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On Multiobjective Duality For Variational Problems

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Abstract: In this paper two types of duals are considered for a class of variational problems involving higher order derivatives. The duality results are derived without any use of optimality conditions. One set of results is based on Mond-Weir type dual that has the same objective functional as the primal problem but different constraints. The second set of results is based on a dual of an auxiliary primal with single objective function. Under various convexity and generalized convexity assumptions, duality relationships between primal and its various duals are established. Problems with natural boundary values are considered and the analogs of our results in nonlinear programming are also indicated.

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1. INTRODUCTION

Calculus of Variations offers a powerful technique for the solution of various important problems appearing in dynamics of rigid bodies, optimization of orbits, theory of vibrations and many areas of science and engineering. The subject of calculus of variation primarily concerns with finding optimal value of a definite integral involving a certain function subject to fixed point boundary conditions. Mond and Hanson [5] were the first to represent the problem of calculus of variation as a mathematical programming problem in infinite dimensional space. Since that time many researches contributed to this subject extensively. For somewhat comprehensive list of references, one may consult Husain and Jabeen [1] and Husain and Rumana [2]. The treatment in [1] has been for the real valued objective function while in [2] for vector valued function.

In this research, we consider a vector valued function for the primal problem and its minimality in the Pareto sense. Both equality and inequality constraints are considered in the formulations. In establishing duality results we consider two types of dual problems to the primal problem. The first one has vector valued objective whereas the second set of results are based on the duality relations between an auxiliary problems and its associated dual as defined in Mond and Hanson [7]. Duality theorems, unlike in case of classical mathematical programming, are not based on optimality criteria but on certain types of convexity and generalized convexity requirements. Finally multiobjective variational problems with natural boundary values rather than fixed end points are mentioned and the analogs of our results in nonlinear programming are pointed out.

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2. PRE-REQUISITES

In the treatment of the following problem (VP), by minimality we mean Pareto minimality. Now consider the following multiobjective variational problem involving higher order derivatives.

(VP) Minimize

$$\left(\int_I f^1(t,x,\dot{x},\ddot{x})\,dt, \ldots, \int_I f^p(t,x,\dot{x},\ddot{x})\,dt \right)$$

Subject to

$$x(a) = 0 = x(b) \quad (1)$$

$$\dot{x}(a) = 0 = \dot{x}(b) \quad (2)$$

$$g(t,x,\dot{x},\ddot{x}) \leqq 0 \ , \ t \in I \quad (3)$$

$$h(t,x,\dot{x},\ddot{x}) = 0 \ , \ t \in I \quad (4)$$

where **(i)** for $I = [a,b] \subseteq R$, $f : I \times R^n \times R^n \times R^n \to R$, $g : I \times R^n \times R^n \times R^n \to R^m$ and $h : I \times R^n \times R^n \times R^n \to R^k$ are continuously differentiable functions, and

(ii) X designates the space of piecewise smooth function $x : I \to R^n$ having its first and second order derivatives $\dot{x}$ and $\ddot{x}$ respectively equipped with the norm.

$$\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2 x\|_\infty ,$$

where the differentiation operator D is given by

$$\omega = Dx \Leftrightarrow x(t) = \int_a^t \omega(s)\,ds$$

Thus $D \equiv \frac{d}{dt}$ except at discontinuities.

We denote the set of feasible solutions of the problem (VP) by K_P, i.e.,

$$K_P = \left\{ x \in X \left| \begin{array}{l} x(a) = 0 = x(b), \ g(t,x,\dot{x},\ddot{x}) \leqq 0, \ t \in I \\ \dot{x}(a) = 0 = \dot{x}(b), \ h(t,x,\dot{x},\ddot{x}) = 0, \ t \in I \end{array} \right. \right\}$$

The following convention for inequalities for vectors in R^n given in Mangasarian [3] will be used throughout the development of the theory:

If $x, y \in R^n$, *then*

$x \geqq y \Leftrightarrow x_i \geqq y_i \quad i = 1,...,n$

$x \geq y \Leftrightarrow x_i \geqq y_i, \quad and \ x \neq y$

$x > y \Leftrightarrow x_i > y_i, i = 1,...,n.$

Definition 2.1: A feasible solution of the problem (VP) i.e., $\bar{x} \in K_P$ is said to be Pareto minimum if there exists no $\hat{x} \in K_p$ such that

$$\left(\int_I f^1(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}})dt, \ldots, \int_I f^p(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}})dt \right)$$
$$\leq \left(\int_I f^1(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt, \ldots, \int_I f^p(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt \right)$$

Pareto maximality can be defined in the same way except that the inequality in the above definition is reversed.

In the subsequent analysis the following result plays a significant role.

