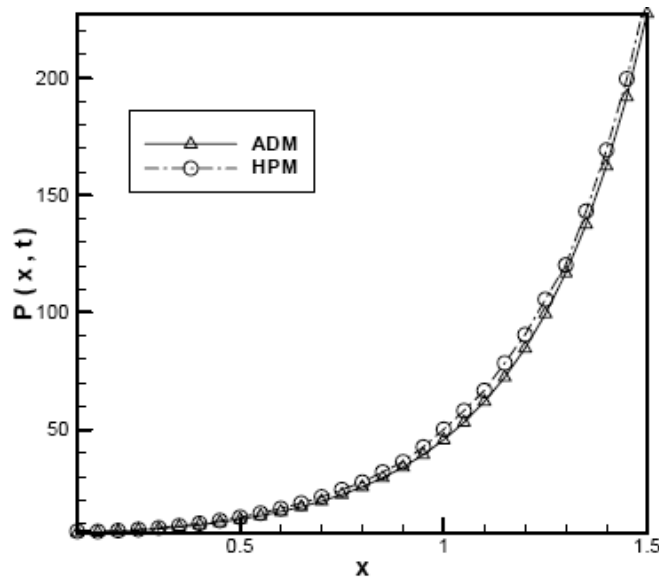

Fig. (3). The comparison of the results of $P(x,t)$ the two methods, at t=0.05.

The behavior of approximate solution of the probability density function is computed numerically and shown graphically. In this work, MAPLE package was used in the mathematical calculations.

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