Real Spatial Shapes and Phase Space Structures of 3-D Nonlinear Electrostatic Plasma Wave

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Abstract: 3-D nonlinear electrostatic plasma wave is periodic along its longitudinal direction. By strictly analyzing a universal equation set of charged particles system, we find that such a longitudinal periodicity has a severe constraint on the transverse shape of a 3-D electrostatic structure. Only few allowed transverse shapes could warrant the longitudinal periodicity. This longitudinal periodicity in real space shapes can lead to a corresponding periodic structure in phase space.

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1. INTRODUCTION

The generation of high-energy charged particles from plasmas is an issue of long history in plasma physics. In 1970s, authors have found, from their computer simulation on two-stream instability [1-5], that electron phase-space distribution function could display a hole structure when self-consistent field is set up within plasmas. Such a hole structure reflects the population of some lower energy electrons being suppressed while that of some higher energy electrons being elevated, and hence is a signal of the generation of high-energy charged particles, or of particle acceleration. It is also described by some authors with ``negative temperature'' conception [6]. Some authors have noticed that a temperature profile, which is time-space varying, is more appropriate than a constant temperature to describe plasmas [7]. All of these earlier works have clearly indicated that plasma is a effective matrix for generating high-energy charged particles.

On the other hand, at the end of 1970s, Tajima and Dawson definitely proposed a notion: plasma-based particle acceleration [8]. This notion stresses that plasma density wave could play a role of traditional accelerator. Because the plasma density wave is closely related with self-consistent electrostatic field within plasmas, this stimulates a lot of investigations on how to set up large-amplitude electrostatic wave within plasmas *via* various stimulus [9-31]. Two familiar conceptions, laser wakefield [13] and plasma wakefield [12,14], are typical examples of such an large-amplitude electrostatic wave. In 1980s, authors have set up basic 1-D theories on these two conceptions [12-14]. Then, during following several decades, a lot of investigations have been addressed to various wakefield-related problems [15-31].

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Despite so many related investigations on so-called wakefield, however, there exists still a basic question, whether or not does a realistic 3-D electrostatic plasma wave exist? Some authors have found, from PIC simulation, that the driven plasma density wave is accompanied by a similar magnetic energy density wave [32]. Because earlier 1-D theories [12-14] cannot include magnetic fields effect [12-14], this implies, to some extent, that we should set up a stricter theory on wakefields of various stimulus rather than simply treat them as electrostatic structures. Moverover, even though we ignore this basic question, we should be aware of that the stimulus to excite these wakefields usually do not correspond to zero self-consistent magnetic field. For example, laser pulse, (the stimulus driving laser wakefield,) has a laser magnetic field and hence is a ``magnetized'' stimulus. Because in realistic situation an electron beam (the stimulus of plasma wakefield) is usually stored in magnetic apparatus such as storage-ring, it is also often a magnetized electron beam. These ``magnetized'' stimulus also force us to carefully treat their wakefields. Some authors have noticed that these wakefields are electromagnetic and set up a related nonlinear theory based on fluid approximation [27,39]. Also, some effort have been paid to experimentally probe the magnetic fields structure of wakefields [33]. But the stress of their approximated fluid theory [27] is not focused on magnetic structure of every wake and hence does not predict those latter results found from PIC simulation [32, 37, 38].

Indeed, those earlier investigations displaying phase space holes [1-6] have revealed that electromagnetic self-consistent field could also lead to high-energy charged particles. Moreover, high-energy particles generated from magnetic reconnection [32,34-36] also suggest that particle acceleration should not merely be related with electrostatic structure within plasmas. Therefore, even the wakefield is electromagnetic, particle acceleration is still available. The particle acceleration, or the generation of high-energy particles, from electromagnetic wakefield is a part of the purpose of next work. Strictly speaking, for a realistic

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``magnetized" stimulus, if its wake is ``automatically" taken as an electrostatic one, the strength of such an electrostatic wake might be greatly over-evaluated and hence the related estimation on some aspects of acceleration quality might be very optimistic.

As the first step of the whole investigation, this work is focused on 3-D nonlinear plasma electrostatic wave. Collective motion mode of numerous charged particles is a traditional subject in plasma physics. People have realized that it is of more practical value to study these collective motion modes on 3-D model. For instance, many theories have been devoted to 3-D nonlinear plasma electromagnetic wave [40-45]. However, the access of most of these theories [40-45] are mainly focused on the exact functional relationships among various physical quantities. For example, two equations of radial and axial components of electron momentum, $p_\perp$ and $p_\parallel$, are presented in ref. [45] but these two equations, as well as $p_\perp$ and $p_\parallel$, are both independent of transverse coordinates r and θ. Similar approximation in which all related physical quantities are independent of r and θ is also widely adopted in a few of related theories [46-52]. Only in few theories [53, 54], transverse dynamics is really studied because the dependence of related physical quantities on the radial coordinates r is taken into account. Likewise, even though recently there are some works [55-62] addressing to 3-D electrostatic structure, authors still do not seriously taken into account the dependence of related physical quantities on the transverse coordinates r and θ, and hence transverse dynamics of 3-D electrostatic structure is still not yet really studied. In short, when dealing with a 3-D question, we need not only to treat all vectors as having three components but also to view every component as depending on both transverse and axial coordinates. A true three-dimensional system should consider spatial variations in three orthogonal directions; for example, r, z, and θ. In most of above-mentioned works, the importance of the latter requirement, i.e. to view every component as depending on both transverse and axial coordinates, seems to be not fully appreciated.

For a nonlinear wave which is periodic along its longitudinal direction, whether or not its transverse shape could warrant this longitudinal periodicity seems to be not noticed by researchers. Here, our following detailed investigations reveals that such a longitudinal periodicity has a severe constraint on transverse shape. In other words, a 3-D nonlinear electrostatic plasma wave cannot have arbitrary transverse shape. Instead, only few allowed transverse shapes could ensure the longitudinal periodicity. The paper is organized as follows: Our theory is presented in details in section II. Section III is for related numerical experiments. We summarize the importance of this newly revealed property of 3-D nonlinear electrostatic plasma wave to plasma-based acceleration in section IV.

2 THEORY

2.1. Starting model Equations

It is well known that plasma is a system of charged particles, which are interacting through their self-consistent fields. Such a classic particles system, according to statistic mechanics and classic mechanics [63-69], could be strictly described by Liouville theorem and Hamilton's equations

$$d_t f\left(\vec{r}(t), \vec{p}(t), t\right) = 0; \tag{1}$$

$$d_t \vec{r}(t) = \frac{\partial H}{\partial \vec{p}(t)} = \vec{\upsilon}(t); d_t \vec{p}(t) = -\frac{\partial H}{\partial \vec{r}(t)}; \tag{2}$$

They will lead to well-known Vlasov equation (VE). Maybe someone will find that according to Klimontovich-Dupree method [67], a functional

$$N(X,V,t) \equiv \sum_i \delta\left(X - x_i(t)\right)\delta\left(V - d_t x_i(t)\right) \tag{3}$$

in which X and V are independent of t, meets VE and hence conclude that the VE is defined over (X,V,t)-space. However, Klimontovich-Dupree method could also be extended to following functional

$$N\left(x(t), d_t x(t), t\right) \equiv \sum_i \delta\left(x(t) - x_i(t)\right)\delta\left(d_t x(t) - d_t x_i(t)\right) \tag{4}$$

One can find that it also meets a VE defined over $\left(x(t), \upsilon(t), t\right)$-space. Therefore, for generality, we take VE as being defined over $\left(x(t), \upsilon(t), t\right)$-space.

This fundamental fact reminds us that the VE is for an element whose trajectory in phase space is $\left[\vec{r}(t), \vec{p}(t)\right]$. Strict expression of VE should outstand time-dependence of $\vec{r}(t)$ and $\vec{\upsilon}(t)$

$$0 = \partial_t f\left(\vec{r}(t), \vec{\upsilon}(t), t\right) + d_t \vec{r}(t) \cdot \partial_{\vec{r}(t)} f\left(\vec{r}(t), \vec{\upsilon}(t), t\right)$$
$$+ d_t \vec{\upsilon}(t) \cdot \partial_{\vec{\upsilon}(t)} f\left(\vec{r}(t), \vec{\upsilon}(t), t\right) \tag{5}$$

In contrast, Maxwell equations (MEs) are for fields of physical quantities and are defined over 4-D $\left(\vec{R}, t\right)$-space. In term of fluid mechanics, VE and its fluid derivations are expressed by Lagrangian variables while MEs are by Eulerian variables. According to any fluid mechanics textbook [69], components of the Lagrangian variables $\left(x(t), t\right)$ are not independent mutually and hence $\frac{dx(t)}{dt}$ is not always $= 0$. In contrast, components of the Eulerian variables (X, t) are independent mutually and hence there always exists $\frac{dX}{dt} = 0$.