PROPOSITION 2.1: Suppose $\lambda > 0$, $\lambda \in R^p$ such that Let $\bar{x}(t) \in K_P$ is an optimal solution of the problem,

$$(P_\lambda): \ \underset{x(t)\in K_P}{Min} \int_I \lambda^T f(t,x,\dot{x},\ddot{x})dt$$

Then $\bar{x}(t)$ is an optimal solution of (MP) in the Pareto sense.

Proof: Assume $\bar{x}(t)$ is not a Pareto optimal of (MP). Then there exists an $\hat{x}(t) \in K_P$ such that

$$\int_I f^i(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}})dt \leqq \int_I f^i(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt, \ i = 1,2,...,p.$$

$$\int_I f^j(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}})dt < \int_I f^j(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt \quad , \ i \neq j.$$

Hence

$$\int_I \lambda^T f(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}})dt < \int_I \lambda^T f(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt \quad .$$

This contradicts the assumption that $\bar{x}$ minimizes $\int_I \lambda^T f(t,x,\dot{x},\ddot{x})dt$ over K_P.

In the subsequent sections some duality results by introducing two types of duals to (VP) will be established.

3. MOND-WEIR TYPE MULTIOBJECTIVE DUALITY

Consider the following Mond-Weir [4] dual to (VP)

(M-WD): Maximize

$$\left(\int_I f^1(t,u,\dot{u},\ddot{u})dt, \ldots, \int_I f^p(t,u,\dot{u},\ddot{u})dt \right)$$

Subject to

$$u(a) = 0 = u(b), \tag{5}$$

$$\dot{u}(a) = 0 = \dot{u}(b), \tag{6}$$

$$\begin{aligned} &\lambda^T f_u(t,u,\dot{u},\ddot{u}) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \\ &-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u})\right) \\ &+D^2 \left(\begin{array}{l} \lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + \\ z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \end{array} \right) = 0 \ , \ t \in I \end{aligned} \tag{7}$$

$$\int_I \left(y(t)^T g(t,u,\dot{u},\ddot{u}) + z(t)^T h(t,u,\dot{u},\ddot{u}) \right) dt \geqq 0 \ , \tag{8}$$

$$\lambda > 0 \ , \ \lambda \in R^p, \ y(t) \geqq 0 \ , \ t \in I \tag{9}$$

Let K_D be the set of the feasible solutions of (M-WD).

Theorem 3.1: Suppose

(A_1): $\bar{x}(t) \in K_P$

(A_2): $(\lambda, u, y(t), z(t)) \in K_D$

(A_3): $\int_I \lambda^T f(t,.,.,.)dt$ is pseudo-convex

(A_4): $\int_I \left\{ y(t)^T g(t,.,.,.) + z(t)^T h(t,.,.,.) \right\} dt$ is quasiconvex.

Then $\int_I \lambda^T f(t,x,\dot{x},\ddot{x})dt \geqq \int_I \lambda^T f(t,u,\dot{u},\ddot{u})dt$.

Proof: Since $y(t) \geqq 0$, $t \in I$, $g(t,x,\dot{x},\ddot{x}) \leqq 0$, $t \in I$ and $h(t,x,\dot{x},\ddot{x}) = 0$, $t \in I$, we have

$$\int_I \left(y(t)^T g(t,x,\dot{x},\ddot{x}) + z(t)^T h(t,x,\dot{x},\ddot{x}) \right) dt \leqq 0 \tag{10}$$

Combining this inequality with (8)

We have,

$$\int_I y(t)^T g(t,x,\dot{x},\ddot{x}) + z(t)^T h(t,x,\dot{x},\ddot{x})dt$$
$$\leqq \int_I \left(y(t)^T g(t,u,\dot{u},\ddot{u}) + z(t)^T h(t,u,\dot{u},\ddot{u}) \right) dt$$

By the hypothesis (A_4), this yields

$$0 \geqq \int_I \Big[(x-u)^T \left(y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \right)$$
$$+(\dot{x}-\dot{u})^T \left(y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right)$$
$$+(\ddot{x}-\ddot{u})^T \left(y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big] dt$$

$$= \int_I (x-u)^T \left(y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \right) dt$$
$$+(x-u)^T \left(y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big|_{t=a}^{t=b}$$
$$-\int_I (x-u)^T D\left(y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right) dt$$
$$+(\dot{x}-\dot{u})^T \left(y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big|_{t=a}^{t=b}$$
$$-\int_1 (\dot{x}-\dot{u})^T D\left(y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) dt$$

(By integration by parts)

$$= \int_I (x-u)^T \Big[\left(y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \right)$$
$$-D\left(y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big] dt$$
$$-(x-u)^T D\left(y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big|_{t=a}^{t=b}$$
$$+\int_1 (x-u)^T D^2 \left(y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) dt$$

