Comparing Fundamentals of Additive and Multiplicative Aggregation in Ratio Scale Multi-Criteria Decision Making

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Abstract: Additive and multiplicative aggregations of ratio scale preferences are frequently used in multi-criteria decision making models. In this paper, we compare the advantages and limitations of these two aggregation rules by exploring only their fundamental properties after ratio scaled local priorities and criteria weights have been successfully generated from the decision maker. The comparisons of these properties are therefore independent of ancillary procedures such as interactive elicitations from decision makers, pairwise comparisons and calculations of local priorities and criteria weights. We compare six fundamental properties of the two aggregation rules. The criteria weights used in the multiplicative aggregation have complicated meanings which are not well understood and often mixed up in the ambiguous notion of "criteria importance". As the scaling factors of the local preference values do not appear explicitly in the computations of the relative ratios of the overall preferences in the multiplicative aggregation model, the relative ratios remain unchanged when the scaling factors are changed or an alternative is added or deleted. Furthermore, the relative ratios in the multiplicative aggregation do not depend on similar local preference values which cancel each other out mathematically. It is quite evident that the additive aggregation model is superior and easier for decision makers to use and understand. We recommend the additive aggregation rule over the multiplicative aggregation rule.

Keywords: Preferences, ratio scale, unit of measure, criteria weights, additive aggregation, multiplicative aggregation.

1. INTRODUCTION

The basic MCDM problem in most decision making is to evaluate competing alternatives under multiple conflicting criteria. Even though rank ordering (an ordinal scale) of the alternatives is the most common form of solution sought by most decision makers (DM), it is always desirable to know the relative standings of the alternatives measured on a scale containing more information. Valuation on a ratio scale is preferred because it provides the DM with a relative measure of alternatives on each criterion as well as the overall ratio preference across all criteria. A ratio scale measure of overall preference value is also useful for the allocation of resources among all the alternatives. With a ratio scale, it is meaningful to reach conclusions such as "alternative A_j is r times preferred to A_k relative to all criteria". Additive aggregation and multiplicative aggregation of the local ratio preferences of each alternative into an overall preference are frequently used in multi-criteria decision making models [1-3].

In this paper, we compare the advantages and limitations of additive aggregation and multiplicative aggregation rules by exploring their fundamental properties. We do this without investigating the different approaches to ancillary and peripheral procedures such as interactive elicitations from decision makers, pairwise comparisons, and calculations of local priorities and criteria weights. By assuming that local priorities and criteria weights have been correctly derived, we are able to focus on the aggregation procedures. From the comparisons of the fundamental properties presented, it is quite evident that the additive aggregation rule is superior with simpler interpretations which are more readily understood by decision makers.

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Fundamental basic elements of the MCDM framework are first depicted without any specific interpretations imposed on these elements. It is assumed that no relevant criterion is missed and each criterion is autonomous. In section 3, the measures of criteria weight, local and overall preferences are assumed to be in ratio scale. Some necessary conditions and the role of normalization are discussed. We then give a brief literature review, with particular attention to the different ancillary procedures and contradicting opinions in model interpretations. Additive and multiplicative aggregation rules are formally introduced in Section 5. In Section 6, we elaborate and compare the fundamental properties of these aggregation rules. Finally, we summarize and give some conclusions.

2. BASIC ELEMENTS OF MCDM MODEL

The basic elements of a typical MCDM model include a set $\mathbf{A}=\{A_1,A_2,\ldots,A_n\}$ of n alternatives $A_1,A_2,\ldots,A_n$ and a set $\mathbf{C}=\{C_1,C_2,\ldots,C_m\}$ of m criteria $C_1,C_2,\ldots,C_m$. The effect of the criteria $C_1,C_2,\ldots,C_m$ in $\mathbf{C}$ is represented by positive numbers $w_1,w_2,\ldots,w_m$ respectively. The vector $\mathbf{w}=[w_1,w_2,\ldots,w_m]$ is called the criteria weight vector of the criteria $C_1,C_2,\ldots,C_m$ in $\mathbf{C}$. The criteria weight vector $\mathbf{w}$ is derived from questioning the DM. The alternatives $A_1,A_2,\ldots,A_n$ can be evaluated under each individual criterion C_p, p=1,2,…,m. For each criterion C_p (p=1,2,…,m), the local preference of the alternatives $A_1,A_2,\ldots,A_n$ in $\mathbf{A}$ with respect to C_p is represented by positive numbers $x_{1p},x_{2p},\ldots,x_{np}$, respectively. The vector $\mathbf{x}_p=[x_{1p},x_{2p},\ldots,x_{np}]$ is called the local preference vector of the alternatives $A_1,A_2,\ldots,A_n$ in $\mathbf{A}$ with respect to C_p. The local preference vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m$ are derived from questioning the DM.

It is important to note that at this rudimentary level, we only assume that the numerical values in the criteria weight vector $\mathbf{w}$ and the local preference vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m$ exist and are used in MCDM model without stipulating any specific properties, interpretations and restrictions. The properties of the scale of these numerical values and the methods of evaluating them are regarded as additional model assumptions which could vary widely among different models [4, 5]. These numerical values can represent mea-sures in ordinal scale [6], interval scale [7] or ratio scale [8, 9]. We note that the outranking relationship [10] is not a complete order and thus may not be represented by such numerals directly.