According to strict theoretical results [70], one can derived an equation for fluid velocity u from Eq.(5).

$$0 = \partial_t \frac{\vec{u}\left(\vec{r}(t), t\right)}{\sqrt{1 - |\vec{u}|^2\left(\vec{r}(t), t\right)}} + \vec{u}\left(\vec{r}(t), t\right) * \nabla_{\vec{r}(t)} \frac{\vec{u}\left(\vec{r}(t), t\right)}{\sqrt{1 - |\vec{u}|^2\left(\vec{r}(t), t\right)}}$$
$$+ \vec{E}\left(\vec{r}(t), t\right) + \vec{u}\left(\vec{r}(t), t\right) \times \vec{B}\left(\vec{r}(t), t\right), \tag{6}$$

which is very alike to the Eulerian equation in fluid mechanics [69]. Note that it is expressed by Lagrangian variables $\left(\overleftarrow{r}(t), t\right)$. In addition, MEs are expressed by Eulerian variables $\left(\overleftarrow{R}, t\right) = (X, Y, Z, t)$, where $\overleftarrow{R}$ and t are independent variables

$$\partial_t \overleftarrow{E} = n\overleftarrow{u} + \nabla \times \overleftarrow{B}; \tag{7}$$

$$\nabla \cdot \overleftarrow{E} = -n + ZN_i; \tag{8}$$

$$\nabla \times \overleftarrow{E} = -\partial_t \overleftarrow{B}; \tag{9}$$

$$\nabla \cdot \overleftarrow{B} = 0. \tag{10}$$

N_i is ion density and n is electron density.

2.2. How to Uniformly Express Starting Model Equations

Obviously, to solve Eqs. (6-10), we should express all of them uniformly by Lagrangian variables or Eulerian variables. From the formula connecting Lagrangian variables $\left(x_0(\tau), \tau\right)$ and Eulerian variables (X, t) (see Pg.35 of ref. [68]. Here, we loyally use the original symbol for Lagrangian variable in [68] $x_0(\tau)$, which is just our symbol for Lagrangian variable $x(\tau)$, from Eq.(11) to Eq.(18))

$$\tau \equiv t, x_0(\tau) \equiv X - \int_0^\tau d\tau' u\left(x_0(\tau), \tau'\right) \tag{11}$$

one can obtain (see Pg.35 of ref. [68])

$$\partial_X = \frac{d\tau}{dX}\partial_\tau + \frac{dx_0(\tau)}{dX}\partial_{x_0(\tau)} = \left[1 + \int_0^\tau d\tau' \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)\right]^{-1} \partial_{x_0(\tau)}; \tag{12}$$

$$\begin{aligned} \partial_t &= \frac{d\tau}{dt}\partial_\tau + \frac{dx_0(\tau)}{dt}\partial_{x_0(\tau)} = \\ &\partial_\tau - u\left(x_0(\tau), \tau\right)\left[1 + \int_0^\tau d\tau' \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)\right]^{-1} \partial_{x_0(\tau)}. \end{aligned} \tag{13}$$

Note that the formula (11) implies implicitly a relation

$$d_\tau x_0(\tau) \equiv -u\left(x_0(\tau), \tau\right) - \int_0^\tau d\tau' d_\tau x_0(\tau) \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)$$

$$\text{or} \, d_\tau x_0(\tau) \equiv -\left[1 + \int_0^\tau d\tau' \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)\right]^{-1} u\left(x_0(\tau), \tau\right), \tag{14}$$

which has been applied when deducing the formula (13). Therefore, we have

$$\begin{aligned} &\partial_\tau + u\left(x_0(\tau), \tau\right)\partial_{x_0(\tau)} \\ &= \partial_\tau - \frac{u\left(x_0(\tau), \tau\right)}{\left[1 + \int_0^\tau d\tau' \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)\right]}\partial_{x_0(\tau)} + \\ &u\left(x_0(\tau), \tau\right)\frac{\left[2 + \int_0^\tau d\tau' \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)\right]}{\left[1 + \int_0^\tau d\tau' \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)\right]}\partial_{x_0(\tau)} \\ &= \partial_t - d_\tau x_0(\tau)\left[2 + \int_0^\tau d\tau' \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)\right]\partial_{x_0(\tau)} \end{aligned}$$

$$\begin{aligned} &= \partial_t - d_\tau x_0(\tau)\left[2 + \int_0^\tau d\tau' \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)\right] \\ &\left[1 + \int_0^\tau d\tau' \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)\right]\partial_X. \end{aligned} \tag{15}$$

and thus

$$\begin{aligned} &Force\left(x_0(\tau), \tau\right) \\ &= \left[\partial_\tau + u\left(x_0(\tau), \tau\right)\partial_{x_0(\tau)}\right] Q\left(x_0(\tau), \tau\right) \\ &= \left\{ \begin{aligned} &\partial_t - d_\tau x_0(\tau)\left[2 + \int_0^\tau d\tau' \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)\right] \\ &\left[1 + \int_0^\tau d\tau' \partial_{x_0(\tau)} u\left(x_0(\tau), \tau'\right)\right]\partial_X. \end{aligned} \right\} \\ &Q\left(x_0(\tau), \tau\right). \end{aligned} \tag{16}$$

When $d_\tau x_0(\tau) = 0$, we will have

$$\partial_t Q(X, t) = Force(X, t). \tag{17}$$

Here, we should note that above formula connecting Lagrangian variables and Eulerian variables are for 1-D case. In more complicated 3-D case, it should be a component of u, $u_{\parallel}$, that connects R and $r_0(\tau)$

$$\tau \equiv t, r_0(\tau) \equiv R - \int_0^\tau d\tau' u_{\parallel}\left(r_0(\tau), \tau'\right). \tag{18}$$

Here, the subindex $\parallel$ means being parallel to the trajectory. Another component of u, $u_\perp$, will have contribution to Lorentz force if $B \neq 0$.

As stressed in fluid mechanics [69], the Eulerian equation, $\partial_t \overleftarrow{u} + (\overleftarrow{u} \cdot \nabla)\overleftarrow{u} = Force$, is for a specified fluid element whose trajectory is $\left(x(t), y(t), z(t)\right)$ (because the variation of any physical quantity in this element is represented by $Q\left(x(t+\delta t), y(t+\delta t), z(t+\delta t), t+\delta t\right) -$) $Q\left(x(t), y(t), z(t), t\right)$. When writing

$$\begin{aligned} \frac{du_x}{dt} &= \frac{\partial u_x}{\partial t} + \frac{\partial u_x}{\partial x}\frac{dx}{dt} + \frac{\partial u_x}{\partial y}\frac{dy}{dt} + \frac{\partial u_x}{\partial z}\frac{dz}{dt} = \frac{\partial u_x}{\partial t} + \left(\overleftarrow{u} \cdot \nabla\right)u_x; \\ \frac{du_y}{dt} &= \frac{\partial u_y}{\partial t} + \frac{\partial u_y}{\partial x}\frac{dx}{dt} + \frac{\partial u_y}{\partial y}\frac{dy}{dt} + \frac{\partial u_y}{\partial z}\frac{dz}{dt} = \frac{\partial u_y}{\partial t} + \left(\overleftarrow{u} \cdot \nabla\right)u_y; \\ \frac{du_z}{dt} &= \frac{\partial u_z}{\partial t} + \frac{\partial u_z}{\partial x}\frac{dx}{dt} + \frac{\partial u_z}{\partial y}\frac{dy}{dt} + \frac{\partial u_z}{\partial z}\frac{dz}{dt} = \frac{\partial u_z}{\partial t} + \left(\overleftarrow{u} \cdot \nabla\right)u_z. \end{aligned} \tag{19}$$

we have indeed taken for granted that $u_{x,y,z}$ are expressed by Lagrangian variables $\left(x(t), y(t), z(t), t\right)$ and meet a relation

$$u_{x,y,z}\left(x(t), y(t), z(t), t\right) = d_t x, y, z. \tag{20}$$

Namely, $u_{x,y,z}\left(x(t), y(t), z(t), t\right)$ is the velocity of a specified fluid element. Now, to solve Eqs.(6-10), we must look for an equation for $u(X, Y, Z, t)$.