(By integration by parts)

Using (7), we have,

$$0 \leq \int_I (x-u)^T \left\{ \lambda^T f_u(t,u,\dot{u},\ddot{u}) - D\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right.$$
$$\left. +D^2 \lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right\} dt$$

$$0 \leq \int_I \left\{ (x-u)^T \lambda^T f_u(t,u,\dot{u},\ddot{u}) + (\dot{x}-\dot{u})^T \lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right\} dt$$
$$-(x-u)^T \lambda^T f_u(t,u,\dot{u},\ddot{u}) \Big|_{t=a}^{t=b}$$
$$-(\dot{x}-\dot{u})^T D\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \Big] dt + (\dot{x}-\dot{u})^T D\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \Big|_{t=a}^{t=b}$$

$$\int_I \left\{ (x-u)^T \lambda^T f_u(t,u,\dot{u},\ddot{u}) + (\dot{x}-\dot{u})^T \lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right.$$
$$\left. +(\ddot{x}-\ddot{u})^T D\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right\} dt \geqq 0$$

Thus, by integration by parts using the boundary conditions, we have,

$$\int_I \left\{ (x-u)^T \lambda^T f_u(t,u,\dot{u},\ddot{u}) + (\dot{x}-\dot{u})^T \lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right.$$
$$\left. +(\ddot{x}-\ddot{u})^T \lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right\} dt \geqq 0$$

This, because of the hypothesis (A_3) implies,

$$\int_I \lambda^T f(t,x,\dot{x},\ddot{x}) dt \geqq \int_I \lambda^T f(t,u,\dot{u},\ddot{u}) dt \quad .$$

Theorem 3.2: Assume

(B_1): $\bar{x}(t) \in K_P$

(B_2): $\left(\lambda, \bar{u}(t), \bar{y}(t), \bar{z}(t) \right) \in K_D$

(B_3): $\int_1 \left(y(t)^T g(t,.,.,.) + z(t)^T h(t,.,.,.) \right) dt$ is quasi-convex

(B_4): $\int_I \lambda^T f(t,.,.,.) dt$ is pseudoconvex

(B_5): $\int_I \bar{\lambda}^T f(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) dt = \int_I \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) dt$

Then $\bar{x}(t)$ is an optimal solution of (VP) and $\left(\lambda, \bar{u}(t), \bar{y}(t), \bar{z}(t) \right)$ is an optimal solution of the problem (M-WD).

Proof: Assume that $\bar{x}$ is not Pareto-optimal of (VP). Then there exists an $\hat{x}(t) \in K_P$ such that

$$\int_I f^i\left(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}}\right) dt \leq \int_I f^i\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right) dt \quad , \quad \text{for all } i$$

and $\int_I f^j\left(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}}\right) dt < \int_I f^j\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right) dt$ for some j, $1 \leq j \leq p$

Since $\bar{\lambda} > 0$, this implies,

$$\int_I \bar{\lambda}^T f\left(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}}\right) dt < \int_I \bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right) dt$$

By the hypothesis (B_5), this inequality implies,

$$\int_I \bar{\lambda}^T f\left(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}}\right) dt < \int_I \bar{\lambda}^T f\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right) dt \quad .$$

This contradicts the conclusion of Theorem 3.1 thus establishing the Pareto optimality of $\bar{x}(t)$ for (VP). Similarly we can show that $\left(\lambda, \bar{u}(t), \bar{y}(t), \bar{z}(t) \right)$ is Pareto optimal for (M-WD).

We state the following theorem without proof as it is similar to Theorem 3.4 of [6].

Theorem 3.3: Assume,

(C_1): $\bar{x}(t) \in K_P$; $\left(\lambda, \bar{u}(t), \bar{y}(t), \bar{z}(t) \right) \in K_D$;

(C_2): $\int_I \bar{\lambda}^T f(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) dt = \int_I \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) dt$;

(C_3): $\int_1 \left(y(t)^T g(t,.,.,.) + z(t)^T h(t,.,.,.) \right) dt$ is convex;

(C_4): $\int_I \lambda^T f(t,.,.,.) dt$ is quasiconvex.

Then $\bar{x}(t) = \bar{u}(t), \ t \in I$.

4. WOLFE TYPE MULTIOBJECTIVE DUALITY

To establish duality results similar to the preceding ones but under different convexity and generalized convexity assumptions, we formulate the following Wolfe type dual to the problem (P_λ) stated in the Preposition 2.1.

We assume that $\bar{\lambda}$ is known and $\bar{\lambda} > 0$.