The ultimate goal of all MCDM problems is to aggregate the evaluation of the n alternatives with respect to the m criteria into overall preference measures $v_1,v_2,\ldots,v_n$ of the alternatives $A_1,A_2,\ldots,A_n$ respectively. The vector $\mathbf{v}=[v_1,v_2,\ldots,v_n]$ is called the overall preference vector of the alternatives $A_1,A_2,\ldots,A_n$ in **A** with respect to all the criteria in **C**. The best alternative is the one with the largest overall preference value. The fundamental assumption in MCDM models is that the overall preference vector $\mathbf{v}=[v_1,v_2,\ldots,v_n]$ is a function of the criteria weight vector $\mathbf{w}=[w_1,w_2,\ldots,w_m]$ and the local preference vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m$.

Consider five houses A_1,A_2,A_3,A_4,A_5 to be evaluated with respect to three criteria C_1,C_2,C_3. The basic elements are shown in Table **1**. For each criterion C_p, we first compute the vector of local preference $\mathbf{x}_p=[x_{1p},x_{2p},x_{3p},x_{4p},x_{5p}]$ (p=1,2,3). Then we determine the criteria weights w_1,w_2,w_3. Finally the overall preference vector $\mathbf{v}=[v_1,v_2,v_3,v_4,v_5]$ is derived by combining the local priority vectors $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3$ with the criteria weights w_1,w_2,w_3 according to some aggregation rule.

Table 1. Basic Elements of MCDM: Alternatives A_1,A_2,A_3,A_4,A_5 and Criteria C_1,C_2,C_3

Alternatives	C_1 Price	C_2 Location	C_3 Condition	Overall
House A_1	x_{11}	x_{12}	x_{13}	v_1
House A_2	x_{21}	x_{22}	x_{23}	v_2
House A_3	x_{31}	x_{32}	x_{33}	v_3
House A_4	x_{41}	x_{42}	x_{43}	v_4
House A_5	x_{51}	x_{52}	x_{53}	v_5
Criteria Weight	w_1	w_2	w_3	

To avoid solving the wrong problem, we assume that all relevant criteria are included in **C** so that no new criterion will be added. To avoid redundancy, we assume that no criterion can be deleted and the criteria in C are disjoint with each other in the sense that no criterion is double counted by another criterion. Each criterion C_p (p=1,2,…,m) is assumed to be "autonomous" in the sense that the performance of each alternative A_j (j=1,2,…,n) with respect to C_p is independent of the other alternatives. In particular, none of the criteria in **C** is a measure of scarcity, abundance or some ranking of the alternatives in **A**. It follows that the evaluation of A_j (j=1,2,…,n) with respect to C_p remains the same regardless of whether any other alternative is deleted or added to **A**.

In MCDM models, the numerical values in the local preference vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m$ and the criteria vector **w** are generated so that they can be aggregated to overall preferences. Some necessary measurement scale underlying the numerical values must be established and maintained so that the aggregated results are meaningful measures. For example, the sum of ordinal numerical values such as rankings is not a valid measure of magnitudes. Nor is it meaningful to add or multiply numerical values in commensurate interval scale such as 25°C+34°C for temperature in Celsius. Thus, $(a+bx_1)+(a+bx_2) = 2a+b(x_1+x_2)$ and $(a+bx_1)(a+bx_2) = a^2+ab(x_1+x_2)+b^2x_1x_2$ are not meaningful as interval scales because they do not have the form a+bx. However, the arithmetic average or weighted average of numerical values in commensurate interval scale is meaningful. For example, $[(a+bx_1)+(a+bx_2)]/2 = a+b(x_1+x_2)/2 = a+b(x_3)$ which is of the form a+bx. Similarly, a weighted average keeps the form a+bx if the sum of the weights equals 1 – e.g. $0.3(a+bx_1)+0.7(a+bx_2) = a+b(0.3x_1+0.7x_2) = a+b(x_3)$.

For ratio scales, the situation is different – both addition and multiplication are permissible. Thus, $bx_1+bx_2 = b(x_1+x_2) = b(x_3)$ belongs to the same ratio scale if they are commensurate. Note that ax_1 and bx_2 cannot be added because they are in different scales. For multiplication of values on the same ratio scale, $(bx_1)(bx_2) = b^2x_1x_2 = c(x_3)$ which is a new ratio scale with a squared unit of measure. As well, multiplication of values from different ratio scales produces a new ratio scale: e. g. $(ax_1)(bx_2) = abx_1x_2 = c(x_3)$. Raising different ratio values to powers also produces a new ratio scale – e. g. $(ax_1)^w(bx_2)^{(1-w)} = [a^wb^{(1-w)}][x_1{}^wx_2{}^{(1-w)}] = c(x_3)$. Unlike addition, multiplication of ratio scales produces a new ratio scale with a different unit of measure.