2.3. Alternative Consideration Based on Integral Equation

In last subsection, we have repeated a strict procedure of ``translating'' any equation expressed by Lagrangian variables to that by Eulerian variables [68]. Actually, this ``translation'' procedure can be illustrated in a more intuitive way. Maybe the integral form is easier for us to understand Eq.(6)

$$\overleftarrow{p}\left(\overleftarrow{r}(t),t\right)=\int_0^t \overleftarrow{F}\left(\overleftarrow{r}\left(t'\right),t'\right)dt'+\overleftarrow{p}\left(\overleftarrow{r}(0),0\right). \tag{21}$$

where the work done by $\overleftarrow{F}$ can be expressed as the summation of two terms

$$\int_0^{\delta t}\overleftarrow{F}\left(x+u\tau',t+\tau'\right)d\tau'=Q(x+u\delta t,t+\delta t)-Q(x,t)$$

$$=\left[Q(x+u\delta t,t+\delta t)-Q(x,t+\delta t)\right]+\left[Q(x,t+\delta t)-Q(x,t)\right] \tag{22}$$

By Taylor expanding $\overleftarrow{p}$ and $\overleftarrow{F}$ around time-independent space coordinate $\overleftarrow{R}$

$$\overleftarrow{p}\left(\overleftarrow{r}(t),t\right)=\sum_{i=0}^{\infty}\frac{1}{i!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^i\partial^i_{\overleftarrow{R}^i}\overleftarrow{p}\left(\overleftarrow{R},t\right); \tag{23}$$

$$\overleftarrow{F}\left(\overleftarrow{r}\left(t'\right),t'\right)=\sum_{i=0}^{\infty}\frac{1}{i!}\left[\overleftarrow{r}\left(t'\right)-\overleftarrow{R}\right]^i\partial^i_{\overleftarrow{R}^i}\overleftarrow{F}\left(\overleftarrow{R},t'\right), \tag{24}$$

we obtain

$$\begin{aligned}&\sum_{i=0}^{\infty}\frac{1}{i!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^i\partial^i_{\overleftarrow{R}^i}\overleftarrow{p}\left(\overleftarrow{R},t\right)\\&=\int_0^t\sum_{i=0}^{\infty}\frac{1}{i!}\left[\overleftarrow{r}\left(t'\right)-\overleftarrow{R}\right]^i\partial^i_{\overleftarrow{R}^i}\overleftarrow{F}\left(\overleftarrow{R},t'\right)dt'+\overleftarrow{p}\left(\overleftarrow{r}(0),0\right).\end{aligned} \tag{25}$$

Making d_t on this equation

$$\begin{aligned}&d_t\left\{\sum_{i=0}^{\infty}\frac{1}{i!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^i\partial^i_{\overleftarrow{R}^i}\overleftarrow{p}\left(\overleftarrow{R},t\right)\right\}\\&=\sum_{i=0}^{\infty}\frac{1}{i!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^i\partial^i_{\overleftarrow{R}^i}\overleftarrow{F}\left(\overleftarrow{R},t\right),\end{aligned} \tag{26}$$

we obtain

$$\begin{aligned}&\sum_{i=0}^{\infty}\frac{1}{i!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^i\partial^i_{\overleftarrow{R}^i}\overleftarrow{F}\left(\overleftarrow{R},t\right)-\left\{\sum_{i=0}^{\infty}\frac{1}{i!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^i d_t\partial^i_{\overleftarrow{R}^i}\overleftarrow{p}\left(\overleftarrow{R},t\right)\right\}\\&=\left\{\sum_{i=0}^{\infty}\frac{1}{(i-1)!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^{i-1}d_t\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]\partial^i_{\overleftarrow{R}^i}\overleftarrow{p}\left(\overleftarrow{R},t\right)\right\}\\&=.d_t\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]\left\{\sum_{i=0}^{\infty}\frac{1}{(i-1)!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^{i-1}\partial^i_{\overleftarrow{R}^i}\overleftarrow{p}\left(\overleftarrow{R},t\right)\right\}\\&=d_t\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]\partial_{\overleftarrow{R}}\overleftarrow{p}\left(\overleftarrow{r}(t),t\right)\\&=0.\end{aligned} \tag{27}$$

Here, we have used a fact that the Taylor expansion $\left\{\sum_{i=0}^{\infty}\frac{1}{(i-1)!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^{i-1}\partial^i_{\overleftarrow{R}^i}\overleftarrow{p}\left(\overleftarrow{R},t\right)\right\}$ is just $\partial_{\overleftarrow{R}}\overleftarrow{p}\left(\overleftarrow{r}(t),t\right)$, which is obviously $=0$

$$\left\{\sum_{i=0}^{\infty}\frac{1}{(i-1)!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^{i-1}\partial^i_{\overleftarrow{R}^i}\overleftarrow{p}\left(\overleftarrow{R},t\right)\right\}=\partial_{\overleftarrow{R}}\overleftarrow{p}\left(\overleftarrow{r}(t),t\right)=0.28 \tag{28}$$

Finally, Eq.(27) means

$$\begin{aligned}0&=\sum_{i=0}^{\infty}\frac{1}{i!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^i\partial^i_{\overleftarrow{R}^i}\overleftarrow{F}\left(\overleftarrow{R},t\right)-\left\{\sum_{i=0}^{\infty}\frac{1}{i!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^i d_t\partial^i_{\overleftarrow{R}^i}\overleftarrow{p}\left(\overleftarrow{R},t\right)\right\}\\&=\sum_{i=0}^{\infty}\frac{1}{i!}\left[\overleftarrow{r}(t)-\overleftarrow{R}\right]^i\partial^i_{\overleftarrow{R}^i}\left[\overleftarrow{F}\left(\overleftarrow{R},t\right)-d_t\overleftarrow{p}\left(\overleftarrow{R},t\right)\right].\end{aligned} \tag{29}$$

Because Eq.(29) is valid for any trajectory $\overleftarrow{r}(t)$, therefore, there should be

$$0=\overleftarrow{F}\left(\overleftarrow{R},t\right)-d_t\overleftarrow{p}\left(\overleftarrow{R},t\right)=\overleftarrow{F}\left(\overleftarrow{R},t\right)-\partial_t\overleftarrow{p}\left(\overleftarrow{R},t\right) \tag{30}$$

2.4. Uniformly Expressed Equations

Thus, equations to be solved can be uniformly expressed by Eulerian variables (X,Y,Z,t)

$$0=\partial_t\frac{\overleftarrow{u}}{\sqrt{1-|u|^2}}+\overleftarrow{E}+\overleftarrow{u}\times\overleftarrow{B}; \tag{31}$$

$$\partial_t\overleftarrow{E}=n\overleftarrow{u}+\nabla\times\overleftarrow{B}; \tag{32}$$

$$\nabla\cdot\overleftarrow{E}=-n+ZN_i; \tag{33}$$

$$\nabla\times\overleftarrow{E}=-\partial_t\overleftarrow{B}; \tag{34}$$

$$\nabla\cdot\overleftarrow{B}=0.35 \tag{35}$$

In previous paragraph, in order to discriminate between Lagrangian variables and Eulerian ones, we have denoted them with lowercase letters x,y,z and uppercase letters X,Y,Z respectively. In following paragraph, we only need to deal with Eulerian variables. For simplicity in symbols, we use lowercase letters x,y,z to denote the Eulerian variables in following paragraph. Namely, in following paragraph, lowercase letters x,y,z are no longer to denote functions of t and instead independent of t.