(WCD_λ): Maximize:

$$\int_I \left(\bar{\lambda}^T f(t,x,\dot{x},\ddot{x}) + y(t)^T g(t,x,\dot{x},\ddot{x}) + z(t)^T h(t,x,\dot{x},\ddot{x}) \right) dt$$

Subject to:

$$x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b) \tag{11}$$

$$\begin{aligned} &\left(\lambda^T f_u(t,u,\dot{u},\ddot{u}) + y(t)^T g_u(t,u,\dot{u},\ddot{u})\right) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \\ &-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u})\right) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u})\Big) \\ &+D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u})\Big) \\ &= 0 \ , \ t \in I \end{aligned} \tag{12}$$

$$y(t) \geqq 0 \, , \, t \in I \tag{13}$$

In the following L_P represents the set of feasible solutions of (P_λ) and L_D the set of feasible solutions of (WCD_λ).

Theorem 4.1: Assume

(H_1) $\bar{x}(t) \in L_P$; $\left(\bar{u}(t), \bar{y}(t), \bar{z}(t)\right) \in L_D$

(H_2) $\int_I \bar{\lambda}^T f(t,.,.,.) dt$ and $\int_I \left(y(t)^T g(t,.,.,.) + z(t)^T h(t,.,.,.)\right) dt$ are convex.

Then,

$$\int_I \bar{\lambda}^T f(t,x,\dot{x},\ddot{x}) dt \geqq \int_I \begin{pmatrix} \bar{\lambda}^T f(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u}) \\ +z(t)^T h(t,u,\dot{u},\ddot{u}) \end{pmatrix} dt.$$

Proof: By the convexity of $\int_I \bar{\lambda}^T f(t,.,.,.) dt$, we have

$$\begin{aligned} \int_I \bar{\lambda}^T f(t,x,\dot{x},\ddot{x}) dt \geqq \int_I \bar{\lambda}^T f(t,u,\dot{u},\ddot{u}) dt + \int_I \Big[(x-u)^T \bar{\lambda}^T f_u(t,u,\dot{u},\ddot{u}) \\ + (\dot{x}-\dot{u})^T \bar{\lambda}^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + (\ddot{x}-\ddot{u})^T \bar{\lambda}^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \Big] dt \end{aligned} \tag{14}$$

From the dual constraint (12), we have,

$$\begin{aligned} &\int_I (x-u)^T \Big[\left(\lambda^T f_u(t,u,\dot{u},\ddot{u}) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \right) \\ &-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right) \\ &+D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big] dt = 0 \end{aligned}$$

This, by integrating by parts and using the boundary conditions as earlier, implies

$$\begin{aligned} &\int_I (x-u)^T \Big[\lambda^T \left(f_u(t,u,\dot{u},\ddot{u}) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \right) \\ &+(\dot{x}-\dot{u})^T \left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right) \\ &+(\ddot{x}-\ddot{u})^T \left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big] dt = 0 \end{aligned}$$

Using this, in (14) we have

$$\begin{aligned} &\int_I \bar{\lambda}^T f(t,x,\dot{x},\ddot{x}) dt \geq \int_I \bar{\lambda}^T f(t,u,\dot{u},\ddot{u}) dt \\ &-\int_I \Big[(x-u)^T \left(y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \right) \\ &+(\dot{x}-\dot{u})^T \left(y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right) \\ &+(\ddot{x}-\ddot{u})^T \left(y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big] dt \end{aligned}$$

By the hypothesis (H_2), this implies

$$\begin{aligned} &\int_I \bar{\lambda}^T f(t,x,\dot{x},\ddot{x}) dt \geq \int_I \bar{\lambda}^T f(t,u,\dot{u},\ddot{u}) dt \\ &+\int_I \left(y(t)^T g(t,u,\dot{u},\ddot{u}) + z(t)^T h(t,u,\dot{u},\ddot{u}) \right) dt \\ &-\int_I \left(y(t)^T g(t,x,\dot{x},\ddot{x}) + z(t)^T h(t,x,\dot{x},\ddot{x}) \right) dt, \end{aligned}$$

Since $x \in L_P$, this implies

$$\int_I \lambda^T f(t,x,\dot{x},\ddot{x}) dt \geqq \int_I \begin{pmatrix} \bar{\lambda}^T f(t,u,\dot{u},\ddot{u}) \\ +y(t)^T g(t,u,\dot{u},\ddot{u}) \\ +z(t)^T h(t,u,\dot{u},\ddot{u}) \end{pmatrix} dt.$$

This proves the theorem.

The following theorem gives a situation in which a Pareto optimal solution of (VP) exists.