3. RATIO SCALE MEASURES AND SCALING FACTORS

To be more focused in our investigation, we emphasize on ratio scales that allow both addition and multiplication for aggregation. We assume that the overall preferences $v_1,v_2,\ldots,v_n$ exist in ratio scale such that A_j is v_j/v_k times preferred to A_k, $1\le j,k\le n$. The overall preference vector $\mathbf{v}=[v_1,v_2,\ldots,v_n]$ is unique up to a positive scaling constant and we write $\mathbf{v}=\theta[u_1,u_2,\ldots,u_n]$ where θ can be any positive number. The relative ratios of $v_1,v_2,\ldots,v_n$ constitute the overall results sought in the model without any specific unit of measure. For convenience, we use the normalization constraint $v_1+v_2+\ldots+v_n=1$ to get rid of θ and specify the overall preference values.

For each criterion C_p (p=1,2,…,m), we assume that the local preferences $x_{1p},x_{2p},\ldots,x_{np}$ exist in ratio scale such that A_j is x_{jp}/x_{kp} times preferred to A_k under C_p, $1\le j,k\le n$. Since the relative ratios x_{jp}/x_{kp} $(1\le j,k\le n)$ are uniquely determined by the values in the local preference vector $\mathbf{x}_p=[x_{1p},x_{2p},\ldots,x_{np}]$ which can be arbitrarily scaled by any positive constant, we write $\mathbf{x}_p=\beta_p[y_{1p},y_{2p},\ldots,y_{np}]$ where β_p is a positive number. We note that, even when there exists original natural measurement for the performance of $A_1,A_2,\ldots,A_n$ under C_p, the local preference values $x_{1p},x_{2p},\ldots,x_{np}$ need not have linear functional relationship with these natural measurements. However, if the local preference values are expressed in well established unit of measure u (e.g. u=$1,000 or u=kg), then we write $\mathbf{x}_p=\beta_p u[y_{1p},y_{2p},\ldots,y_{np}]$. As $\beta_1,\beta_2,\ldots,\beta_m$ are arbitrary positive numbers, the values of $\beta_p y_{jp}$ and $\beta_q y_{jq}$ are arbitrary for criteria C_p and C_q. In particular, comparing the magnitudes of $\beta_p y_{jp}$ and $\beta_q y_{jq}$ under different criteria C_p and C_q is not sufficient because they do not have a common unit of measurement. Also, for each j (j=1,2,…,n), the alternative A_j is not fully described by $[\beta_1 y_{j1},\beta_2 y_{j2},\ldots,\beta_m y_{jm}]$ without some knowledge of the units of measure.

We assume that criteria weights $w_1,w_2,\ldots,w_m$ exist in ratio scale such that the weight of C_p is w_p/w_q times the weight of C_q, $1\le p,q\le m$. The criteria weight vector $\mathbf{w}=[w_1,w_2,\ldots,w_m]$ is unique up to a positive scaling constant and we write $\mathbf{w}=\alpha[w_1',w_2',\ldots,w_m']$ where α can be any positive number. For convenience, we assume that α has been chosen such that $w_1+w_2+\ldots+w_m=1$.

The scaling factors $\beta_1,\beta_2,\ldots,\beta_m$ and their relationship with the criteria weights have caused much ambiguity and confusion in Analytic Hierarchical Process (AHP) type models [4, 5, 11, 12]. The local preference values in $\mathbf{x}_p$ are represented as $\beta_p u[y_{1p},y_{2p},\ldots,y_{np}]$ but the role of unit is obscure because the unit of measure u used in measuring the local preference values is usually not explicitly specified and β_p is an arbitrary positive number. In most AHP type models, $\beta_1,\beta_2,\ldots,\beta_m$ are not explicitly shown as arbitrary positive numbers. This obscurity has led to some misunderstanding [12]. A common technique used to determine $\beta_1,\beta_2,\ldots,\beta_m$ and hence fixing the numerical values of $\mathbf{x}_p$ is to impose some artificial normalization constraint such as $x_{1p}+x_{2p}+\ldots+x_{np}=1$. Even though $x_{1p}+x_{2p}+\ldots+x_{np}=1$ is most frequently used, it is not the only one that can be used to specify the scaling constant β_p. Other normalization constraints such as $x_{jp}=1$ for some j may be used as in the linking Pin AHP [9]. However, the DM must be aware of the role of the normalization constraint in the questioning procedure of the model. In this paper, we shall show the scaling factors $\beta_1,\beta_2,\ldots,\beta_m$ explicitly in the model development until the units for measuring the local preferences are adequately specified and used in model calculations.

We now give closer scrutiny to the role of scaling constant β_p by considering the prices in Canadian dollars C\$200, C\$300 and C\$400 of three objects. The prices are represented by $\beta_p u[2,3,4]$ where u="C\$1" and $\beta_p=100$. Prices in different currencies are represented by different β_p and u which depend on the exchange rate that is applied. For pairwise evaluations of the relative ratios of the prices of these three objects, no particular currency u has to be specified and it is sufficient to use $\beta_p[2,3,4]$ without specifying β_p and u is ignored. However, it is impossible to evaluate the impact of the prices of these objects solely from the knowledge of $\beta_p[2,3,4]$ which captures only the relative ratios of the prices. The knowledge of one of the prices in a known currency is absolutely necessary to recover all the actual prices. It is important to realize the obvious fact that "the relative prices in $\beta_p[2,3,4]$" does not contain the crucial information $\beta_p u$=C\$100 or "the price of the first object is C\$200". $\beta_p[2,3,4]$ can also represents the relative prices of [C\$2,C\$3,C\$4] and thus it can not be used as a complete representation of [C\$200,C\$300, C\$400]. Put it in another way, $\beta_p[2,3,4]$ are pure numbers – they do not have complete information in terms of some well established unit of measure. For the derivation of criteria weights, the actual prices are needed to determine the impact of the price attributes on the overall preferences. When the actual prices are unknown, the importance of the actual prices in deriving the criteria weights can be emphasized by $\beta_p u[2,3,4]$ or u[1,1.5,2] where $\beta_p=1/2$ and u="price of object 1".

