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Adaptive Wavelet Galerkin Solution of Some Elastostatics Problems on Irregularly Spaced Nodes

S.M. Quraishi, R. Gupta and K. Sandeep*

Department of Mechanical Engineering, Institute of Technology, Banaras Hindu University, Varanasi -2210 05, India

Abstract: In this paper the second generation wavelets are applied as a basis in finite element method. The wavelet basis is constructed over typical nonequispaced nodes and on boundaries. In addition the wavelet bases are tailored to the Poisson's operator. The wavelet basis is lifted to enforce operator orthogonality, this eliminates coupling between coarse and detail parts of the stiffness matrix. The scale decoupled stiffness matrix permits optimal $O(N)$ computation. The Lagrangian second order wavelets are chosen for demonstration purposes, and a Poisson equation is solved. The potential application of this method in simulating a heterogenous material is outlined.

Keywords: Finite element method, multiscale modeling, second generation wavelets, non-equispaced grids.

1. INTRODUCTION

The partial differential equations (PDEs) are approximately solved in the finite element method by expressing the solution as a linear combination of basis functions. The projection coefficients on this finite basis could be determined from minimization of functional associated with a weak form of the PDE. Several possible choices exist for a basis function like piecewise polynomials [1], moving least squares (MLS) [2], reproducing kernel functions [3], partition of unity functions [4]. Choosing a suitable basis expansion is crucial and it is mostly determined by the nature of solution.

One of the difficulties with classical FEM is that it provides a single scale solution to a PDE, but it is an established fact that most of the natural phenomena occur over a range of length and time scales, and hence the governing PDEs are multiscale in nature. An interesting multiscale phenomena is elastostatic response of a solid with non homogeneities, multiple phases, or defects. The solution exhibits multi scale features like coarse solution in the whole domain and details near singularities. In the conventional FEM the coarse and detail part of the solution are coupled together, this necessitates recalculation of solution on a finer mesh to get the desired convergence. The costs associated with remeshing the domain and recomputing the solution makes it unattractive to use FEM for problems involving multiscale features and singularities.

To address the multiscale problems, Yserentant [5, 6] introduced the idea of hierarchical basis (HB) in FEM, these basis functions provide multiresolution properties so they can be interpreted as a primitive kind of wavelets. The HB function could be refined adaptively [7], that is we can refine by augmenting the basis with narrowly supported functions at a finer scale (Fig. (**1**)). Adaptive refinement eliminates the need to remesh the whole domain. The major shortcoming of

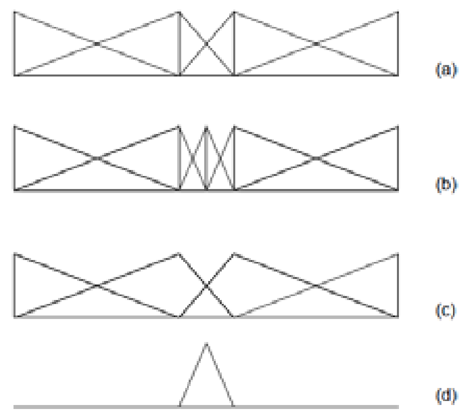

Fig. (1). A comparision of the nodal basis and hierarchical basis. Fig. (**a**, **b**) demonstrate the element refinement in traditional nodal basis while (**c**, **d**) demonstrate adaptive refinement in hierarchical basis.

HB FEM is that in general it can not achieve scale decoupling.

The multiresolution capabilities of wavelets resulted in the development of wavelet Galerkin method [8, 9], in this method the solution is expanded into a basis comprising of scaling function and wavelets, the solution to the PDE is thus obtained hierarchically at scales provided by the scaling functions and wavelets. Compactly supported wavelets perform well in resolving high gradients in the solution, e.g. stress concentrations, crack tip stress field and other material or geometric non-linearities. The problem in this approach is that the wavelets like other spectral methods can provide solution over infinite or periodic systems. These wavelets can not be constructed on bounded domains and on non-equispaced grids.

Second generation wavelets [10-12] are a generalization of wavelets with additional properties and customizability. We can tailor the properties of wavelets by choosing suitable lifting coefficients and stable completions. The lifting scheme is used to enforce vanishing moments, while stable completions provide other useful properties like compact

*Address for correspondence to this author at the Department of Mechanical Engineering, Institute of Technology, Banaras Hindu University, Varanasi - 2210 05, India; E-mails: dr.sandeep.kumar@gmail.com, sandeep@bhu.ac.in

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support. We have demonstrated in our results that the vanishing moment condition enforces operator orthogonality which decouples the stiffness matrix. The scale-decoupled system is easier to solve adaptively, as we do not need to update the coarser parts of the solutions while the details are added to the solution. This decoupling could facilitate development of distributed algorithms for solution of multiscale PDE's.