2.5. Electrostatic Wave in Real Space

We are interested in Eqs.(31-35) at $\overleftarrow{B}=0$ case. In particular, we wish to find solutions which is static in a moving frame of a constant velocity $\frac{1}{\eta}$

$$\xi=\eta z-t;\ p=\frac{u}{\sqrt{1-[u]^2}};\ u(z,t,r,\theta)=u(\xi,r,\theta), \tag{36}$$

where r and θ stand for transverse coordinate and z for axial coordinate in the cylindric frame. Sometime the term ``electrostatic" is understood loosely as referring to a time-independent $B=\overline{B}(r,z,\theta)\neq 0$. However, such a time-independent $B=\overline{B}(r,z,\theta)\neq 0$, which is ``time-dependent" relative to the $\frac{1}{\eta}$-frame, does not favor the presence of a plasma electrostatic wave whose E is static relative to the $\frac{1}{\eta}$-frame. Unless such a time-dependent B is also z--independent, otherwise, such a running wave form $E=E(r,\eta z-t,\theta)$ will not appear. This could be verified by strictly analyzing Eqs.(31-35). For transverse inhomogeneous static $B=\overline{B}(r,\theta)\neq 0$, we could find that there are three corresponding static quantities: $\overline{E}$, $\overline{n}$ and $\overline{u}$ which meet $\overline{E}+\overline{u}\times\overline{B}=0$, $\nabla\cdot\overline{E}=\overline{n}-\alpha N$ (α is a constant coefficient) and $\nabla\times\overline{B}=\overline{n}\overline{u}$. An equation of $p\cdot\ \overline{B}$ could be derived in a same way of deriving Eq.(50) (see below). Then, because $\overline{B}$ is ξ-independent, we could obtain an equation of p which depends on $\overline{B}$. But we could find that because such a $B=\overline{B}(r,\theta)\neq 0$ does not couple with $\partial_\xi p$, it will not affect periodicity requirement $\beta\propto\frac{1}{r}$, which will be presented below. Namely, a severe constraint on transverse shape for warranting longitudinal periodicity still holds in $B=\overline{B}(r,\theta)\neq 0$ case. Detailed investigations on such a $B=\overline{B}(r,\theta)\neq 0$ case will be presented in other works.

In the 3-D case, we introduce two functions β and λ to denote the ratio between velocity components along different directions

$$u_r=\beta u_z;\, p_r=\beta p_z, \tag{37}$$

$$u_\theta=\lambda u_z;\, p_\theta=\lambda p_z, \tag{38}$$

and Eqs.(31-35) yield following formulas

$$E_z=-\partial_t p_z=\partial_\xi p_z; \tag{39}$$

$$E_r=\partial_\xi p_r=\partial_\xi(\beta p_z)=\beta\partial_\xi p_z+(\partial_\xi\beta)p_z, \tag{40}$$

$$E_\theta=\partial_\xi p_\theta=\partial_\xi(\lambda p_z)=\lambda\partial_\xi p_z+(\partial_\xi\lambda)p_z \tag{41}$$

$$-\partial_t B_\theta=0=\left[\eta\partial_\xi E_r-\partial_r E_z\right]=\eta\partial_\xi\left[\beta\partial_\xi p_z+(\partial_\xi\beta)p_z\right]-\partial_r E_z$$

$$=(2\eta\partial_\xi\beta)\partial_\xi p_z+\eta\beta\partial_{\xi\xi}p_z+\eta(\partial_{\xi\xi}\beta)p_z-\partial_r\partial_\xi p_z. \tag{42}$$

$$-\partial_t B_r=0=\left[\frac{1}{r}\partial_\theta E_z-\eta\partial_\xi E_\theta\right]=\frac{1}{r}\partial_\theta E_z-\eta\partial_\xi\left[\lambda\partial_\xi p_z+(\partial_\xi\lambda)p_z\right]$$

$$=\frac{1}{r}\partial_\theta\partial_\xi p_z-(2\eta\partial_\xi\lambda)\partial_\xi p_z-\eta\lambda\partial_{\xi\xi}p_z-\eta(\partial_{\xi\xi}\lambda)p_z \tag{43}$$

$$-\partial_t B_z=0=\frac{1}{r}\left[\partial_r(rE_\theta)-\partial_\theta E_r\right]$$

$$=\frac{1}{r}\left\{\begin{array}{l} r\partial_r\left[\lambda\partial_\xi p_z+(\partial_\xi\lambda)p_z\right]+\left[\lambda\partial_\xi p_z+(\partial_\xi\lambda)p_z\right]- \\ \partial_\theta\left[\beta\partial_\xi p_z+(\partial_\xi\beta)p_z\right]\end{array}\right\}$$

$$=\frac{1}{r}\left\{\begin{array}{c}\left[r\lambda\partial_r\partial_\xi p_z-\beta\partial_\theta\partial_\xi p_z\right]+ \\ \left[r\partial_r\partial_\xi\lambda+\partial_\xi\lambda-\partial_\theta\partial_\xi\beta\right]p_z \\ +\left[\begin{array}{l} r(\partial_r\lambda)\partial_\xi p_z+r(\partial_\xi\lambda)\partial_r p_z+\lambda\partial_\xi p_z- \\ (\partial_\theta\beta)\partial_\xi p_z-(\partial_\xi\beta)\partial_\theta p_z\end{array}\right]\end{array}\right\} \tag{44}$$

Because of Eqs.(42,43), we could rewrite Eq.(44) as

0

$$=\left\{\begin{array}{c}\left[\begin{array}{c} r\lambda(2\eta\partial_\xi\beta)\partial_\xi p_z+r\lambda\eta\beta\partial_{\xi\xi}p_z+r\lambda\eta(\partial_{\xi\xi}\beta)p_z \\ -\beta r(2\eta\partial_\xi\lambda)\partial_\xi p_z-\beta r\eta\lambda\partial_{\xi\xi}p_z-\beta r\eta(\partial_{\xi\xi}\lambda)p_z\end{array}\right] \\ +\left[r(\partial_r\lambda)\partial_\xi p_z+r(\partial_\xi\lambda)\partial_r p_z+\lambda\partial_\xi p_z-(\partial_\theta\beta)\partial_\xi p_z-(\partial_\xi\beta)\partial_\theta p_z\right] \\ +\left[r\partial_r\partial_\xi\lambda+\partial_\xi\lambda-\partial_\theta\partial_\xi\beta\right]p_z\end{array}\right\}$$

$$=\left\{\begin{array}{c}\left[\begin{array}{c} r\lambda(2\eta\partial_\xi\beta)\partial_\xi p_z-\beta r(2\eta\partial_\xi\lambda)\partial_\xi p_z+r(\partial_r\lambda)\partial_\xi p_z \\ +r(\partial_\xi\lambda)\partial_r p_z+\lambda\partial_\xi p_z-(\partial_\theta\beta)\partial_\xi p_z-(\partial_\xi\beta)\partial_\theta p_z\end{array}\right] \\ +\left[r\lambda\eta(\partial_{\xi\xi}\beta)-\beta r\eta(\partial_{\xi\xi}\lambda)+r\partial_r\partial_\xi\lambda+\partial_\xi\lambda-\partial_\theta\partial_\xi\beta\right]p_z\end{array}\right\}, \tag{45}$$

in which all second-order derivative terms of p_z disappear. Likewise, Eqs.(33,34) can be written as

$$\left[ZN_i-n\right]$$

$$=\eta\frac{1}{r}\partial_\xi\left[rE_z\right]+\frac{1}{r}\partial_r\left[rE_r\right]+\frac{1}{r}\partial_\theta E_\theta$$

$$=\eta\partial_{\xi\xi}p_z+\partial_r\left[\beta\partial_\xi p_z+(\partial_\xi\beta)p_z\right]+\frac{1}{r}\left[\beta\partial_\xi p_z+(\partial_\xi\beta)p_z\right]$$

$$+\frac{1}{r}\partial_\theta\left[\lambda\partial_\xi p_z+(\partial_\xi\lambda)p_z\right]$$

$$=\left[\eta\partial_{\xi\xi}p_z+\beta\partial_r\partial_\xi p_z+\frac{\lambda}{r}\partial_\theta\partial_\xi p_z\right]+$$

$$\left[\left(\partial_r\beta+\frac{\beta}{r}+\frac{1}{r}\partial_\theta\lambda\right)\partial_\xi p_z+(\partial_\xi\beta)\partial_r p_z+\frac{1}{r}(\partial_\xi\lambda)\partial_\theta p_z\right]$$

$$+\left[\partial_r\partial_\xi\beta+\frac{\partial_\xi\beta}{r}+\frac{1}{r}\partial_\theta\partial_\xi\lambda\right]p_z. \tag{46}$$

$$\left[\nabla\times B\right]|_z=0=\partial_t E_z-nu_z=-\partial_\xi E_z-nu_z; \tag{47}$$

$$\left[\nabla\times B\right]|_r=0=\partial_t E_r-nu_r=-\partial_\xi E_r-nu_r; \tag{48}$$

$$\left[\nabla\times B\right]|_\theta=0=\partial_t E_\theta-nu_\theta=-\partial_\xi E_\theta-nu_\theta; \tag{49}$$