Theorem 4.2. Suppose

(F_1): $\bar{x}(t) \in L_P$, $\left(\bar{u}(t), \bar{y}(t), \bar{z}(t)\right) \in L_D$;

(F_2): $\int_I \bar{\lambda}^T f(t,.,.,.) dt$; and $\int_I \left(y(t)^T g(t,.,.,.) + z(t)^T h(t,.,.,.) \right) dt$ are convex,

(F_3):

$$\int_I \bar{\lambda}^T f(t,x,\dot{x},\ddot{x}) dt = \int_I \begin{pmatrix} \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + y(t)^T g(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) \\ +z(t)^T h(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) \end{pmatrix} dt$$

Then $\bar{x}(t)$ and $\left(\bar{y}(t), \bar{z}(t), \bar{u}(t)\right)$ are optimal solutions of (P_λ) and (WCD_λ). Hence $\bar{x}(t)$ is a Pareto optimal solution of (VP).

The last part of the conclusion follows from Proposition 2.1.

Proof: Suppose $\bar{x}(t)$ does not minimize (P) then there exist, $x^*(t) \in L_P$ such that

$$\begin{aligned} &\int_I \bar{\lambda}^T f\left(t, x^*(t), \dot{x}^*(t), \ddot{x}^*(t)\right) dt < \int_I \bar{\lambda}^T f\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) dt \\ &= \int_I \left(\bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + y(t)^T g(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + z(t)^T h(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) \right) dt \end{aligned}$$

This contradicts the conclusion of Theorem 4.1. Hence $\bar{x}(t)$ minimizes (P_λ).

We can similarly prove that $\left(\bar{y}(t),\bar{z}(t),\bar{u}(t)\right)$ maximizes (WCD_{λ}).

Theorem 4.3 Assume

(G1): $\bar{x}(t)\in L_P$, $\left(\bar{y}(t),\bar{z}(t),\bar{u}(t)\right)\in L_D$;

(G2):

$$\int_I \bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt=\int_I \begin{pmatrix}\bar{\lambda}^T f\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+y(t)^T g\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +z(t)^T h\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}dt$$

(G3): $\int_1 \left(\bar{\lambda}^T f(t,.,.,.)+y(t)^T g(t,.,.,.)+z(t)^T h(t,.,.,.)\right)dt$ is convex

Then

$$\int_I y(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt=0\ ,\ t\in I$$

Proof: By hypotheses (G2) and (G3), we have

$$\begin{aligned}
\int_I \bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt &=\int_I \begin{pmatrix}\bar{\lambda}^T f\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+y(t)^T g\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +z(t)^T h\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}dt\\
&\leq \int_I \begin{pmatrix}\bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\\ +\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\end{pmatrix}dt\\
&-\int_I \Bigg[\left(\bar{x}-\bar{u}\right)^T \begin{pmatrix}\bar{\lambda}^T f_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\\
&+\left(\dot{\bar{x}}-\dot{\bar{u}}\right)^T \begin{pmatrix}\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\\
&+\left(\ddot{\bar{x}}-\ddot{\bar{u}}\right)^T \begin{pmatrix}\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\Bigg]dt\\
&=\int_I \begin{pmatrix}\bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\\ +\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\end{pmatrix}dt\\
&-\int_I \Bigg[\left(\bar{x}-\bar{u}\right)^T \begin{pmatrix}\bar{\lambda}^T f_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\\
&-\left(\bar{x}-\bar{u}\right)^T \begin{pmatrix}\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\Bigg|_{t=a}^{t=b}\\
&-\int_I \left(x-u\right)^T D\begin{pmatrix}\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\Bigg]dt\\
&+\left(\dot{\bar{x}}-\dot{\bar{u}}\right)^T \begin{pmatrix}\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\Bigg|_{t=a}^{t=b}\\
&-\int_I \left(\dot{\bar{x}}-\dot{\bar{u}}\right)^T D\begin{pmatrix}\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}dt\\
&+\left(\ddot{\bar{x}}-\ddot{\bar{u}}\right)^T \begin{pmatrix}\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\Bigg]dt
\end{aligned}$$

(Integrating by parts)

$$\begin{aligned}
&=\int_I \begin{pmatrix}\bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\\ +\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\end{pmatrix}dt\\
&+\int_I \Bigg[\left(\bar{x}-\bar{u}\right)^T \begin{pmatrix}\bar{\lambda}^T f_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\\
&-\left(\bar{x}-\bar{u}\right)^T D\begin{pmatrix}\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h\dot{u}_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\Bigg]dt\\
&-\left(\bar{x}-\bar{u}\right)^T D\begin{pmatrix}\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\Bigg|_{t=a}^{t=b}\\
&+\int_I .\left(\bar{x}-\bar{u}\right)^T D^2\begin{pmatrix}\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}dt
\end{aligned}$$

