

Theoretical Study on Fluid Velocity for Viscous Fluid in a Circular Cylindrical Shell

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Abstract: A theoretical algorithm by united Lagrangian-Eulerian method for the problem in dealing with viscous fluid and a circular cylindrical shell is presented. In this approach, each material is described in its preferred reference frame. Fluid flows are given in Eulerian coordinates whereas the elastic circular cylindrical shell is treated in a Lagrangian framework. The fluid velocity in a two-dimensional uniform elastic circular cylindrical shell filled with viscous fluid is studied under the assumption of low Reynolds number. The coupling between the viscous fluid and the elastic circular cylindrical shell shows kinematic conditions at the shell surface. Also, the radial velocity and axial velocity of the fluid are discussed with the help of graphs.

Keywords: Elastic circular cylindrical shell, fluid velocity, kinematic condition, united lagrangian-eulerian method, viscous fluid.

1. INTRODUCTION

This paper deals with the mathematical analysis of problem for viscous fluid and a circular cylindrical shell by a new theoretical method. These phenomena are of major importance for aerospace, ocean engineering, mechanical or biomedical applications, *etc.* and thus have been studied by many authors over the past few years [1-4]. The situation has mainly been analysed by numerical methods [5, 6]. Our focus is the velocity of the fluid when the elastic circular cylindrical shell filled with viscous fluid vibrates.

Typically, the viscous fluid and a circular cylindrical shell are given in different coordinate systems making a common solution. Fluid flows are given in Eulerian coordinates whereas the circular cylindrical shell is treated in a Lagrangian framework. United Lagrangian-Eulerian method is used to present the flow velocity of a viscous and incompressible fluid in a circular cylindrical shell. It is a new method of where fluid and circular cylindrical shell equations are given in their preferred reference frames. Coupling between the fluid and the circular cylindrical shell domains represents kinematic conditions at the shell surface. The two-dimensional problem is that of an elastic circular cylindrical shell in which waves of lateral displacement are propagated.

2. BASIC EQUATIONS FOR VISCOUS FLUID

In this study, the analysis of the fluid velocity of an incompressible viscous fluid in a complete circular cylindrical shell of radius R is conducted. The shell is horizontal. In order to describe the geometry of the shell, a

cylindrical coordinates system $(O;r,\theta,z)$ is considered, where O is the origin placed at the centre of the shell, r is the radial and z is the axial coordinate. The displacement field of the middle surface of the shell is given by the following components: u_r, u_θ and u_z ; in the radial, circumferential and axial directions, respectively.

In cylindrical coordinates system, the Navier-Stokes equations of incompressible viscous fluid can be expressed as

$$\begin{cases} \rho \left(\frac{dv_r}{dt} - \frac{v_\theta^2}{r} \right) = \rho F_r - \frac{\partial p}{\partial r} + \mu \left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right) \\ \rho \left(\frac{dv_\theta}{dt} + \frac{v_r v_\theta}{r} \right) = \rho F_\theta - \frac{\partial p}{r \partial \theta} + \mu \left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right) \\ \rho \frac{dv_z}{dt} = \rho F_z - \frac{\partial p}{\partial z} + \mu \nabla^2 v_z \end{cases} \quad (1)$$

where μ is the coefficient of viscosity, ρ is the mass density of the fluid, p is the pressure, v_r, v_θ and v_z are the projections of the velocity vector, F_r, F_θ and F_z are the projections of the unit mass force vector, t is time. In equations (1), the operators are defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + v_\theta \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

The Navier-Stokes equations of incompressible viscous fluid are almost equal to the Stokes equations under the assumption of low Reynolds number. The Stokes equations can be written as

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$$\begin{cases} \frac{\partial p}{\partial r} = \mu \left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right) \\ \frac{\partial p}{r \partial \theta} = \mu \left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right) \\ \frac{\partial p}{\partial z} = \mu \nabla^2 v_z \end{cases} \quad (2)$$

The continuity equation for the viscous fluid can be described as

$$\frac{1}{r} \frac{\partial (rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \quad (3)$$

3. UNITED LAGRANGIAN-EULERIAN METHOD

The fixed point m on the shell surface coincides with the space point M at $t=0$ whereas m coincides with the space point M^* at $t>0$. Adhesion condition for viscous fluid can be simplified as [7]

$$\mathbf{V} = \frac{\partial \mathbf{u}}{\partial t} \quad (4)$$

where \mathbf{u} is the displacement vector of the fixed point m , \mathbf{V} is velocity vector of space point M^* .