Unlike the currencies that are tangible and transparent to the DM, it is more challenging when dealing with local preferences which are intangible measures with no simple choice of β_p. It is vital to note that $\beta_p[y_{1p},y_{2p},\ldots,y_{np}]$ contain the relative ratios only and do not contain the original performances of $A_1,A_2,\ldots,A_n$ under C_p which are necessary for deriving the overall preferences of $A_1,A_2,\ldots,A_n$. This implies that the pairwise matrices in AHP type models do not contain the crucial information of the original performances necessary for evaluating the alternatives with respect to all criteria. To rectify this, the criteria weights $w_1,w_2,\ldots,w_m$ must capture and transform the original performances of $A_1,A_2,\ldots,A_n$ under C_p, p=1,2,…,m, into overall preferences. This requirement must be clearly stated in the questioning procedures for eliciting the criteria weights from the DM.

In order to focus only on the more fundamental properties of the aggregation rules, we have assumed that all the relevant criteria are already identified and properly formulated. It follows from the autonomy of the criteria that A_j is always x_{jp}/x_{kp} time preferred to A_k ($1\le j,k\le n$) with respect to C_p regardless of whether any other alternative is deleted or added to **A**. In this paper, we compare the advantages and limitations of additive and multiplicative aggregation of $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m$ with **w** based only on their fundamental mathematical properties. We do not consider the different impacts of numerous distinct methods proposed for the auxiliary problems of deriving the local priorities $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m$ and the criteria weights in **w**.

4. LITERATURE REVIEW

Local preferences under individual criteria are the basic elements used in most MCDM models. The criteria weights are usually modeled in ratio scale which allows an arbitrary positive scaling constant [11]. Local preferences are assumed to be measured in ordinal scale in ranking models [6], in interval scale in multiple attribute utility models [7] and in ratio scale in AHP type models [8, 9]. There are many methods proposed for deriving local preferences from pairwise comparison matrices [4, 5, 13-16]. The local preference vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m$ and the criteria weight vector **w** are then aggregated into overall preference vector $\mathbf{v}=[v_1,v_2,\ldots,v_n]$ by the additive aggregation [9, 17] or the multiplicative aggregation [2]. Goal programming and data development analysis are also used in some MCDM models [14, 16, 18, 19]. Most of these methods have been extensively analyzed and compared [1, 4, 5, 15, 20].

There are conflicting viewpoints and disagreements in many aspects of these MCDM models. The criteria weight w_p can have many different interpretations. It is problematic when the generic term "criteria weight of importance" is used in questioning the DM without explaining its specific intended meaning. The criteria weights in **w** elicited from the DM may not be consistent with the associated mathematical operations used in the model [11, 12]. The local preferences $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m$ are subjective evaluations from the DM with reference to the performance of the alternatives under individual criteria. Even when there are convenient original performance measures of the alternatives such as the prices in dollars, the subjective local preferences need not be linearly related. To simplify these subjective evaluations, different discrete intensity scales of 1-9 with verbal judgment are framed in different models for the DM to select one of the preference intensities [2, 17]. There is also disagreement on the validity of rank preservation of alternatives after some alternative is added or deleted [3, 21]. Another disagreement is whether additive or multiplicative aggregation is the better way to aggregate [22]. In the remainder of this paper, we compare the fundamental properties of the additive and multiplicative aggregation rules for ratio scales MCDM.

5. ADDITIVE AND MULTIPLICATIVE AGGREGATION RULES

In Additive aggregation model, the overall preferences $v_1,v_2,\ldots,v_n$ of $A_1,A_2,\ldots,A_n$ are estimated by the weighted Arithmetic means $f_1,f_2,\ldots,f_n$ where

$$f_j = w_1\beta_1 y_{j1} + w_2\beta_2 y_{j2} + \ldots.. + w_m\beta_m y_{jm},\ j = 1,2,\ldots,n \quad (1)$$

In multiplicative aggregation model, the overall preferences $v_1,v_2,\ldots,v_n$ of $A_1,A_2,\ldots,A_n$ are estimated by the weighted Geometric mean $g_1,g_2,\ldots,g_n$ where

$$g_j = (\beta_1 y_{j1})^{w_1}(\beta_2 y_{j2})^{w_2}\ldots..(\beta_m y_{jm})^{w_m},\ j = 1,2,\ldots,n \quad (2)$$

The scaling constants $\beta_1,\beta_2,\ldots,\beta_m$ are included in (1) and (2) to show their impact, or lack thereof, on the overall preferences explicitly. While $\beta_1,\beta_2,\ldots,\beta_m$ must be explicitly specified in (1), it will be shown later that $\beta_1,\beta_2,\ldots,\beta_m$ can be deleted in (2) with no change in the relative ratios of $g_1,g_2,\ldots,g_n$. However, since $\beta_1,\beta_2,\ldots,\beta_m$ are associated to units of measure, they must be known and specified before determining the criteria weights $w_1,w_2,\ldots,w_m$.