The second generation wavelets can be constructed over irregularly spaced grids and on bounded domains. The wavelets are constructed in spatial domain using lifting scheme, so they are no longer translates and dilates of each other. However, they inherit the properties of first generation wavelets and can be constructed on general settings like irregular nodes and on boundaries. These additional propeties makes them ideal candidate for solving PDEs in the wavelet Galerkin framework.

The wavelet basis has following advantages over alternative basis sets:

- The basis set can be locally enriched, one can add more wavelets in regions where more information is required.
- Different resolutions can be used in different regions of space, one can add higher resolution wavelets in regions where the solution is rapidly varying. So we have a framework to zoom in to regions where some interesting phenomena is happening at a smaller scale, and zoom out at smoother regions to save computational effort.
- The second generation wavelets generalizes the wavelets to irregular grids and on boundaries.
- The wavelets can be customized in the second generation framework to get scale decoupling in the solution.
- The computational cost scales linearly with respect to the system size.

In this paper, we have generated second order Lagrangian scaling functions and wavelets over irregularly spaced nodes. The node spacings were chosen *apriori*, such that the nodes are closely spaced in regions of high gradient, e.g. material non-homogeneity. In the homogenization [13, 14] based on first generation wavelets non uniform node distribution were not permissible. In this research we demonstrate that choosing irregular nodes can result in faster convergence to solution. In our adaptive scheme we choose higher density of nodes in regions where sharp change or gradient is expected, the oracle determines which nodes are to be kept and which to be killed. After increasing resolution to a higher level the solution starts to converge. The adaptive scheme is demonstrated on non-homogeneous two phase materials, our solution adapts well to these irregularities. The large wavelet coefficients points to the critical regions of the domain where more wavelets should be added. By increasing the resolution we have found solution to desired accuracy.

2. BASIS REFINEMENT USING SECOND GENERATION WAVELETS

In finite element the usual approach for adaptive calculation is to refine the mesh using a single level basis functions, but this approach leads to poorly conditioned stiffness matrix. The better approach is to use a Hierarchical or a multiscale basis consisting of wavelets, the resulting multiscale stiffness matrices are well conditioned.

Fig. (**1a**) shows a nodal basis used in FEM. Approximation space V_j, is spanned by single level basis of linear Lagrangian scaling functions at level j. Fig. (**1c**) depicts a hierarchical basis, the essential feature of this representation is that spaces V_{j+1} can be partitioned as: $V_{j+1} = V_j \oplus W_j$, where W_j is the detail-space, which is spanned by wavelets at level j. Scaling functions can be further decomposed into coarser functions and details until the coarsest level is reached, this results in a full multiresolution decomposition of the space V_j:

$$V_j = W_{j-1} \oplus V_{j-1} = W_{j-1} \oplus W_{j-2} \oplus ...W_o \oplus V_o. \tag{1}$$

In this representation, coarse basis functions covers the full domain while details could be added selectively to enrich specific nodes.

Sweldens [10] demonstrated that one can build scaling functions and wavelets in a general setting, such as on boundaries, irregular samples etc. These second generation wavelets inherit powerful properties of first generation wavelets like fast transforms, localization and good approximation.

The approximation spaces V_j are spanned by scaling functions. Any function u can be approximately represented at level j as a linear combination of scaling functions at scale j as

$$u^j(x) = \sum_k u_k^j \phi_k^j. \tag{2}$$

where u_k^j are expansion coefficients.

The scaling functions satisfy refinement relation

$$\phi_k^j = \sum_l h_{k,l}^j \phi_l^{j+1}. \tag{3}$$

where, $h_{k,l}^j$ is a low pass filter associated with the scaling function. The summation is taken over l-nodes in the immediate neighborhood of the node k. For irregular sampling the filter coefficients are different for each scaling function. Since the scaling functions are interpolating the above equation could be simplified as

$$\phi_k^j = \phi_k^{j+1} + \sum_m h_{k,m}^j \phi_m^{j+1}. \tag{4}$$

The complimentary detail spaces W_j are spanned by wavelets. The wavelets are constructed using lifting scheme, in this scheme a wavelet is built by adding "lifted" neighboring scaling functions to a more primitive wavelet, which is chosen to be a simple scaling functions from a finer level.

$$\psi_m^j = \phi_m^{j+1} + \sum_k s_{k,m}^j \phi_k^j. \tag{5}$$

The lifting coefficients are obtained by the condition of vanishing moments over wavelets. The HB functions are a special case of this more general second generation framework, HB results when all the lifting coefficients are taken as zero in above equation.