(Using boundary conditions and Integrating by parts)

$$\begin{aligned}
&=\int_I \begin{pmatrix}\bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\\ +\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\end{pmatrix}dt\\
&+\int_I \Bigg[\left(\bar{x}-\bar{u}\right)^T \Bigg\{\begin{pmatrix}\bar{\lambda}^T f_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\\
&-D\begin{pmatrix}\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h\dot{u}_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\\
&+D^2\begin{pmatrix}\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\\ +\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\end{pmatrix}\Bigg\}\Bigg]dt\\
&=\int_I \left(\bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt
\end{aligned}$$

(Using (13))

This implies

$$\int_I \left(\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt \geqq 0 \qquad (15)$$

But since $\bar{y}(t)\geq 0$, $g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\leq 0$ and $h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)=0$, $t\in I$ yield,

$$\int_I \left(\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt \leq 0 \qquad (16)$$

Combining (15) and (16), we have

$$\int_I \left(\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt = 0$$

This, because of $h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)=0$, $t\in I$, gives

$$\int_I \bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt=0.$$

This, together with $\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) \leqq 0,\ t \in I$, implies $\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) = 0,\ t \in I$

Theorem 4.4. Suppose

(R_1): $(\bar{y}(t),\bar{z}(t),u(t)) \in L_D$ and $\bar{u}(t) \in L_P$;

(R_2): $\bar{y}(t)^T g(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) = 0$, $t \in I$;

(R_3): $\int_I \bar{\lambda}^T f(t,.,.,.)\,dt$ and

$\int_I \left(\bar{y}(t)^T g(t,.,.,.) + \bar{z}(t)^T h(t,.,.,.)\right) dt$ are convex;

Then $\bar{u}(t)$ is an optimal solution of (P_λ) and hence of (VP).

Proof: If $\bar{u}(t)$ is the only feasible solution of (P_λ), the conclusion is self evident. So, assume that $\bar{x}(t)$ is another feasible solution of (P_λ). Then by the hypotheses (R_1) and (R_3), we have

$$\int_I \bar{\lambda}^T f(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})\,dt \geqq \int_I \bar{\lambda}^T f_u(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\,dt$$
$$\int_I \Big[(\bar{x}-\bar{u})^T \left(\bar{\lambda}^T f_u(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right) + (\dot{\bar{x}}-\dot{\bar{u}})^T \left(\bar{\lambda}^T f_{\dot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)$$
$$+ (\ddot{\bar{x}}-\ddot{\bar{u}})^T \left(\bar{\lambda}^T f_{\ddot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)\Big] dt$$

Now integrating by parts, we have,

$$= \int_I \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\,dt - \int_I \Big[(\bar{x}-\bar{u})^T \bar{\lambda}^T f_u(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\,dt$$
$$-(\bar{x}-\bar{u})^T \left(\bar{\lambda}^T f_{\dot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)\Big|_{t=a}^{t=b} - \int_I (x-u)^T D\left(\bar{\lambda}^T f_{\dot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)\Big] dt$$
$$+(\dot{\bar{x}}-\dot{\bar{u}})^T \left(\bar{\lambda}^T f_{\ddot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)\Big|_{t=a}^{t=b} - \int_I (\dot{\bar{x}}-\dot{\bar{u}})^T D\left(\bar{\lambda}^T f_{\ddot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right) dt$$
$$= \int_I \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\,dt + \int_I (\bar{x}-\bar{u})^T D\left(\bar{\lambda}^T f_{\dot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right) dt$$
$$+ \int_I (\dot{\bar{x}}-\dot{\bar{u}})^T D\left(\bar{\lambda}^T f_{\ddot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right) dt$$

(Using boundary conditions (11)

$$= \int_I \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\,dt + \int_I (\bar{x}-\bar{u})^T \Big[\bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})$$
$$-D\left(\bar{\lambda}^T f_{\dot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right) + D^2\left(\bar{\lambda}^T f_{\ddot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)\Big] dt$$
$$= \int_I \left(\bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right) dt$$
$$-\int_I (\bar{x}-\bar{u})^T \Big[\bar{y}(t)^T g_u(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + \bar{z}(t)^T h_u(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})$$
$$-D\left(\bar{y}(t)^T g_{\dot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + \bar{z}(t)^T h\dot{u}_u(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)$$
$$+D^2\left(\bar{y}(t)^T g_{\ddot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + \bar{z}(t)^T h_{\ddot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)\Big] dt$$