Dynamic condition is

$$\mathbf{Z} = \mathbf{p}_3 - \mathbf{p}'_3 \quad (5)$$

where \mathbf{p}_3 and \mathbf{p}'_3 are the pressures in different directions at the space point M^* , \mathbf{Z} is the force vector. In the cylindrical coordinate, V_m can be expressed as

$$V_m = v_m + u_r \frac{\partial v_m}{\partial r} + u_\theta \frac{\partial v_m}{r \partial \theta} + u_z \frac{\partial v_m}{\partial z} \quad (6)$$

where V_m ($m=r,\theta,z$) are the projections of the velocity at the space point M^* . v_m ($m=r,\theta,z$) are the projections of the velocity at the space point M . Taking into account the minor deformation of the shell, the third item of Taylor expansions will not be considered.

Expression (4) can be written as

$$\frac{\partial u_r}{\partial t} - V_r = 0, \quad \frac{\partial u_\theta}{\partial t} - V_\theta = 0, \quad \frac{\partial u_z}{\partial t} - V_z = 0 \quad (7)$$

where V_r , V_θ and V_z can be obtained from equation (6).

4. VELOCITY OF FLUID

In order to simplify the analysis, only an infinite length circular cylindrical shell was considered. The waving surface of the circular cylindrical shell is is represented by:

$$u_r = b \sin(\alpha z - \omega t), \quad u_z = 0 \quad (8)$$

where $\alpha = 2\pi/\lambda$, $\omega = c\alpha$, b is the amplitude of the traveling wave, λ is the wavelength, c is the velocity of the wave.

Considering the Reynolds number $Re \ll 1$, the Stokes equations in two dimensions can be written as

$$\frac{\partial p}{\partial r} = \mu \left(\nabla^2 v_r - \frac{v_r}{r^2} \right), \quad \frac{\partial p}{\partial z} = \mu \nabla^2 v_z \quad (9)$$

in which

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2}$$

The continuity equation of the incompressible viscous fluid can be described as

$$\frac{1}{r} \frac{\partial (rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \quad (10)$$

The stream function ψ satisfies the conditions

$$v_r = \frac{\partial \psi}{r \partial z}, \quad v_z = -\frac{\partial \psi}{r \partial r} \quad (11)$$

The kinematic conditions can be written as

$$\begin{cases} \frac{\partial u_r}{\partial t} = v_r + u_r \frac{\partial v_r}{\partial r} + u_\theta \frac{\partial v_r}{\partial \theta} + u_z \frac{\partial v_r}{\partial z} \\ \frac{\partial u_\theta}{\partial t} = v_\theta + u_r \frac{\partial v_\theta}{\partial r} + u_\theta \frac{\partial v_\theta}{\partial \theta} + u_z \frac{\partial v_\theta}{\partial z} \\ \frac{\partial u_z}{\partial t} = v_z + u_r \frac{\partial v_z}{\partial r} + u_\theta \frac{\partial v_z}{\partial \theta} + u_z \frac{\partial v_z}{\partial z} \end{cases}$$

Thus the following expressions for ψ can be written

$$\begin{cases} \frac{\partial \psi}{r \partial r} + u_r \frac{\partial}{\partial r} \left(\frac{\partial \psi}{r \partial r} \right) = 0 \\ \frac{\partial \psi}{r \partial z} + u_r \frac{\partial}{\partial r} \left(\frac{\partial \psi}{r \partial z} \right) = \frac{\partial u_r}{\partial t} \end{cases} \quad (r=R) \quad (12)$$

Making use of equations (9), (10) and (11), expression for stream function ψ can be written as

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} \right)^2 \psi = 0 \quad (13)$$

The solution of the equation (13) is

$$\psi = B_0 r^2 + \sum_{n=1}^N [A_n r^2 I_0(n\alpha r) + B_n r I_1(n\alpha r)] b \sin n(\alpha z - \omega t) \quad (14)$$

where $I_0(n\alpha r)$, $I_1(n\alpha r)$ are the Bessel functions of the first kind [8].