Mathematically, there is a one-one correspondence between the result $\mathbf{f}=[f_1,f_2,\ldots,f_n]$ from the additive model and the result of the multiplicative model applied to the exponentially transformed local preferences. Specifically, by taking exponential of (1), we have

$$e^{f_j} = (e^{\beta_1 y_{j1}})^{w_1}(e^{\beta_2 y_{j2}})^{w_2}\ldots..(e^{\beta_m y_{jm}})^{w_m},\ j = 1,2,\ldots,n \quad (3)$$

Thus the additive overall preference vector $\mathbf{f}=[f_1,f_2,\ldots,f_n]$ can be derived from the equations in (3) which is the multiplicative aggregation of the exponentially transformed local preferences. Thus, the same answer can be derived from (1) and (3). In similar fashion, the multiplicative overall preference vector $\mathbf{g}=[g_1,g_2,\ldots,g_n]$ in (2) can be derived from (4)

$$ln(g_j) = w_1 ln(\beta_1 y_{j1}) + w_2 ln(\beta_2 y_{j2}) + \ldots.. + w_m ln(\beta_m y_{jm}),\ j = 1,2,\ldots,n \quad (4)$$

Thus the result of the additive model can be obtained from the multiplicative model and vice versa. However, the additive and multiplicative aggregation models in equations (1) and (2) yield different results for the same data. Accordingly, they are regarded as different MCDM models, particularly due to the many very different interpretations of the intermediate terms $w_p x_{jp}$, x_{jp}^(w_p) and the criteria weights $w_1,w_2,\ldots,w_m$. This leads to the need for different questioning procedures to match the associated mathematical operations used in each model and different instructional protocols to ensure that the questioning procedures are well understood by the DM [1, 9].

6. FUNDAMENTAL PROPERTIES OF ADDITIVE AND MULTIPLICATIVE AGGREGATION RULES

We now consider the fundamental properties (**FP**) of the additive model (**FPA**) and the multiplicative model (**FPM**). These properties are not affected by ancillary peripheral procedures such as interactive elicitations from decision makers, pairwise comparisons, and calculations of local priorities and criteria weights.

(1) Partial Values

(**FPA**1) In the additive model, $w_p\beta_p y_{jp}$ is a partial value of A_j under C_p.

The right hand side of (1) is the sum of $w_1\beta_1 y_{j1}$, $w_2\beta_2 y_{j2},\ldots,w_m\beta_m y_{jm}$ which are ratio scale measurements in commensurate units of overall preferences. Thus the $w_1\beta_1 y_{j1}$, $w_2\beta_2 y_{j2},\ldots,w_m\beta_m y_{jm}$ components can be regarded as partial values in overall preference units. The partial value $w_p\beta_p y_{jp}$ represents the portion of the overall preference of A_j contributed from the criterion C_p. The estimated overall preference f_j of A_j is the sum of all these partial values.

(**FPM**1) In the multiplicative model, it is difficult to interpret $(\beta_p y_{jp})$^w_p, p=1,2,…,m.

The right hand side of (2) is a product of complicated terms which have no tractable units of measure. The contribution of A_j under C_p is not explicitly identified in (2). As factors of a product, the $(\beta_1 y_{j1})$^w_1, $(\beta_2 y_{j2})$^w_2,…, $(\beta_m y_{jm})$^w_m components are not portions that come together as g_j. It is a mistake to treat $(\beta_1 y_{j1})$^w_1, $(\beta_2 y_{j2})$^w_2,…, $(\beta_m y_{jm})$^w_m as partial values measured in commensurate units and it is meaningless to compare their magnitudes.

(2) Criteria Weights

(**FPA**2) In the additive model, the criterion weights $w_1,w_2,\ldots,w_m$ are conversion factors.

The local preferences $\beta_1 y_{j1},\beta_2 y_{j2},\ldots,\beta_m y_{jm}$ are converted to partial values $w_1\beta_1 y_{j1},w_2\beta_2 y_{j2},\ldots,w_m\beta_m y_{jm}$. These partial values are in commensurate units and sum to f_j which is an estimation of the overall preference v_j of A_j. The criteria weights depend on the performances of the n alternatives with respect to each of the m criteria. When β_p is specified by imposing the normalization constraint $\beta_p y_{1p}+\beta_p y_{2p}+\ldots+\beta_p y_{np}=1$ as in the conventional AHP [11, 17], then $w_p=w_p\beta_p y_{1p}+w_p\beta_p y_{2p}+\ldots+w_p\beta_p y_{np}$ represents one unit equivalence of local preference in C_p measured in units of overall preference. This is a difficult concept because it may not be easy to conceive an alternative (real or imaginary) that achieves a local preference of $\beta_p y_{1p}+\beta_p y_{2p}+\ldots+\beta_p y_{np}=1$ under C_p. When $\beta_p y_{jp}=1$ is imposed to specify β_p as in Linking Pin AHP [9], then $w_p=w_p\beta_p y_{jp}$ represents the equivalence of the performance of A_j in C_p measured in units of overall preference.

(**FPM**2) In the multiplicative model, there is no simple interpretation for criteria weights.