From Eqs.(39-49), we can obtain

$$\partial_{\xi\xi} p_z = -nu_z$$

$$= \left\{ \begin{array}{l} \left[\eta \partial_{\xi\xi} p_z + \beta \partial_r \partial_\xi p_z + \frac{\lambda}{r} \partial_\theta \partial_\xi p_z \right] + \\ \left[\begin{array}{l} \left(\partial_r \beta + \frac{\beta}{r} + \frac{1}{r} \partial_\theta \lambda \right) \partial_\xi p_z + \\ \left(\partial_\xi \beta \right) \partial_r p_z + \frac{1}{r} \left(\partial_\xi \lambda \right) \partial_\theta p_z \end{array} \right] \\ + \left[\partial_r \partial_\xi \beta + \frac{\partial_\xi \beta}{r} + \frac{1}{r} \partial_\theta \partial_\xi \lambda \right] p_z \end{array} \right\}$$

$$\frac{p_z}{\sqrt{1 + \left(1 + \beta^2 + \lambda^2\right) p_z^2}}$$

$$= \left\{ \begin{array}{l} \left[\eta \left(1 + \beta^2 + \lambda^2\right) \partial_{\xi\xi} p_z \right] + \\ \left[\begin{array}{l} \partial_r \partial_\xi \beta + \frac{\partial_\xi \beta}{r} + \beta \eta \partial_{\xi\xi} \beta + \\ \frac{1}{r} \partial_\theta \partial_\xi \lambda + \lambda \eta \partial_{\xi\xi} \lambda \end{array} \right] p_z \\ + \left[\begin{array}{l} \left(\begin{array}{l} \partial_r \beta + \frac{\beta}{r} + \frac{1}{r} \partial_\theta \lambda + \\ 2 \eta \beta \partial_\xi \beta + 2 \eta \lambda \partial_\xi \lambda \end{array} \right) \partial_\xi p_z + \\ \left(\partial_\xi \beta \right) \partial_r p_z + \frac{1}{r} \left(\partial_\xi \lambda \right) \partial_\theta p_z \end{array} \right] \end{array} \right\}$$

$$\frac{p_z}{\sqrt{1 + \left(1 + \beta^2 + \lambda^2\right) p_z^2}}. \tag{50}$$

Likewise, two similar equations for $p_r = \beta p_z$ and $p_\theta = \lambda p_\theta$ exist

$$-n\beta u_z = \partial_{\xi\xi} \left(\beta p_z \right) = \beta \partial_{\xi\xi} p_z + 2 \partial_\xi \beta \partial_\xi p_z + p_z \partial_{\xi\xi} \beta; \tag{51}$$

$$-n\lambda u_z = \partial_{\xi\xi} \left(\lambda p_z \right) = \lambda \partial_{\xi\xi} p_z + 2 \partial_\xi \lambda \partial_\xi p_z + p_z \partial_{\xi\xi} \lambda, \tag{52}$$

and hence there are

$$2 \partial_\xi \beta \partial_\xi p_z + p_z \partial_{\xi\xi} \beta = 0; \tag{53}$$

$$2 \partial_\xi \lambda \partial_\xi p_z + p_z \partial_{\xi\xi} \lambda = 0, \tag{54}$$

which yields

$$\partial_\xi \beta = \frac{C_1 \left(r, \theta\right)}{p_z^2} \text{or} \partial_\xi \beta = 0; \tag{55}$$

$$\partial_\xi \lambda = \frac{C_2 \left(r, \theta\right)}{p_z^2} \text{or} \partial_\xi \beta = 0; \tag{56}$$

where $C_{1,2} \left(r\right)$ are two binary functions of r and θ.

Obviously, if p_z is a periodic function of ξ, the equation of p_z should be able to be transformed into a first integral. Because β and λ appear in Eq.(50), if β (and λ) meets the former case $\partial_\xi \beta = \frac{C_1 \left(r, \theta\right)}{p_z^2}$ (and $\partial_\xi \lambda = \frac{C_2 \left(r, \theta\right)}{p_z^2}$), Eq.(50) will be very complicated and cannot warrant a first integral of p_z, which implies p_z being a periodic function of ξ, existing. Therefore, for finding periodic solutions of Eq.(50), we only need to consider the latter case $\left(\partial_\xi \beta, \partial_\xi \lambda \right) = \left(0, 0\right)$ in which $\beta = \beta \left(r, \theta\right)$, as well as $\lambda = \lambda \left(r, \theta\right)$, is a binary function of r and θ. Thus, we rewrite Eq.(50) as

$$\left[1 - \frac{\eta \left(1 + \beta^2 + \lambda^2\right) p_z}{\sqrt{1 + \left(1 + \beta^2 + \lambda^2\right) p_z^2}} \right] \partial_{\xi\xi} p_z$$

$$= \left\{ \left[\left(\partial_r \beta + \frac{\beta}{r} + \frac{1}{r} \partial_\theta \lambda \right) \partial_\xi p_z \right] - ZN_i \right\} \frac{p_z}{\sqrt{1 + \left(1 + \beta^2 + \lambda^2\right) p_z^2}}. \tag{57}$$

It is well-known that such a general form

$$f_2 \left(y\right) y'' + f_1 \left(y\right) y' + f_0 \left(y\right) = 0 \tag{58}$$

which contains a linear term of y', cannot correspond to a first integral unless $f_1 \left(y\right) = 0$. Therefore, a periodic solution of p_z implies β and λ meeting

$$\partial_r \beta + \frac{\beta}{r} + \frac{\partial_\theta \lambda}{r} = 0. \tag{59}$$

Moreover, in the latter case $\left(\partial_\xi \beta, \partial_\xi \lambda \right) = \left(0, 0\right)$, Eq.(46) will lead to $\left[r \left(\partial_r \lambda \right) + \lambda - \left(\partial_\theta \beta \right) \right] \partial_\xi p_z = 0$ or

$$\partial_r \lambda + \frac{\lambda}{r} - \frac{\partial_\theta \beta}{r} = 0. \tag{60}$$

From Eqs.(59,60), we could find that if p_z is a periodic solution, β must meet

$$\left[\partial_r + \frac{1}{r} \right]^2 \beta + 2 \frac{1}{r} \left[\partial_r + \frac{1}{r} \right] \beta + \frac{1}{r^2} \partial_{\theta\theta} \beta = 0. \tag{61}$$

Likewise, λ meets a same equation

$$\left[\partial_r + \frac{1}{r} \right]^2 \lambda + 2 \frac{1}{r} \left[\partial_r + \frac{1}{r} \right] \lambda + \frac{1}{r^2} \partial_{\theta\theta} \lambda = 0. \tag{62}$$

Note that Eq.(61) only contains linear terms of β and hence the well-known *method of separation of variables* is applicable. After obtaining β from Eq.(61), we can obtain λ through Eq.(59) and hence we have (where D is a constant and ν meets $\nu \left(\nu - 3\right) + 2 - k^2 = 0$)

$$\beta = \frac{1}{r^\nu} D \cos \left(k\theta\right); \lambda = \frac{\nu - 1}{r^\nu k} D \sin \left(k\theta\right) \tag{63}$$

Likewise, after obtaining λ from Eq.(62), we could obtain β through Eq.(60) and hence we have

$$\lambda = \frac{1}{r^\nu} D \sin\left(k\theta\right); \beta = \frac{\nu - 1}{r^\nu k} D \cos\left(k\theta\right) \tag{64}$$

Obviously, one could directly verify that if these two sets of solutions are equivalent, one of β and λ must be 0. Moreover, from Eqs.(59,60) and Eqs.(61,62), we can have other two sets of possible solutions

$$\beta = \frac{1}{r^\nu} D \sin\left(k\theta\right); \lambda = \frac{1-\nu}{r^\nu k} D \cos\left(k\theta\right), \tag{65}$$

$$\lambda = \frac{1}{r^\nu} D \cos\left(k\theta\right); \beta = \frac{1-\nu}{r^\nu k} D \sin\left(k\theta\right). \tag{66}$$

We can also directly verify that if these two sets of solutions are equivalent, one of β and λ must be 0. Therefore, the solutions of β and λ must be

$$\beta = \frac{1}{r} D; \lambda = 0. \tag{67}$$

$$\text{or} \beta = 0; \lambda = \frac{1}{r} D. \tag{68}$$

The former solution implies transverse isotropy (i.e., θ-independence) and the latter implies rotating around z-axis.