The solutions of the Bessel equation $x^2 y'' + xy' + (x^2 - v^2)y = 0$ ($v \geq 0$) is v order Bessel function. A special solution of Bessel function can be written as

$$y(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(k+\rho)!} \left(\frac{x}{2} \right)^{2k+\rho} \quad (\rho = v)$$

This is v order Bessel function of the first kind. We get

$$I_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

$$I_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(k+1)!} \left(\frac{x}{2}\right)^{2k+1}$$

$$I_0(n\alpha r) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k!)^2} \left(\frac{n\alpha r}{2}\right)^{2k}$$

$$I_1(n\alpha r) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(k+1)!} \left(\frac{n\alpha r}{2}\right)^{2k+1}$$

$$\frac{\partial I_0(n\alpha r)}{\partial r} = -n\alpha I_1(n\alpha r), \quad \frac{\partial [rI_1(n\alpha r)]}{\partial r} = n\alpha r I_0(n\alpha r)$$

Substituting the equations (8) and (14) into (12), the following expressions can be written as

$$\begin{aligned} & 2B_0 + \sum_{n=1}^N [2A_n I_0(n\alpha r) - n\alpha r A_n I_1(n\alpha r) + n\alpha B_n I_0(n\alpha r)] \\ & \times \sin n(\alpha z - \omega t) + b \sin(\alpha z - \omega t) \{ 2B_0 + \\ & \sum_{n=1}^N [2A_n I_0(n\alpha r) - 3n\alpha r A_n I_1(n\alpha r) - (n\alpha r)^2 A_n I_0(n\alpha r) \\ & + n\alpha B_n I_0(n\alpha r) - n^2 \alpha^2 r B_n I_1(n\alpha r)] \sin n(\alpha z - \omega t) \} = 0 \\ & \sum_{n=1}^N (n\alpha) [A_n r I_0(n\alpha r) + B_n I_1(n\alpha r)] \cos n(\alpha z - \omega t) - \\ & b \sin(\alpha z - \omega t) \sum_{n=1}^N (n\alpha) \left[-A_n I_0(n\alpha r) + \frac{B_n}{r} I_1(n\alpha r) \right. \\ & \left. + n\alpha r A_n I_1(n\alpha r) - n\alpha B_n I_0(n\alpha r) \right] \cos n(\alpha z - \omega t) + \\ & b\omega \cos(\alpha z - \omega t) = 0 \end{aligned}$$

Because of z arbitrary, the following expressions can be written

$$B_0 = 0$$

$$\sum_{n=1}^N [2A_n I_0(n\alpha r) - n\alpha r A_n I_1(n\alpha r) + n\alpha B_n I_0(n\alpha r)] = 0$$

$$\sum_{n=1}^N [2A_n I_0(n\alpha r) - 3n\alpha r A_n I_1(n\alpha r) - (n\alpha r)^2 A_n I_0(n\alpha r) + n\alpha B_n I_0(n\alpha r) - (n\alpha)^2 r B_n I_1(n\alpha r)] = 0$$

$$\sum_{n=1}^N (n\alpha) [r A_n I_0(n\alpha r) + B_n I_1(n\alpha r)] = 0$$

$$-\alpha A_1 I_0(\alpha r) + \alpha^2 r A_1 I_1(\alpha r) + \frac{\alpha B_1}{r} I_1(\alpha r) - \alpha^2 B_1 I_0(\alpha r) - 2\omega = 0$$