It is difficult to interpret $(\beta_p y_{jp})$^w_p in (2). The criteria weight w_p, used as an exponent, does not have any direct apparent practical interpretation.

Looking at the logarithmic transformed values of (2) as given in (4), it is clear that the criterion weights $w_1,w_2,\ldots,w_m$ are the conversion factors that convert the logarithmic transformed values of local preferences $ln(\beta_1 y_{j1}),ln(\beta_2 y_{j2}),\ldots,ln(\beta_m y_{jm})$ into partial values $w_1 ln(\beta_1 y_{j1}),w_2 ln(\beta_2 y_{j2}),\ldots,w_m ln(\beta_m y_{jm})$. These partial values are in commensurate units and sum to $ln(g_j)$. However, it is much more complicated than the conversion factors in the additive model due to the logarithmic transformation.

(3) Scaling Factors and Normalization Constraint

(**FPA**3) In the additive model, the criteria weights $w_1,w_2,\ldots,w_m$ are dependent on the normalization constraints and the associated scaling factors of $\beta_1,\beta_2,\ldots,\beta_m$.

The units of measure for local preferences must be specified before they can be meaningfully converted to units of overall preferences. The local preferences $\beta_1 y_{j1},\beta_2 y_{j2},\ldots,\beta_m y_{jm}$ are specified by fixing the unit of measure *via* $\beta_1,\beta_2,\ldots,\beta_m$ with some normalization constraints. Then the criteria weights $w_1,w_2,\ldots,w_m$, as conversion factors, can be meaningfully elicited from the DM. We note that after $w_1,w_2,\ldots,w_m$ are determined, the normalization constraint and the scaling factors $\beta_1,\beta_2,\ldots,\beta_m$ can not be arbitrarily changed without altering the unit of measure of the local priorities. If a different normalization constraint is used and β_p is changed to β_p', then the criteria weight w_p must be adjusted to $w_p'=w_p\beta_p/\beta_p'$ to maintain the same results generated from the original units. Otherwise, unwarranted rank reversals may occur [12].

(**FPM**3) In the multiplicative model, the scaling factors $\beta_1,\beta_2,\ldots,\beta_m$ do not appear explicitly in the relative ratios g_j/g_k. Thus g_j/g_k is independent of $\beta_1,\beta_2,\ldots,\beta_m$.

The multiplicative model (2) can be written equivalently as

$$g_j = (\beta_1^{w_1}\beta_2^{w_2}.....\beta_m^{w_m})(y_{j1}^{w_1}y_{j2}^{w_2}.....y_{jm}^{w_m}),\ j=1,2,...,n \qquad (2a)$$

The term

$$\beta_1^{w_1}\beta_2^{w_2}.....\beta_m^{w_m}$$

on the right hand side of (2a) is a common factor which can be deleted in the ratios g_j/g_k, $1\le j,k\le n$. The relative ratios of $g_1,g_2,\ldots,g_n$ are unchanged when the scaling factors $\beta_1,\beta_2,\ldots,\beta_m$ and the normalization constraints are changed.

The criteria weights $w_1,w_2,\ldots,w_m$ are dependent on the actual performances underlying the local preference values. This is easily neglected since $\beta_1,\beta_2,\ldots,\beta_m$ do not appear explicitly in the expressions for the relative ratios g_j/g_k ($j,k=1,2,\ldots,n$). The DM is usually asked to provide $w_1,w_2,\ldots,w_m$ as "relative criteria importance" without additional guidance. Even though it is complicated, correct guidance can be discerned from equation (4) and the criteria weights are converting factors that convert the logarithm of local priorities into commensurate units.

(4) Criteria Importance

(**FPA**4) In the additive model, there exists special form of normalization constraint for the criteria weights $w_1,w_2,\ldots,w_m$ to be interpreted as weights of importance.

For each criterion C_p ($p=1,2,\ldots,m$), $\mathbf{x}_p=\beta_p u[y_{1p},y_{2p},\ldots,y_{np}]$ where $y_{1p},y_{2p},\ldots,y_{np}$ are some specified local preferences under C_p, β_p is a positive number and u specifies an unit of measure if available. Let t_p be a typical value of all local preferences $y_{1p},y_{2p},\ldots,y_{np}$ under C_p. In practice, t_p is some standard value or aspiration value elicited from the DM or, it may simply be the median or average of $y_{1p},y_{2p},\ldots,y_{np}$. By imposing the normalization constraint $\beta_p t_p=1$, we can say that $w_p=w_p\beta_p t_p$ is the amount of overall preference equivalent to a typical local preference under C_p. Interpreted as the contribution of a typical local preference, w_p can be regarded as a measure of criteria importance of C_p [11].

(**FPM**4) In the multiplicative model, suppose A_j and A_k differ only in C_p and $g_j/g_k \approx \delta_{jk}$, then the criteria weight w_p is approximately $ln(\delta_{jk})/ln(y_{jp}/y_{kp})$.