Thus, we finally obtain an equation

$$\left[1 - \frac{\eta\left(1+\beta^2+\lambda^2\right) p_z}{\sqrt{1+\left(1+\beta^2+\lambda^2\right) p_z^2}}\right] \partial_{\xi\xi} p_z = -ZN_i \frac{p_z}{\sqrt{1+\left(1+\beta^2+\lambda^2\right) p_z^2}}, \tag{69}$$

which corresponds to a first integral of following general form

$$\left(\partial_\xi p_z\right)^2 + f_0\left(r,\theta,p_z\right) = G\left(r,\theta\right), \tag{70}$$

where $G\left(r,\theta\right)$ is a binary function of r and θ, and f_0 stands for well-known Sagdeev potential.

After solving a periodic solution of p_z, we can calculate a periodic density profile according to following formula

$$\begin{aligned} n &= ZN_i - \eta\left(1+\beta^2+\lambda^2\right)\partial_{\xi\xi} p_z \\ &= ZN_i + \frac{\eta\left(1+\beta^2+\lambda^2\right) p_z}{\sqrt{1+\left(1+\beta^2+\lambda^2\right) p_z^2} - \eta\left(1+\beta^2+\lambda^2\right) p_z} ZN_i \\ &= \frac{\sqrt{1+\left(1+\beta^2+\lambda^2\right) p_z^2}}{\sqrt{1+\left(1+\beta^2+\lambda^2\right) p_z^2} - \eta\left(1+\beta^2+\lambda^2\right) p_z} ZN_i. \end{aligned} \tag{71}$$

Note that the condition $n \geq 0$ will lead to a constraint on p_z

$$\sqrt{1+\left(1+\beta^2+\lambda^2\right) p_z^2} - \eta\left(1+\beta^2+\lambda^2\right) p_z > 0, \tag{72}$$

$$\text{or} p_z < \frac{1}{\sqrt{\eta^2\left(1+\beta^2+\lambda^2\right)^2 - \left(1+\beta^2+\lambda^2\right)}} < \frac{1}{\sqrt{\eta^2 - 1}} \text{if} \eta > 1. \tag{73}$$

More important, for the case of $\left(\beta,\lambda\right) = \left(\frac{D}{r}, 0\right)$, because of $\beta\left(r=0\right) = \infty$, Eqs.(67,68) will yield on-axis density $n_{3D}\left(r=0,\xi\right) = 0$, which differs greatly from its counterpart in the 1-D case, $n_{1D}\left(\xi\right) = \frac{1}{1-u\eta} ZN_i$. This result also holds for the case of $\left(\beta,\lambda\right) = \left(0, \frac{D}{r}\right)$. This implies that 3-D effect can result in more drastic density variation.

Two functions in Eq.(71), $f_0\left(r,\theta,p_z\right)$ and $G\left(r,\theta\right)$, read

$$f_0\left(r,\theta,p_z\right) = \frac{2}{c}\left\{\begin{array}{l} -\sqrt{1+\left(1+\beta^2+\lambda^2\right) p_z^2} \\ -\frac{1}{2}\frac{\eta\left(1+\beta^2+\lambda^2\right)}{\sqrt{c}} \\ \ln\frac{\sqrt{1+\left(1+\beta^2+\lambda^2\right) p_z^2} - \frac{\eta\left(1+\beta^2+\lambda^2\right)}{\sqrt{c}}}{\sqrt{1+\left(1+\beta^2+\lambda^2\right) p_z^2} + \frac{\eta\left(1+\beta^2+\lambda^2\right)}{\sqrt{c}}} \\ -\eta\left(1+\beta^2+\lambda^2\right) p_z \\ +\frac{1}{2}\frac{\eta\left(1+\beta^2+\lambda^2\right)}{\sqrt{c}} \\ \ln\frac{\sqrt{c}\, p_z - 1}{\sqrt{c}\, p_z + 1} \end{array}\right\} ZN_i \tag{74}$$

$$G\left(r,\theta\right) = f_0\left(r,\theta,p_z=0\right) + \left(\partial_\xi p_z\right)^2 |_{p_z=0}, \tag{75}$$

$$\text{where} c = \left[\eta^2\left(1+\beta^2+\lambda^2\right)^2 - \left(1+\beta^2+\lambda^2\right)\right] > 0 \text{if} \eta > 1. \tag{76}$$

We can qualitatively understand the behavior of p_z as the motion of a particle in a ``Sagdeev potential well'' $f_0\left(r,\theta,p_z\right)$. Obviously, at different $r$ position, such a ``Sagdeev potential well'' has different shapes, which might mean different r position corresponding different longitudinal behavior. This is the origin for the non-separability reported previously.

If $\partial_\xi p_z |_{u_z=1/\eta}$ is normally calculated according to Eq.(71), there will be a meanful solution $0 < \partial_\xi p_z |_{u_z=1/\eta} < \infty$. This implies that a normal procedure will allow $u_z > 1/\eta$ appearing. However, Eq.(73) indicates that $u_z = \frac{1}{\eta}$ will correspond to $n < 0$. Therefore, even though the normal procedure could yield $0 < \partial_\xi p_z |_{u_z=1/\eta} < \infty$, we must set a constraint on $\partial_\xi p_z |_{u_z=1/\eta}$ in order to agree with the $n \geq 0$ constraint or the constraint Eq.(73)

$$\partial_\xi p_z |_{u_z=1/\eta} = \frac{1}{2}\left[\partial_\xi p_z |_{u_z=(1/\eta)^-} + \partial_\xi p_z |_{u_z=(1/\eta)^+}\right] = 0; \tag{77}$$

where

$$\partial_\xi p_z |_{u_z=(1/\eta)^-} = \sqrt{G - f\left(u_z = 1/\eta\right)}$$

and $\partial_\xi p_z |_{u_z=(1/\eta)^+} = -\sqrt{G - f\left(u_z = 1/\eta\right)}$.

This constraint Eq.(77) implies that $p_z = 1/\sqrt{\eta^2 - 1}$ corresponds to a sharp peak in the $n - \xi$ curve.

2.6. Phase Space Structures

Some authors have found, from low-dimension Vlasov-Maxwell simulation, that phase space profile of a charged particle system agree with real space shape of the self-consistent fields [71]. According to the method presented in ref. [70], the phase space profile can be calculated from solved $\left(\overleftarrow{E}, \overleftarrow{B}\right)$:

$$f = f_{mono} + \sum_{i \leq 1} b_i \left[\left(\overleftarrow{\upsilon} - \overleftarrow{u}\right)\left(|\overleftarrow{\upsilon}|^2 - 1\right)\right]^i; \tag{78}$$

$$f_{mono} = \left\{\int f d^3\upsilon - \int \sum_i b_i \left[\left(\overleftarrow{\upsilon} - \overleftarrow{u}\right)\left(|\overleftarrow{\upsilon}|^2 - 1\right)\right]^i d^3\upsilon\right\}\delta\left(\overleftarrow{\upsilon} - \overleftarrow{u}\right). \tag{79}$$

The equation of b_i can be obtained by comparing terms in VE

$$\partial_t b_i + \overleftarrow{u}\cdot\nabla b_i + \nabla b_{i-1} -$$

$$b_i \nabla\overleftarrow{u} + \frac{1}{\left[\sqrt{1+|\overleftarrow{p}|^2\left(u\right)}\right]^3} b_{i-1}\overleftarrow{B}\times\frac{\left[\overleftarrow{\upsilon} - \overleftarrow{u}\right]}{|\left[\overleftarrow{\upsilon} - \overleftarrow{u}\right]|} = 0. \tag{80}$$

This equation illustrates why there exists an agreement between phase space profile and real space shape. Strict analysis indicates that the function coefficient set $\{b_i; i \geq 1\}$ meeting

$$b_{2i-1} = \left[\frac{1}{\eta} - \overleftarrow{u}\right]c_i; \text{and} b_{2i} = -c_i \tag{81}$$

where the constant set $\{c_i; i \geq 1\}$ is independent of space-time coordinates, is a strict solution of VE in $B = 0$ case.