Thus, by integrating by parts and using boundary conditions, as earlier, we get

$$= \int_I \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\,dt$$
$$-\int_I \Big[(\bar{x}-\bar{u})^T \left(\bar{y}(t)^T g_u(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + \bar{z}(t)^T h_u(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)$$
$$+(\dot{\bar{x}}-\dot{\bar{u}})^T \left(\bar{y}(t)^T g_{\dot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + \bar{z}(t)^T h_{\dot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)$$
$$+(\ddot{\bar{x}}-\ddot{\bar{u}})^T \left(\bar{y}(t)^T g_{\ddot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + \bar{z}(t)^T h_{\ddot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)\Big] dt$$
$$\geqq \int_I \left(\bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right) dt$$
$$+\int_I \Big[\bar{y}(t)^T g(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + \bar{z}(t)^T h(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\Big] dt$$
$$-\int_I \Big[\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) + \bar{z}(t)^T h(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})\Big] dt$$
$$\geqq \int_I \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\,dt$$

(Using hypothesis (A_1), (A_2) and $x \in L_p$).

This implies that $\bar{u}$ minimizes $\int_I \bar{\lambda}^T f(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})\,dt$ over L_p.

Remark: In Theorem 4.4, we assume that a part of feasible solution of (WCD_λ) is a feasible solution of (P_λ). It is a natural question if there is any set of appropriate conditions under which this assumption is true. The following theorem gives one such set of conditions.

Theorem 4.5. Assume

(Q_1): $x(t) \in L_p$ and $(\bar{y}(t),\bar{z}(t),u(t)) \in L_D$;

(Q_2): $g(t,.,.,.)$ and $h(t,.,.,.)$ are differentiable convex functions;

(Q_3): $\int_I \left(g(t,u(t),\dot{u}(t),\ddot{u}(t)) + h(t,u(t),\dot{u}(t),\ddot{u}(t))\right) dt = 0$

(Q_4): $(x-u)^T g_u(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + (\dot{x}-\dot{u})^T \left(g_{\dot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)$

$+(\ddot{x}-\ddot{u})^T g_{\ddot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) \geqq 0$, $t \in I$

(Q_5): $(x-u)^T h_u(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + (\dot{x}-\dot{u})^T \left(h_{\dot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)$

$+(\ddot{x}-\ddot{u})^T h_{\ddot{u}}(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) \geqq 0$, $t \in I$

Then $\bar{u} \in L_p$.

Proof: By the convexity of $g(t,.,.,.)$ and $h(t,.,.,.)$, we have

$$g(t,x,\dot{x},\ddot{x}) \geqq g(t,u,\dot{u},\ddot{u}) + (x-u)^T g_u(t,u,\dot{u},\ddot{u}) + (\dot{x}-\dot{u})^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + (\ddot{x}-\ddot{u})^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \geqq 0 \quad (17)$$

$$h(t,x,\dot{x},\ddot{x}) \geqq h(t,u,\dot{u},\ddot{u}) + (x-u)^T h_u(t,u,\dot{u},\ddot{u}) + (\dot{x}-\dot{u})^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) + (\ddot{x}-\ddot{u})^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \geqq 0 \quad (18)$$

Using (13) and (14) together with the hypotheses (Q_1), (Q_2) and (Q_3), we have

$$g(t,u,\dot{u},\ddot{u}) \leqq 0 , \quad t \in I \tag{19}$$

and

$$h(t,u,\dot{u},\ddot{u}) \leqq 0 , \quad t \in I \tag{20}$$

By (15) and (16), we have

$$g(t,u(t),\dot{u}(t),\ddot{u}(t)) + h(t,u(t),\dot{u}(t),\ddot{u}(t)) \leqq 0 , \quad t \in I \tag{21}$$

The hypothesis (Q_2) with (17) implies

$$g(t,u(t),\dot{u}(t),\ddot{u}(t)) + h(t,u(t),\dot{u}(t),\ddot{u}(t)) = 0 , \quad t \in I \tag{22}$$

But $g(t,u,\dot{u},\ddot{u}) \leqq 0$, $t \in I$. Hence by (22) we have

$$h(t,u,\dot{u},\ddot{u}) \geqq 0 , \quad t \in I . \tag{23}$$

The inequalities (20) and (21) imply

$$h(t,u,\dot{u},\ddot{u}) = 0 , \quad t \in I . \tag{24}$$

The relations (19) and (24) imply that $\bar{u}(t) \in L_p$

5. VARIATIONAL PROBLEMS WITH NATURAL BOUNDARY VALUES

It is possible to construct variational problems with natural boundary values rather than the problem with fixed end point considered in the preceding sections. The problems of Section 2 can be formulated as follows:

$$(VP_N) : Maximize \left(\int_I f^{'}(t,x,\dot{x},\ddot{x})\,dt, ..., \int_I f^{'}(t,x,\dot{x},\ddot{x})\,dt \right)$$