The detail function at level j can be expressed as

$$d^j(x) = \sum_k d_k^j \psi_k^j. \tag{6}$$

The detail function represents the details lost while going from a finer resolution V_{j+1} to coarser resolution V_j,

$$u^{j+1}(x) = u^j(x) + d^j(x) \tag{7}$$

The projection of a function $u(x)$ on space V_{j+1}, can be written in multiresolution format as

$$u^{j+1}(x) = u^0(x) + \sum_{i=0}^{j} d^i(x). \tag{8}$$

where $u^0(x)$ is the projection of function on coarsest space V_0, while $d^i(x)$ are its projection in space W_j. The projection can be further expressed in terms of basis and projection coefficients, the resulting MRA of function $u(x)$ on space V_{j+1} becomes:

$$u^{j+1}(x) = \sum_l u_l^0 \phi_l^0(x) + \sum_{i=0}^{j}\sum_m d_m^i \psi_m^i(x). \tag{9}$$

where u_l^j and d_m^j are projection coefficients of $u(x)$ in the spaces V_j and W_j. The coefficients u_l^j are termed as scaling functions while d_m^j are known as detail coefficients. The detail coefficients provides a guideline for adaptive refinement. We can keep adding details until the approximation is within error bounds. An error estimate ε can be defined as

$$\varepsilon = max\,|d^j| = max\,|u^{j+1} - u^j| \tag{10}$$

In this paper we have chosen second order Lagrangian interpolation functions so the HB can be lifted once to enforces one vanishing moment. Due to inheritance of vanishing moments [15, 16], the wavelets with one vanishing moment is operator orthogonal with respect to the Poisson's operator. So this choice of wavelet scale decouples the stiffness matrix arising from the variational form of the Poisson's PDE.

3. WAVELET GALERKIN METHOD

A bar with axial distributed force $f(x)$ is shown in (Fig. (**3**)), the governing Euler's equation is

$$\frac{d}{dx}\left(E(x)A(x)\frac{du}{dx}\right) + f(x) = 0 \qquad 0 < x < L \tag{11}$$

with boundary conditions $u(0) = 0$ $u(L) = 0$, where $u(x)$ is the axial displacement, $E(x)$ is the modulus of Elasticity and $A(x)$ is area of cross section.

The weak or variational form of above PDE is

$$\int \frac{du}{dx} E(x)A(x)\frac{dv}{dx}\,dx = \int f(x)\,dx \tag{12}$$

where v is the variation in the displacement field. We approximate the displacement field at level j+1 and substitute equation 9 in above equation to obtain MRA of stiffness matrix:

$$\begin{bmatrix} K_{\phi\phi}^{0,0} & K_{\phi\psi}^{0,0} & K_{\phi\psi}^{0,1} & \cdots & K_{\phi\psi}^{0,j} \\ K_{\psi\phi}^{0,0} & K_{\psi\psi}^{0,0} & K_{\psi\psi}^{0,1} & \cdots & K_{\psi\psi}^{0,j} \\ K_{\psi\phi}^{1,0} & K_{\psi\psi}^{1,0} & K_{\psi\psi}^{1,1} & \cdots & K_{\psi\psi}^{1,j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{\psi\phi}^{j,0} & K_{\psi\psi}^{j,0} & K_{\psi\psi}^{j,1} & \cdots & K_{\psi\psi}^{j,j} \end{bmatrix} \begin{bmatrix} u^0 \\ d^0 \\ d^1 \\ \vdots \\ d^j \end{bmatrix} = \begin{bmatrix} f_\phi^0 \\ f_\psi^0 \\ f_\psi^1 \\ \vdots \\ f_\psi^j \end{bmatrix} \tag{13}$$

It can be written as:

$$K^e u^e = f^e \tag{14}$$

where f^e is the nodal force vector of element.

$$f^e = \begin{bmatrix} f_\varphi^0 \\ f_\psi^0 \\ f_\psi^1 \\ \vdots \\ f_\psi^j \end{bmatrix} = l\int_0^1 \begin{bmatrix} \phi^0 \\ \psi^0 \\ \psi^1 \\ \vdots \\ \psi^j \end{bmatrix} f(\xi)d\xi \tag{15}$$

where, l is the length of element, and ξ is the element local coordinate.