The following expressions for $r = R$ and $N = 2$ can be written

$$2I_0(\alpha R)A_1 - \alpha RI_1(\alpha R)A_1 + \alpha I_0(\alpha R)B_1 + 2I_0(2\alpha R)A_2 - 2\alpha RI_1(2\alpha R)A_2 + 2\alpha I_0(2\alpha R)B_2 = 0 \tag{15}$$

$$2I_0(\alpha R)A_1 - 3\alpha RI_1(\alpha R)A_1 - (\alpha R)^2 I_0(\alpha R)A_1 + \alpha I_0(\alpha R)B_1 - \alpha^2 RI_1(\alpha R)B_1 + 2I_0(2\alpha R)A_2 - 6\alpha RI_1(2\alpha R)A_2 - (2\alpha R)^2 I_0(2\alpha R)A_2 \tag{16}$$

$$+ 2\alpha I_0(2\alpha R)B_2 - (2\alpha)^2 RI_1(2\alpha R)B_2 = 0$$

$$\alpha RI_0(\alpha R)A_1 + \alpha I_1(\alpha R)B_1 + 2\alpha RI_0(2\alpha R)A_2 + 2\alpha I_1(2\alpha R)B_2 = 0 \tag{17}$$

$$-\alpha A_1 I_0(\alpha R) + \alpha^2 R A_1 I_1(\alpha R) + \frac{\alpha B_1}{R} I_1(\alpha R) - \alpha^2 B_1 I_0(\alpha R) - 2\omega = 0 \tag{18}$$

The solutions of the set of equations (15)-(18) are

$$A_1 = \frac{D_1}{D}, \quad B_1 = \frac{D_2}{D}, \quad A_2 = \frac{D_3}{D}, \quad B_2 = \frac{D_4}{D}$$

where

$$\begin{aligned} D = & 8(\alpha^5 R^2 - 3\alpha^3) I_0(\alpha R) I_1(\alpha R) I_1^2(2\alpha R) + \\ & 16\alpha^5 R^2 I_0(\alpha R) I_1(\alpha R) I_0^2(2\alpha R) \\ & - 4(\alpha^6 R^3 - 3\alpha^4 R) I_1^2(\alpha R) I_1^2(2\alpha R) - \\ & (4\alpha^6 R^3 + \alpha^4 R) I_1^2(\alpha R) I_0^2(2\alpha R) + \\ & 8(\alpha^5 R^2 - \alpha^3) I_0^2(\alpha R) I_0(2\alpha R) I_1(2\alpha R) + \\ & 4(\alpha^5 R^2 + 4\alpha^3) I_1^2(\alpha R) I_0(2\alpha R) I_1(2\alpha R) \\ & - 8(\alpha^6 R^3 + 2\alpha^4 R) I_0^2(\alpha R) I_1^2(2\alpha R) - \\ & 4\alpha^6 R^3 I_0^2(\alpha R) I_0^2(2\alpha R) + \\ & 8(\alpha^2 / R - \alpha^4 R) I_0(\alpha R) I_1(\alpha R) I_0(2\alpha R) I_1(2\alpha R) \end{aligned}$$

$$\begin{aligned} D_1 = & 4(\alpha^2 \omega - 2\alpha^4 R^2 \omega) I_1(\alpha R) I_0^2(2\alpha R) - \\ & 16\alpha^3 R \omega I_1(\alpha R) I_0(2\alpha R) I_1(2\alpha R) \\ & - 8\alpha^4 R^2 \omega I_1(\alpha R) I_1^2(2\alpha R) + \\ & 8\alpha^4 R^2 \omega I_0(\alpha R) I_0(2\alpha R) I_1(2\alpha R) \\ & + 16\alpha^3 R \omega I_0(\alpha R) I_1^2(2\alpha R) \end{aligned}$$

$$\begin{aligned} D_2 = & 4(-\alpha^2 R \omega + 2\alpha^4 R^3 \omega) I_0(\alpha R) I_0^2(2\alpha R) \\ & - 24\alpha^4 R^3 \omega I_0(\alpha R) I_1^2(2\alpha R) - 16\alpha^3 R^2 \omega I_1(\alpha R) I_0^2(2\alpha R) + \\ & 16\alpha^2 R \omega I_1(\alpha R) I_0(2\alpha R) I_1(2\alpha R) \end{aligned}$$