3. NUMERICAL RESULTS AND DISCUSSION

Fig. (**1**) illustrates the effect of the transverse shape factor β on the density profile. Here, length is in unit of μm. Comparing Fig. (**1a-c**), we could find that when β drops more quickly with respect to r rising, the radial variation in n is milder. For example, for $\beta = \frac{1}{r}$, n varies from 0.9 to 1.35 over a range $0 < r < 0.25$. In contrast, for $\beta = \frac{4}{r}$, n varies from 0.94 to 1.08 over a same range.

As shown in Fig. (**1**), n displays gentle longitudinal variation at low-r region. For higher r, longitudinal variation in n becomes more drastic. In Fig. (**2**), the Fourier spectra of $n\left(r,\theta,\xi\right)$, or $\int n\left(r,\theta,\xi\right)\cos\left(k\xi\right)d\xi$, at different r also suggest the absence of separable form $func1(r,\theta) * func2(\xi)$. Namely, if n is of a separable form, the Fourier spectra at different r should be a common shape in different magnitudes. Obviously, these presented spectra do not have a common shape.

Moreover, $G\left(r,\theta\right)$ defined in Eq.(75) reflects the boundary condition of a 3-D wave. Usually, a physical boundary condition corresponds to a driver of finite radial extension (i.e., when r is $\to\infty$, the driving force or $\partial_\xi p_z |_{p_z=0}$ is 0). Because of this physical boundary condition and the fact that there is $\beta \to 0$ when $r \to \infty$, a 3-D wave will have $u_r\left(r \to 0\right) = 0$ and $u_z\left(r \to 0\right) = 0$ and hence be bound in the radial direction.

Fig. (**3**) indicates that phase space structures display as same periodicity as real space shapes or E-profile. These phase space holes (see Fig. **3**) also suggest that building electrons of an electrostatic plasma wave do not have a monotonic energy distribution, i.e., the number of building electrons of higher energy is not always less than that of lower energy. Therefore, to some extent, these phase space holes are a signature of some electrons being accelerated.

In conclusion, we have demonstrated strictly that because the $B = 0$ condition yield severe constraints on E in any 3-D electrostatic question, a 3-D nonlinear electrostatic plasma wave cannot have arbitrary transverse shape. Only for few allowed transverse shapes, E could be periodic along its longitudinal direction. Periodic longitudinal shape could be analytically described by a first integral depending on the radial coordinate r (see Eq.(71) and Eqs.(74,75)) and must meet the $n \geq 0$ constraint. Namely, in 1-D case, periodic longitudinal motion can be described by a first integral, whereas in 3-D case, such a first integral includes two ``parameters'' r and θ.

Above results are important for us to interpret related experiments or phenomena in term of nonlinear electrostatic waves. For instance, for plasma-based particles acceleration, people often use various stimulus to excite nonlinear plasma

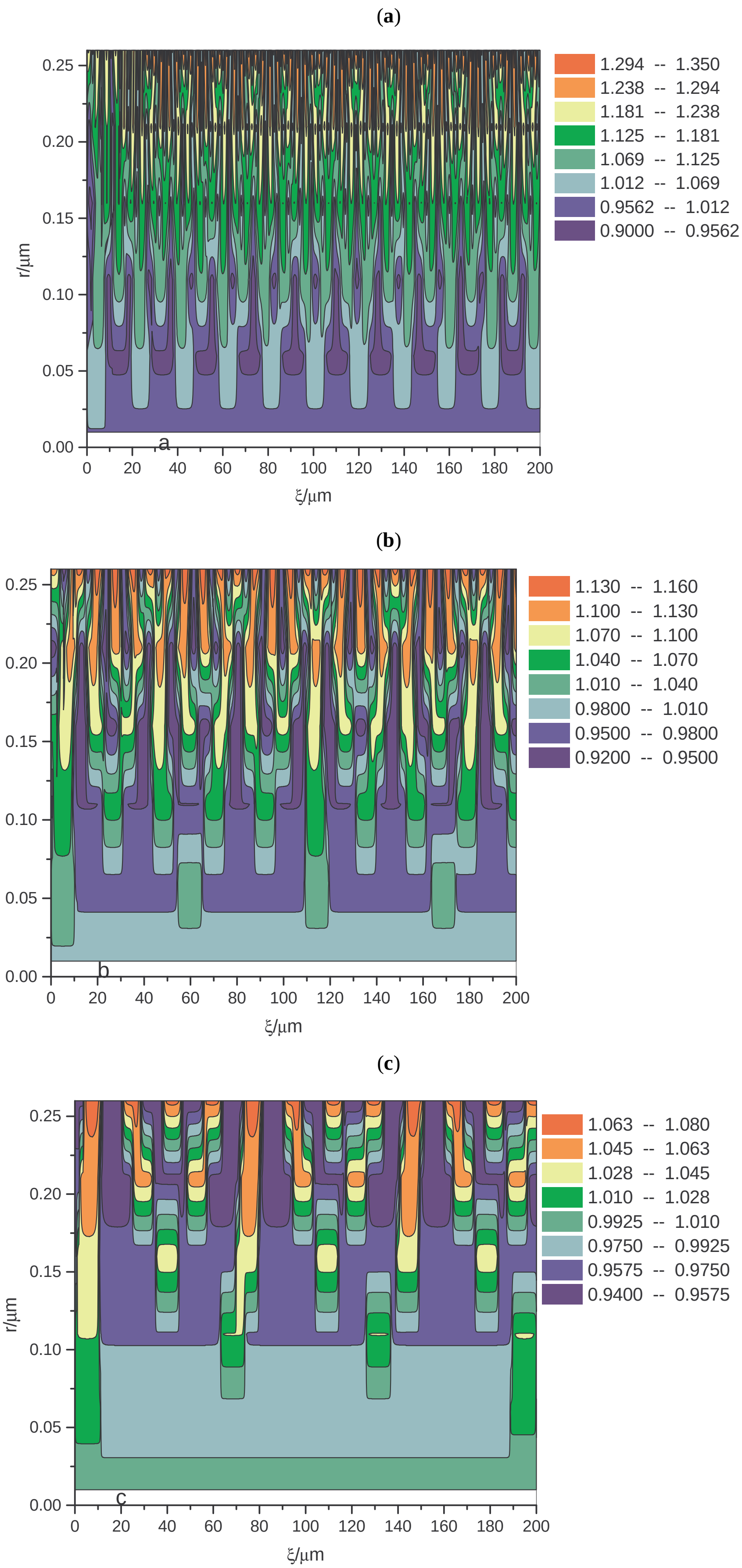

Fig. (1). Examples of contours of (n/N_i), where (**a**, **b**, **c**) are for $\beta = (1/r, 2/r, 4/r)$ and $\lambda = 0$ respectively. All other parameters are same for (**a**, **b**, **c**).