K^e is the multiresolution stiffness matrix of the element. The entries of this matrix can be formulated from the variational form of the PDE (equation 14):

$$K_{\phi\phi}^{0,0}(r,c) = \frac{1}{l}\int_0^1 \frac{d\phi_r^0}{d\xi} E(\xi)A(\xi)\frac{d\phi_c^0}{d\xi}d\xi \tag{16}$$

$$K_{\phi\psi}^{0,k}(r,c) = \frac{1}{l}\int_0^1 \frac{d\phi_r^0}{d\xi} E(\xi)A(\xi)\frac{d\psi_c^k}{d\xi}d\xi \tag{17}$$

$$K_{\psi\phi}^{k,0}(r,c) = K_{\phi\psi}^{0,k}(r,c) \tag{18}$$

$$K_{\psi\psi}^{m,k}(r,c) = \frac{1}{l}\int_0^1 \frac{d\psi_r^m}{d\xi} E(\xi)A(\xi)\frac{d\psi_c^k}{d\xi}d\xi \tag{19}$$

where, (r, c) represents the row and column of the entry, and m, k are the level of resolution.

The wavelets are customized to ensure operator orthogonality w.r.t the Poisson's operator. The second order Lagrangian basis is used, the wavelets arising from the hierarchical basis functions are lifted with one interior scaling function. The lifting coefficient is chosen to ensure one vanishing moment i.e. $\int\psi d\xi = 0$ so this wavelet can kill zeroth order polynomials, the inheritance of vanishing moments provides two vanishing moments to $d\psi/d\xi$ hence it kills first order polynomials $\{1, \xi\}$. This property makes $d\psi/d\xi$ orthogonal to $d\phi/d\xi$, and to the first derivatives of wavelets at other scales. This choice of lifted wavelets ensures that the off diagonal terms like $K_{\psi\phi}^{k,0}(r,c)$, $K_{\phi\psi}^{0,k}(r,c)$, and $K_{\psi\psi}^{m,k}(r,c)$ in the stiffness matrix are zero. The

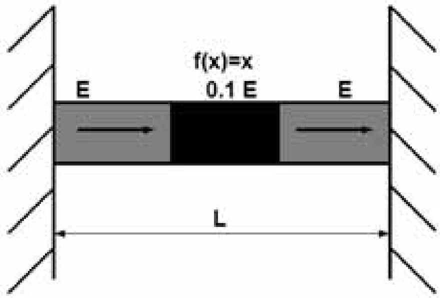

Fig. (2). A beam with a weakened midsection, with linearly varying body force.

customized wavelets scale decouples the multiresolution stiffness matrix and provides a way for adaptive solution, where the details are added incrementally to the solution without remeshing the whole domain.

$$\begin{bmatrix} K_{\phi\phi}^{0,0} & K_{\psi\psi}^{0,0} & K_{\psi\psi}^{1,1} & \cdots & K_{\psi\psi}^{0,j} \end{bmatrix} \begin{bmatrix} u^0 \\ d^0 \\ d^1 \\ \vdots \\ d^j \end{bmatrix} = \begin{bmatrix} f_\phi^0 \\ f_\psi^0 \\ f_\psi^1 \\ \vdots \\ f_\psi^j \end{bmatrix} \tag{20}$$

The assembly cost of the scale decoupled multiresolution matrix is O(N) as we only need to compute the diagonal terms in the stiffness matrix.

4. NUMERICAL EXAMPLES

A beam with a softer midsection (Fig. (**2**)) is chosen to demonstrate the adaptivity and the computational efficiency resulting from non uniformly distributed nodes. The beam has unit cross sectional area and a uniaxial body force $f(x) = x$ is applied to the beam, both ends of the beam are fixed. The governing equation for this problem is:

$$\frac{d^2 E(x)u}{dx^2} + x = 0 \qquad 0 < x < 1 \tag{21}$$

along with boundary conditions:

$$u(0) = 0$$
$$u(1) = 0$$

In the adaptive scheme [17] we use thresholding parameters $\varepsilon_j^u > \varepsilon_j^l > 0$; where superscripts refer to upper threshold and lower threshold at level j. Assuming that the analysis at the resolution level j is finished, to proceed to next finer resolution j+1 following adaptive procedure is used:

(i) Exclude the wavelet ψ_k^j from the basis set if its coefficient $|d_k^j| < \varepsilon_j^l$

(ii) Preserve the wavelet ψ_k^j in the basis set if its coefficient $\varepsilon_j^l < |d_k^j| > \varepsilon_j^u$

(iii) Add child wavelets of ψ_k^j into the basis set in the regions of large detail coefficients $\varepsilon_j^l < |d_k^j| > \varepsilon_j^u$

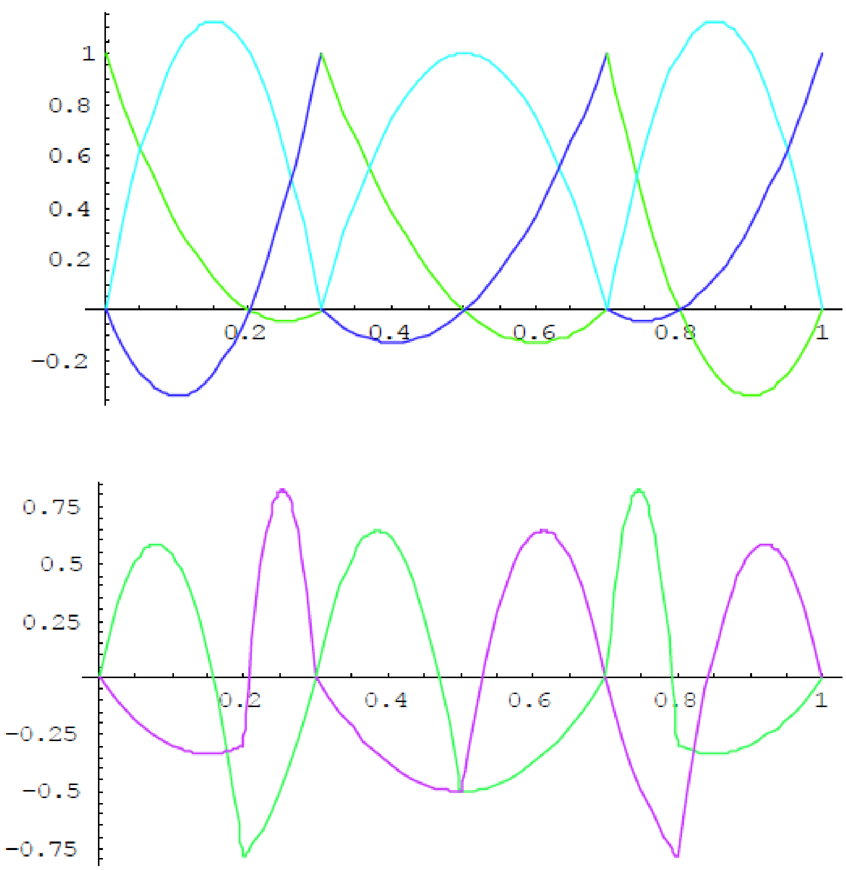

Fig. (3). The scaling functions and wavelets arising from irregular node placement.

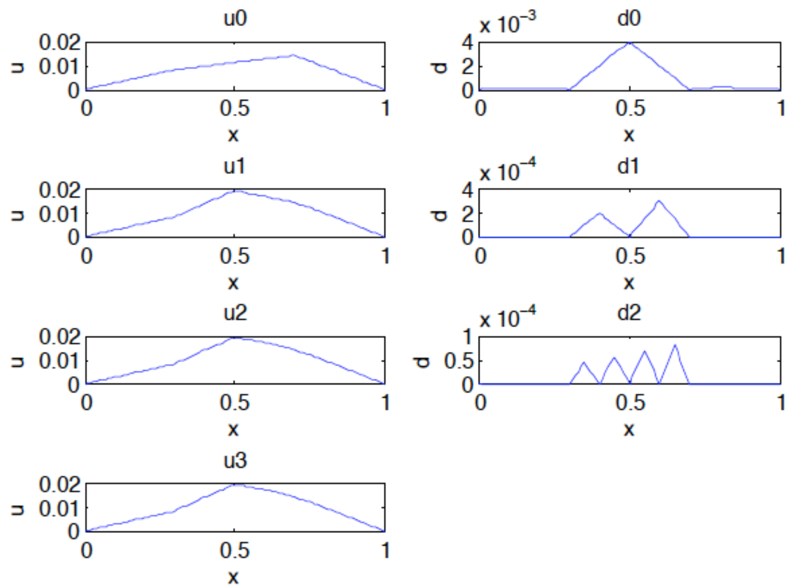

Fig. (4). The approximations and detail solution of Eulers beam equation obtained using linear hierarchical basis (HB).