$$\begin{aligned} D_3 = & 4\alpha^4 R^2 \omega I_0^2(\alpha R) I_1(2\alpha R) \\ & - 8\alpha^3 R \omega I_0(\alpha R) I_1(\alpha R) I_1(2\alpha R) \\ & + 8\alpha^3 R \omega I_1^2(\alpha R) I_0(2\alpha R) \\ & + 4\alpha^4 R^2 \omega I_1^2(\alpha R) I_1(2\alpha R) \end{aligned}$$

$$\begin{aligned}
 D_4 = & -8\alpha^3 R^2 \omega I_0^2(\alpha R) I_1(2\alpha R) \\
 & -4\alpha^4 R^3 \omega I_0^2(\alpha R) I_0(2\alpha R) \\
 & +16\alpha^3 R^2 \omega I_0(\alpha R) I_1(\alpha R) I_0(2\alpha R) + \\
 & 4(4\alpha^2 R \omega - \alpha^4 R \omega) I_0(\alpha R) I_1(\alpha R) I_1(2\alpha R) \\
 & -4(\alpha^4 R \omega + 2\alpha^2 R \omega) I_1^2(\alpha R) I_0(2\alpha R)
 \end{aligned}$$

The following expressions for v_r and v_z can be written as

$$\begin{aligned}
 v_r = & [\alpha A_1 r I_0(\alpha r) + \alpha B_1 I_1(\alpha r)] \cos(\alpha z - \omega t) \\
 & + [2\alpha A_2 r I_0(2\alpha r) + 2\alpha B_2 I_1(2\alpha r)] \cos 2(\alpha z - \omega t) \\
 v_z = & -[2A_1 r I_0(\alpha r) - \alpha A_1 r^2 I_1(\alpha r) + \alpha B_1 r I_0(\alpha r)] \sin(\alpha z - \omega t) \\
 & - [2A_2 r I_0(2\alpha r) - 2\alpha A_2 r^2 I_1(2\alpha r) + 2\alpha B_2 r I_0(2\alpha r)] \sin 2(\alpha z - \omega t)
 \end{aligned}$$

5. NUMERICAL RESULTS AND DISCUSSIONS

In this section, numerical examples are presented. The test circular cylindrical shell and fluid flow have the following characteristics: $c = 8.5$ m/s, $\omega = 6\pi$ rad/s, $R = 0.005$ m. The results are shown in Figs. (1-6).

Figs. (1-3) illustrate the fluid velocity V_z from united Lagrangian-Eulerian method. Fig. (3) displays the variation of V_z for several sets of values of the parameters ω, c, R, r and z (see also Table 1). Moreover, Fig. (3) shows that fluid velocity V_z increases with an increase in ω, R and r . However, it decreases with an increase in c .

Figs. (4-6) illustrate the fluid velocity V_r . Fig. (6) shows that the fluid velocity V_r and V_z have the same rules (see also Table 1). However, the positive and negative values of fluid velocity V_r are asymmetrical.

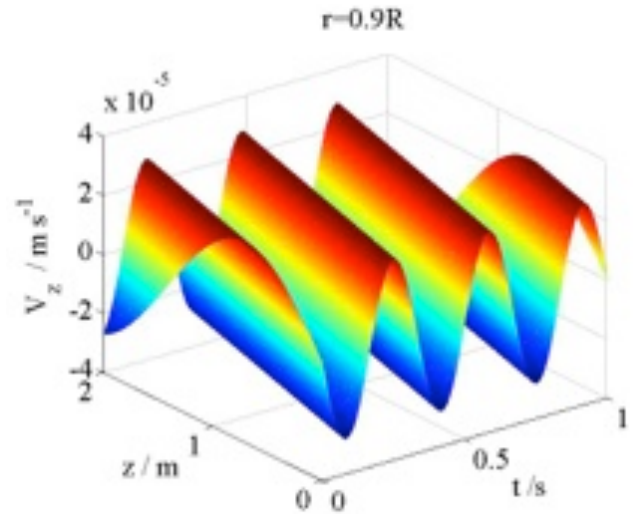


Fig. (2). Fluid velocity V_z at $r = 0.9R$.