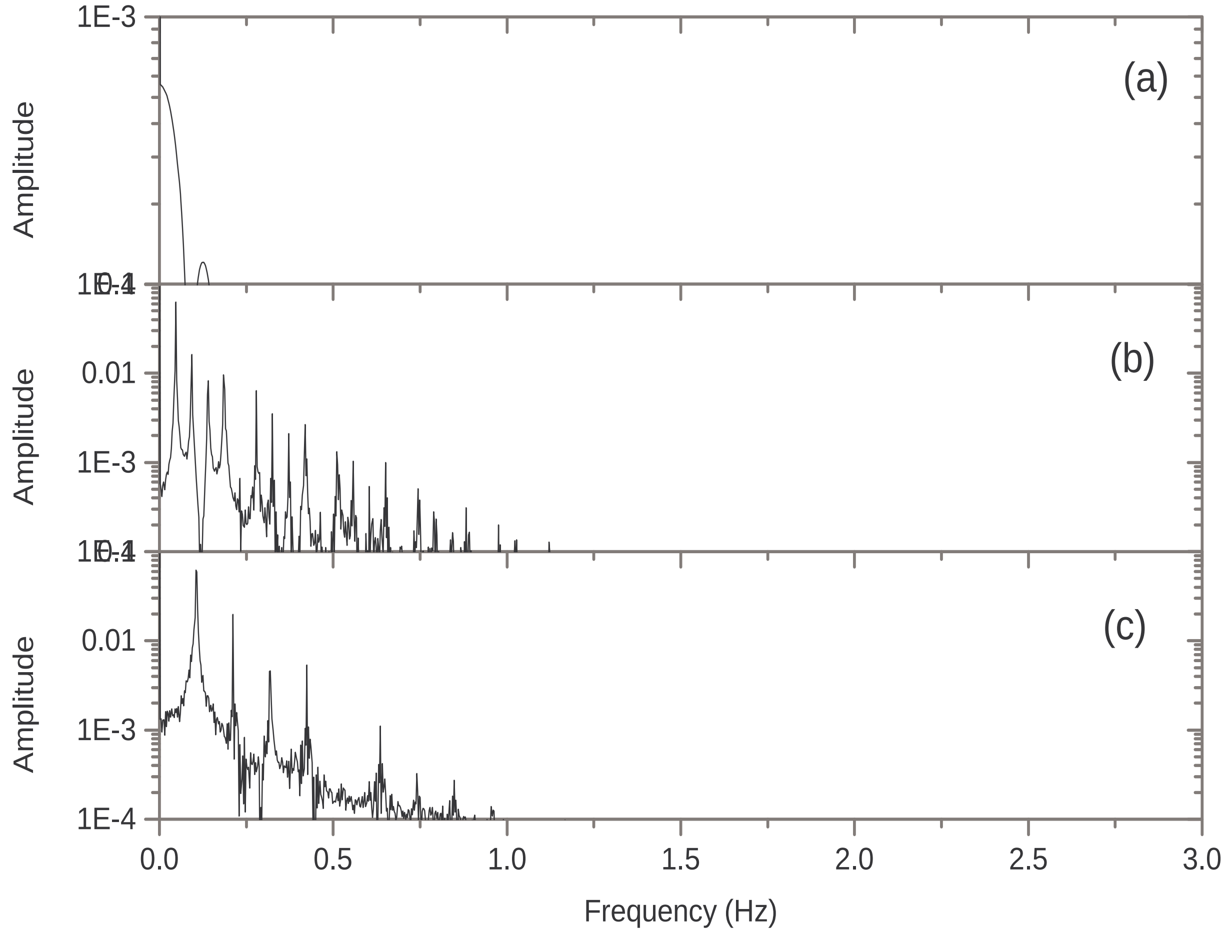

Fig. (2). Examples of Fourior spectrum at different r, where $\beta = 2/r$, $\lambda = 0$, and (**a**, **b**, **c**) are for $r = (0.01, 0.11, 0.21)$ respectively. All other parameters are same for (**a**, **b**, **c**).

electrostatic waves, which will act as ``accelerator'' [16-23]. However, above results indicate that due to 3-D effect, usually such a wave cannot be obtained. For a nonlinear electrostatic wave, its longitudinal periodicity requires its transverse shape to be of some specified forms instead of arbitrary forms. Therefore, we should not automatically believe those excited ``accelerators'' as periodic structure (or wave). For fully interpreting plasma-based particles acceleration, we should take into account those aperiodic ``accelerators''. Moreover, because of the charge quasineutrality condition of neutral plasmas, people often mis-believe that the self-consistent fields should be associated with periodic profile of charge density. Namely, charge density must be >0 at some regions and <0 at other regions. Therefore, when studying many issues about neutral plasmas (which demand the knowledge of the self-consistent fields), authors are accustomed to focus their attention to electrostatic plasma wave or periodic self-consistent electrostatic fields. To some extent, people's intense interest in periodic self-consistent fields might be motivated by this charge quasineutrality condition. However, strictly speaking, this charge quasineutrality condition could also permit aperiodic charge density profile (for instance, along z-direction, charge density profile alternatively takes on peaks and valleys, but the ``height'' (or ``depth'') of those peaks (or valleys) increase with respect to z-value.) and hence does not definitely imply periodicity. Our above theory indeed reminds authors that nonlinear dynamics equations of charged particles system, i.e., Eqs.(31-35), have a more severe requirement on periodic, electrostatic self-consistent fields.

4. SUMMARY

By strictly analyzing a universal equation set of charged particle system, i.e., Vlasov-Maxwell equations, we found that the longitudinal periodicity of an electrostatic plasma wave has a severe constraint on the transverse shape of a 3-D electrostatic structure. Or, one could say in other words, the longitudinal periodic structure (LPS), if exists, should correspond to number-limited transverse shapes (TS). In addition, the phase space structure of such a 3-D electrostatic structure can be expressed as a power series of $(\upsilon - u)$. The expansion coefficients can be clearly calculated through a set of fluid equations [70] and a strict solution of those expansion coefficients is presented in this work.

The significance of the newly revealed property: the longitudinal periodicity of a plasma electrostatic wave has severe requirements on its transverse shape, is obvious. Nonlinear plasma electrostatic wave is taken as a new-

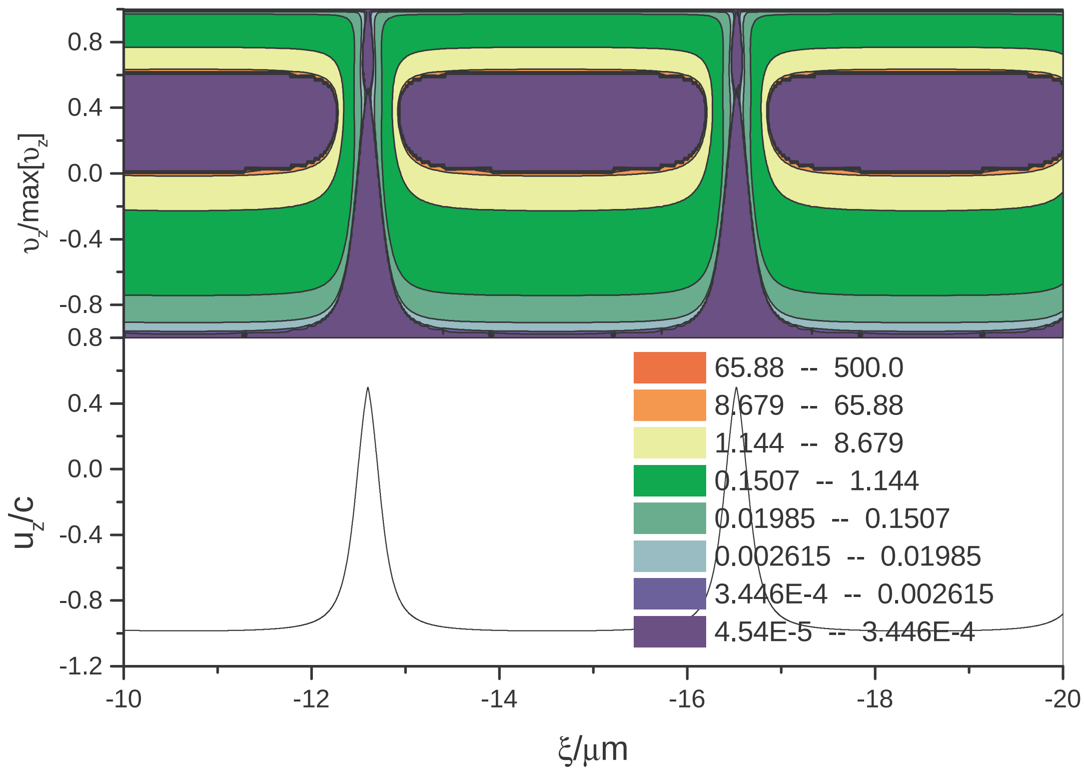

Fig. (3). Examples of contours of $f - f_{mono}$, where $\max\left[\upsilon_z\right] = \frac{c}{\sqrt{1+\beta^2(r,\theta)}}$ and E has a phase velocity $\frac{1}{\eta} = 0.5c$.

generation advanced accelerator and hence has received intensive investigations over past 30 years. However, as commented previously, in many related works, authors do not strictly take into account the dependence of related physical quantities on the transverse coordinates (r,θ) and hence corresponding theories are not really addressed to 3-D physics questions. This makes related researchers being unware of that this new-generation accelerator has severe constraint on its transverse shape. Such a severe constraint on the transverse shape is a very realistic problem for the new-generation accelerator because of its potential, complicated effect on the quality of acceleration. Therefore, how to deal with this realistic problem is an urgent task for related researchers.

Indeed, to merely attribute the plasma-based acceleration to periodic electrostatic structure (or plasma electrostatic wave) is not an overall understanding this phenomenon. Aperiodic electrostatic structure could also lead to high-velocity particles or corresponding phase space peaks. Moreover, electromagnetic 3-D wakefield could also correspond to various coherent phase space structures which manifest particle acceleration. Namely, to overall understand plasma-based acceleration, we should not merely confine ourselves to the electrostatic plasma wave.

CONFLICT OF INTEREST

The author declares that they have no financial/commercial conflicts of interest.

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