Subject to

$$g(t,x,\dot{x},\ddot{x}) \leqq 0, \; t \in I$$
$$h(t,x,\dot{x},\ddot{x}) = 0, \; t \in I$$

$$(M-WD)_N : \; Maximize \left(\int_I f'(t,u,\dot{u},\ddot{u})\,dt, ..., \int f'(t,u,\dot{u},\ddot{u})\,dt \right)$$

$$\begin{aligned}
&\left(\lambda^T f_u(t,u,\dot{u},\ddot{u})\right) + y(t) g_u(t,u,\dot{u},\ddot{u}) \\
&+ z(t)^T h_u(t,u,\dot{u},\ddot{u})) - D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u})\right) \\
&+ y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u})) + \\
&D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \\
&+ z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u})) = 0, \; t \in I, \\
&\int_1 \left(y(t)^T g(t,u,\dot{u},\ddot{u}) + z(t)^T h(t,u,\dot{u},\ddot{u}) \right) dt \geqq 0 \\
&\lambda > 0, \; y(t) \geqq 0, \; t \in I. \\
&\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) \\
&+ z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) = 0, \; at \; t = a, t = b \qquad (\alpha) \\
&\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \\
&+ z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) = 0, \; at\, t = a, t = b \qquad (\beta)
\end{aligned}$$

The conditions (α) and (β) are popularly known as natural boundary conditions in calculus of variation.

Theorems of the section 3 for (VP_N) and $(M-WD_N)$ can easily be proved in view of earlier analysis in this research. The problems of the Section 4 can be written with natural boundary values as follows:

For given $0 < \lambda \in R^p$.

$$(P_\lambda)_N : \qquad Minimize \int_I \lambda^T f(t,x,\dot{x},\ddot{x})dt$$

$$\text{Subject to} \quad \begin{array}{l} g(t,x,\dot{x},\ddot{x}) \leqq 0, \; t \in I. \\ h(t,x,\dot{x},\ddot{x}) = 0, \; t \in I. \end{array}$$

$$(WCD_\lambda)_N : \; Maximize \int_I \begin{pmatrix} \lambda^T f(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u}) \\ + z(t)^T h(t,u,\dot{u},\ddot{u}) \end{pmatrix} dt$$

$$\begin{aligned}
&\left(\lambda^T f_u(t,u,\dot{u},\ddot{u})\right) + y(t) g_u(t,u,u,u) \\
&+ z(t)^T h_u(t,u,\dot{u},\ddot{u})) - D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u})\right) + y(t)^T g_{\dot{u}}(t,u,u,u) \\
&+ z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u})) + D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right) + y(t)^T g_{\ddot{u}}(t,u,u,u) \\
&+ z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u})) = 0, t \in I, \lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) \\
&+ y(t)^T g_{\dot{u}}(t,u,u,u) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) = 0, at\, t = a, t = b \\
&\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,u,u) \\
&+ z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) = 0, at\, t = a, t = b \\
&\lambda > 0, y(t) \geqq 0, t \in I.
\end{aligned}$$

6. MULTIOBJECTIVE NONLINEAR PROGRAMMING PROBLEMS

If all the functions in the problems $(VP)_N$, $(WCD_\lambda)_N$, and $(P_\lambda)_N$ are independent of t, then these problems reduce to the following problems:

$$(VP_0): \qquad Minimize \; (f^1(u), f^2(u), ..., f^p(u))$$
$$Subject \; to$$
$$g(x) \leqq 0, \quad h(x) = 0 \; .$$

$$(M-WP_0): Maximize \; (f^1(u), f^2(u), ..., f^p(u))$$
$$Subject \; to$$
$$\lambda^T f_u(u) + y^T g_u(u) + z^T h_u(u) = 0,$$
$$y^T g(u) + z^T h(u) \geqq 0.$$

For given $\lambda > 0, \lambda \in R^p$, *we have*

$$(VP_\lambda)_0 : \qquad Minimize \; \lambda^T f(u)$$
$$Subject \; to$$
$$g(u) \leqq 0, \quad h(u) = 0 \; .$$

$$(WCD_\lambda)_0 : \qquad Maximize \; \lambda^T f(u) + y^T g(u) + z^T h(u)$$
$$Subject \; to$$
$$\lambda^T f_u(u) + y^T g_u(u) + z^T h_u(u) = 0,$$
$$y^T g(u) + z^T h(u) \geqq 0. \; y \geqq 0.$$

Theorems 3.1-3.3 for the pair of Mond-Weir [4] type dual problems $(VP_0)\,and\,(M-WP_0)$ and Theorems 4.1-4.5 for the pair of Wolf type dual problems $(VP_\lambda)_0$ and $(WCD_\lambda)_0$ are simple to be validated, albeit validations of these theorems are not explicitly mentioned in the literature.

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CONFLICT OF INTEREST

Declared none.

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