It follows from (2a) that

$$\frac{g_j}{g_k} = \frac{y_{jp}^{w_p}}{y_{kp}^{w_p}} \text{ and thus } \frac{y_{jp}^{w_p}}{y_{kp}^{w_p}} \approx \delta_{jk}.$$

Taking logarithm, we get

$$w_p \approx \frac{(\delta_{jk})}{(\frac{y_{jp}}{y_{kp}})}$$

It is conceivable that the DM can compare two (real or imaginary) alternatives which differ only in one criterion C_p and provide δ_{jk} as an estimated value of g_j/g_k. Then w_p is approximately $ln(\delta_{jk})/ln(y_{jp}/y_{kp})$. This provides a method for deriving criteria weights from the original performance of alternatives. It can also be used to check the validity of the model results. Unfortunately, there is no apparent insightful interpretation for $ln(\delta_{jk})/ln(y_{jp}/y_{kp})$.

(5) Local Level Dependency

(**FPA**5) In the additive model, the ratio f_j/f_k depends on the magnitudes of y_{jp} and y_{kp}, even when $y_{jp}=y_{kp}$.

Example 1. $\mathbf{C}=\{C_1,C_2\}$, $\mathbf{A}=\{A_1,A_2,A_3,A_4\}$, $\beta_1=\beta_2=1$, $\mathbf{w}=[0.25,0.75]$, $\mathbf{x}_1=[8,8,1,1]$ and $\mathbf{x}_2=[6,1,6,1]$.

Table 2. f_j/f_k Depends on the Magnitudes of y_{j1} and y_{k1} with $y_{j1}=y_{k1}$

	C_1	C_2	Overall f_j
alternative A_1	8	6	6.5
alternative A_2	8	1	2.75
alternative A_3	1	6	4.75
alternative A_4	1	1	1
Criteria Weight	0.25	0.75	

$f_1/f_2 = 6.5/2.75 = 2.3636$ and $f_3/f_4=4.75$. As can be seen, the same local preferences in C_1 do not cancel with each other. The difference in the ratios f_1/f_2 and f_3/f_4 is 2.3864, caused by the different magnitudes of local preferences 8 and 1 under C_1.

(**FPM**5) In the multiplicative model, the ratio g_j/g_k does not depend on the magnitudes of y_{jp} and y_{kp}, when $y_{jp}=y_{kp}$.

$$g_j = (y_{j1}{}^{w_1})(\prod_{p=2}^{m} y_{jp}{}^{w_p}),\ g_k = (y_{k1}{}^{w_1})(\prod_{p=2}^{m} y_{kp}{}^{w_p}) \quad \text{and}$$

$$\frac{g_j}{g_k} = \frac{\prod_{p=2}^{m} y_{jp}{}^{w_p}}{\prod_{p=2}^{m} y_{kp}{}^{w_p}}.$$

Suppose $y_{j1}=y_{k1}$. Then we can see that the ratio g_j/g_k is independent of the magnitude y_{j1}.

Example 2. **C**=$\{C_1,C_2\}$, **A**=$\{A_1,A_2,A_3,A_4\}$, $\beta_1=\beta_2=1$, **w**=[0.25,0.75], $\mathbf{x}_1$=[8,8,1,1] and $\mathbf{x}_2$=[6,1,6,1].

Table 3. g_j/g_k Independent of the Magnitudes of y_{j1} and y_{k1} with $y_{j1}=y_{k1}$

	C_1	C_2	Overall g_j
alternative A_1	8	6	6.4474
alternative A_2	8	1	1.6818
alternative A_3	1	6	3.8337
alternative A_4	1	1	1
Criteria Weight	0.25	0.75	

$g_1/g_2 = g_3/g_4 = 3.8337$. Same local preferences in C_1 are canceled with each other. Thus the different magnitudes of the local preferences 8 and 1 under C_1 do not lead to difference results for the ratios g_1/g_2 and g_3/g_4.

However, the huge difference between the magnitudes 8 and 1 under C_1 is expected to have significant impact as in (**FPA**5) rather than absolutely no impact in (**FPM**5).

(6) Addition or Deletion of Alternative

(**FPA6**) In the additive model, if the values of $\beta_1,\beta_2,\ldots,\beta_m$ are specified by the normalization $x_{1p}+x_{2p}+\ldots+x_{np}=1$ ($p=1,2,\ldots,m$), then the criteria weights $w_1,w_2,\ldots,w_m$ need to be adjusted when some alternative is added or deleted in order to maintain commensurability. If linking pin normalization $\beta_p y_{jp}=1$ is used, the criteria weights $w_1,w_2,\ldots,w_m$ remain unchanged when some alternative is added or some alternative other than the j-th one is deleted.

When an alternative is added or deleted, the terms in the left hand side of the normalization constraint $\beta_p y_{1p}+\beta_p y_{2p}+\ldots+\beta_p y_{np}=1$ is changed and β_p attains a new value β_p'. The partial value of A_j under C_p is given by $w_p\beta_p y_{jp}$ and also by $w_p'\beta_p' y_{jp}$. Thus $w_p'=w_p\beta_p/\beta_p'$ and the criteria weight w_p needs to be adjusted to w_p'.

(**FPM6**) In the multiplicative model, the scaling factors $\beta_1,\beta_2,\ldots,\beta_m$ do not appear explicitly in the relative ratio g_j/g_k. The criteria weights $w_1,w_2,\ldots,w_m$ may remain unchanged when some alternative is added or deleted

The criteria weights $w_1,w_2,\ldots,w_m$ are dependent on the actual performances of the alternatives underlying the local preference values. Adding or deleting an alternative does not alter the perception of the criteria.