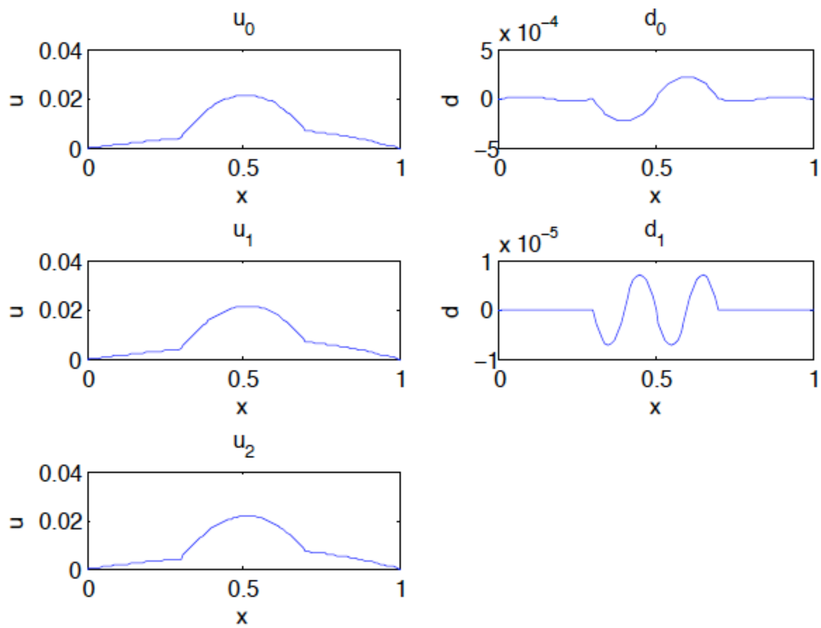

Fig. (5). The approximations and detail solution of Eulers beam equation obtained using quadratic wavelet basis.

In our simulation child wavelets are successively added near the midsection to capture the results. The irregular node placement allows the freedom to add child wavelets near the midsection in arbitrarily. This results in faster convergence to solution.

The nodes are irregularly spaced to get more resolution around the midsection. The scaling functions and wavelets arising from irregular node distribution is shown in Fig. (**3**). These wavelets are enriched according to an adaptive strategy and the resolution is increased for convergence. The results obtained by using linear Lagrangian basis functions using Hierarchical basis (HB) are shown in Fig. (**4**) while those obtained by using quadratic Lagrangian wavelet basis are demonstrated in Fig. (**5**). Although both linear HB and quadratic wavelet basis provides solution in multiresolution format, it is evident that the quadratic wavelet basis outperform linear HB, and error goes down to 10^{-5} in first level of resolution, while the error remains 10^{-4} even in second and third level of resolution of linear HB. The single scale FEM provides solution at first resolution, and to progress to higher resolutions remeshing of the whole domain has to be done.

The quadratic wavelet basis has significant advantages over single scale FEM or multiscale HB. The cost of assembling the multiresolution and its solution is significantly reduced, assembly cost scales linearly with system size. The solution can be enriched in desired regions by adding details, so the costly remeshing and recomputing solution over whole domain is avoided. The proposed method is more efficient and has significant advantages over other methods available for solving heterogenous systems, and in simulating multiscale materials.

5. CONCLUSIONS

The construction of second generation wavelets is demonstrated, the lifting framework provides flexibility to design the wavelets such that solution cost scales linearly with the system size. This is achieved by tailoring wavelets with vanishing moments, so that complete operator orthogonality is achieved in the wavelet Galerkin framework. The resulting system of equation can be easily solved in hierarchical, incremental and adaptive manner. The detail solution can be added to localized regions of sharp transition without the need for remeshing and solving over whole domain. Our results demonstrate the potential application in studying heterogeneous materials with multiscale features like defects, multiple phases, or other discontinuities or singularities. To achieve efficient results in studying heterogeneous media, it is important to use irregularly spaced nodes. We have choosen initial node distribution with higher concentration around regions of sharp gradients, this results in faster convergence to solution by incorporating fewer detail levels. The second generation based wavelet Galerkin procedure results in an efficient, hierarchical and adaptive algorithm which scales linearly with systems size. Its potential application in studying heterogeneous system is demonstrated.

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