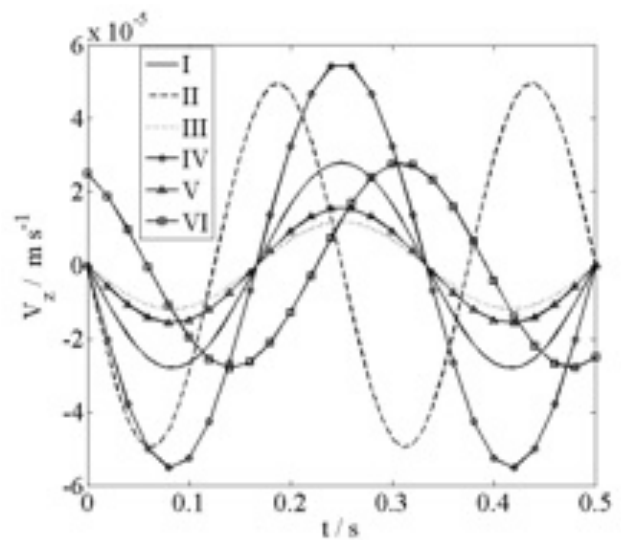


Fig. (3). Fluid velocity V_z .

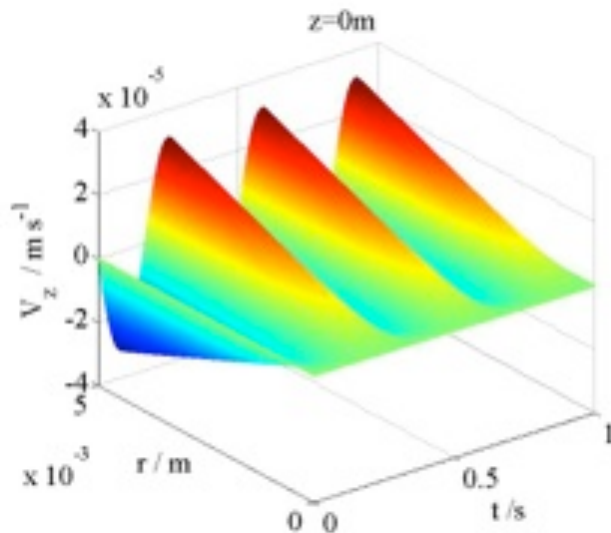


Fig. (1). Fluid velocity V_z at $z = 0$ m.

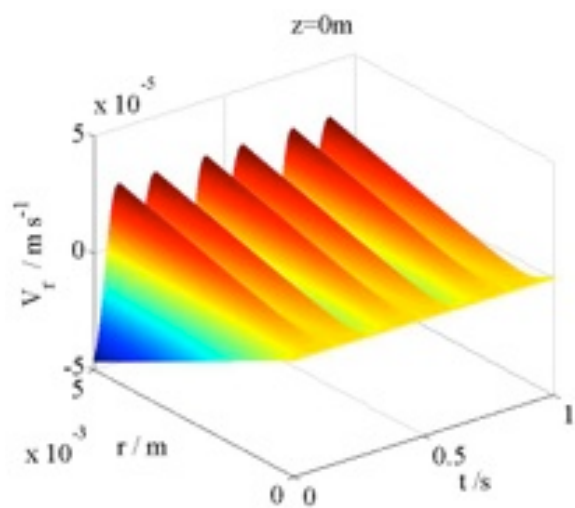


Fig. (4). Fluid velocity V_r at $z = 0$ m.

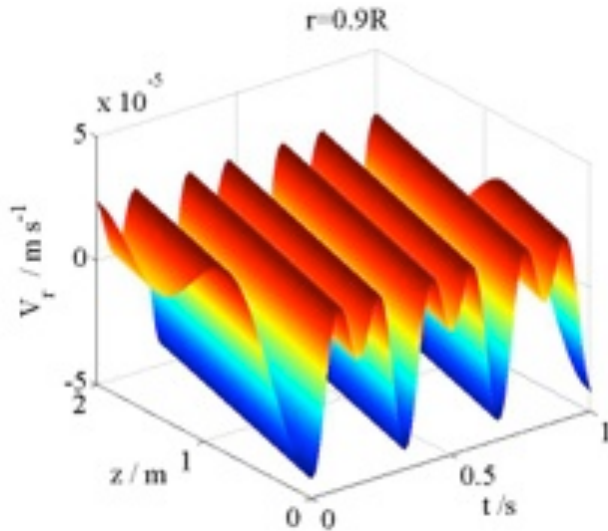


Fig. (5). Fluid Velocity V_r at $r = 0.9R$.