7. SUMMARY AND CONCLUSION

The advantages and limitations of the additive and multiplicative aggregation rules are explored and summarized in Table **4**. The additive model has the useful interpretation of $w_p\beta_p y_{jp}$ as partial value of the performance of A_j under C_p and the simple interpretation of the criteria weights as conversion factors. Incorporating these interpretations would increase the likelihood that the questioning procedures are better understood by the DM in practice.

The partial values in the additive model are converted by the scaling factors (criteria weights) to commensurate values which are summed to the overall preference values $f_1,f_2,\ldots,f_n$. For the additive version of the multiplicative model (equation in (4)) the partial values must undergo the logarithmic transformations before conversion by the scaling factors (criteria weights). The implication is that the weights have to be derived differently. In practice, this is not done. Instead, weights are generated in a similar manner for both methods.

The multiplicative adherents [2, 22] criticize the ambiguous criteria weights used in AHP. Yet, they themselves do not pay attention to the wording necessary for the criteria weights in the multiplicative model. In the additive versions of both models, commensurate values must be generated before any addition takes place. The necessity of commensurate units defines the role of criteria weights as conversion factors.

$\beta_1,\beta_2,\ldots,\beta_m$ do not appear explicitly in the expressions for the relative ratios g_j/g_k ($j,k=1,2,\ldots,n$). Thus the relative ratios are unchanged when the scaling factors $\beta_1,\beta_2,\ldots,\beta_m$ and the normalization constraints are changed or some alternative is added or deleted. However, this leads to the common error of neglecting the actual performances of the alternatives underlying the local preference values in the derivation of criteria weights in the multiplicative model which is misconstrued as an advantage of having fewer elements to manipulate. In actual fact, it is a weakness of the multiplicative model. This is similar to deriving criteria weights $w_1,w_2,\ldots,w_m$ from the prices of three objects given only as β[2,3,4] without the crucial knowledge of the actual prices [C$200,C$300, C$400]. This means the performances of the alternatives are not fully used in the multiplicative model. Furthermore, it is not logical that the ratio g_j/g_k is independent of the magnitudes of y_{jp} and y_{kp} when y_{jp} and y_{kp}. are equal or have similar magnitudes. Thus the results of the overall preference values from the multiplicative model are not as credible as the results from the additive model. Also it is more complicated to estimate the criteria weight w_p from $ln(\delta_{jk})/ln(y_{jp}/y_{kp})$ ($p=1,2,\ldots,m$) where δ_{jk} is elicited from the DM's comparison of two imaginary alternatives which differ only in C_p with local preferences $\beta_p y_{jp}$ and $\beta_p y_{kp}$.

It is evident that the additive aggregation model, based on the six fundamental properties (FPA1)-(FPA6), is superior insofar as it is effective and easier for the DM to understand and use. Proper interpretations and careful handling of the normalization constraints used in the model are essential elements to be incorporated properly into the questioning procedures. The criteria weights used in the multiplicative

Table 4. Fundamental Properties of Additive and Multiplicative Rules

	Additive Model	Multiplicative Model
Partial Values	$w_p\beta_p y_{jp}$ is part of overall value of A_j from C_p in units of overall preferences	$(\beta_p y_{jp})$^w_p is not partial value of A_j No tractable units of measurement
Criteria Weights	Criteria weights are conversion factors, depend on alternatives' performances	Criteria weights have no practical interpretation, no explicit relation with alternatives' performances
Scaling Factors	$\beta_1,\beta_2,...,\beta_m$ fixed before deriving **w**. Criteria weights need to be adjusted for changes in $\beta_1,\beta_2,...,\beta_m$	$\beta_1,\beta_2,...,\beta_m$ are arbitrary after deriving **w**. Relative ratios of $g_1,g_2,...,g_n$ are independent of $\beta_1,\beta_2,...,\beta_m$
Normalization Constraint	Criteria weights depend on the normalization constraints	$\beta_1,\beta_2,...,\beta_m$ do not appear explicitly in the expression for g_j/g_k. The relative ratios g_j/g_k are independent of $\beta_1,\beta_2,...,\beta_m$
Criteria Importance	Criteria weights may be interpreted as weights of importance	Criteria weight w_p are possibly an approximation of $ln(\delta)/ln(y_{jp}/y_{kp})$
Local Level Dependency	f_j/f_k depends on the magnitudes of y_{jp} and y_{kp} when $y_{jp}=y_{kp}$.	g_j/g_k independent of the magnitudes of y_{jp} and y_{kp} when $y_{jp}=y_{kp}$.
Add/Delete Alternative	For sum to one normalization, criteria weights adjusted when alternative is added or deleted For linking pin normalization, criteria weights unchanged when alternative is added or deleted	$\beta_1,\beta_2,\ldots,\beta_m$ do not appear explicitly in the relative ratio g_j/g_k. Criteria weights unchanged when alternative is added or deleted

aggregation have complicated meanings which are not well understood and often mixed up in the ambiguous notion of "criteria importance". Furthermore, the relative ratios in the multiplicative aggregation do not depend on similar local preference values which cancel each other out mathematically. Thus we recommend the additive aggregation rule over the multiplicative aggregation rule.

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