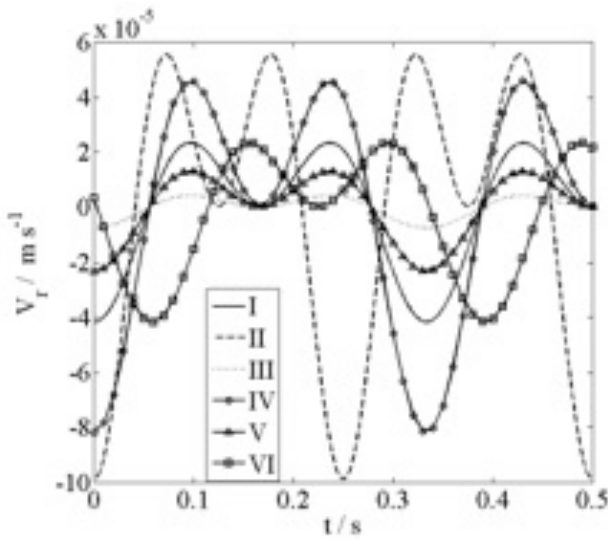


Fig. (6). Fluid Velocity V_r .

It is seen that a stagnation point exists where $V = 0$ and the point $r = 0$ in the circular cylindrical shell. A maximum velocity would exist near the surface of the circular cylindrical shell and the fluid. The fluid velocity is independent of the coefficient of viscosity. This result qualitatively agrees with the conclusions of the reference [9].

Table 1. Sets of parametric values for Figs. (3, 6).

	ω (rad/s)	c (m/s)	R (m)	r (m)	z (m)
I	6π	8.5	0.005	0.0045	0
II	8π	8.5	0.005	0.0045	0
III	6π	20	0.005	0.0045	0
IV	6π	8.5	0.007	0.0045	0
V	6π	8.5	0.005	0.0025	0
VI	6π	8.5	0.005	0.0045	0.5

CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

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REFERENCES

- [1] M.Z. Chen, *Viscous Fluid Dynamics*, Higher Education Press, Beijing, 2001, pp. 111-120.
- [2] Q. Gu, *Method of Mathematical Physics*, Science Press, Beijing, 2012, pp. 395-420.
- [3] M.T. Heil, and J. Pedley, "Large axisymmetric deformation of a cylindrical shell conveying a viscous flow," *Journal of Fluids and Structures*, vol. 3, pp. 237-256, 1995.
- [4] M.A. Iljgajmov, *The Introduction to Nonlinear Hydroelasticity*, Moscow, Nauka, 1991, pp. 15-53.
- [5] Y. Jin, and X. Yuan, "Numerical simulation of fluid-induced vibration in seals by fluid-structure coupling method," *Journal of Engineering Thermophysics*, vol. 3, pp. 395-398, 2003.
- [6] V.N. Kuznetsov, "Interaction of an elastic cylindrical shell with a viscous fluid," *Soviet Applied Mechanics*, vol. 11, pp. 1228-1231, 1978.
- [7] S. Lakovlev, "Submerged fluid-filled cylindrical shell subjected to a shock wave: Fluid-structure interaction effects," *Journal of Fluids and Structures*, vol. 1, pp. 117-142, 2007.
- [8] J.Z. Ma, Y.R. Yang and J.Y. Zhang, "Investigation of dynamical characteristics of a core barrel in still water," *Journal of Vibration Engineering*, vol. 3, pp. 468-473, 2000.
- [9] Y.L. Zhang, J.M. Reese and D.G. Gorman, "Finite element analysis of the vibratory characteristics of cylindrical shells conveying fluid," *Computer Methods in Applied Mechanics and Engineering*, vol. 45, pp. 5207-5231, 2002.