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Hitting Time and Place of Brownian Motion with Drift

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Abstract: We consider a *d*-dimensional Brownian motion in $\mathbb{R}^d$ with drift. The explicit expressions are obtained for the joint density of the hitting time and place to a sphere, when the process starts either from the inside the sphere or from the out of sphere.

Keywords: Brownian motion with drift, hitting time, hitting place, Joint density.

1. INTRODUCTION

Let $\{X(t),\ t\geq 0\}$ be a standard d-dimensional Brownian motion with drift c: $X(t)=B(t)+ct,\ \ t\geq 0,$ where $B(t)$ is the standard d-dimensional Brownian motion, $c\in\mathbb{R}^d (d\geq 2)$ is a fixed vector. Let us denote by $P_x^c(\bullet)$ the probability measure on the path space of X corresponding to initial value $X(0)=x$ and drift vector c, $E_x^c(\bullet)$ the corresponding expectation operator. For simplicity, we shall write $P_x(\bullet)$ and $E_x(\bullet)$ to refer the case $c=0$. For $r>0$, consider the sphere $\partial B_r=\ \{x:\ x\in\mathbb{R}^d,\ |x|=r\}$. The first hitting time of X through ∂B_r is defined as $T_r=\ \inf\{t>0:\ |X(t)|=r\}$. As usual, we take $\inf\{\varnothing\}=+\infty$. The first hitting place is $X(T_r)$. Because of the sample path continuity of the process, $X(T_r)$ lies on ∂B_r.

A Laplace-Gegenbauer transform of the first hitting time and the first hitting place to a sphere centered at the origin was found by Wendel [1]; Betz and Gzyl [2, 3] gave another proof to Wendel's exterior problem. Yin [4] extened Wendel's results to the case of Brownian motion with constant drift. The joint density of the first hitting time and the first hitting place of a sphere by Brownian motion which starts at any point inside the sphere was obtained by Hsu [5]. The aim of this paper is to obtain the joint density of the first hitting time and the first hitting place of a sphere by Brownian motion with or without drift which starts at any point in space.

The following notation can be found in [6]. Let J_υ and N_υ denote the first and second Bessel function of order υ, respectively. Let I_υ and K_υ denote the first and second Bessel function of purely imaginary argument, respectively. Let C_m^υ be the Gegenbauer polynomial of degree m and order υ, which is defined via its generating function: $(1-2\beta t+\beta^2)^{-\upsilon}=\sum_{n=0}^{\infty}C_n^\upsilon(t)\beta^n$. It is customary to take $C_0^0=1,\ C_0^\upsilon=1,\ C_m^0=2T_m/m,$ here T_m is the mth Tchebycheff polynomial: $T_m(\cos\theta)\ =\cos m\theta$. Set $h=(d-2)/2.$ We use $\{q_{m,n},\ n\geq 1\}$ to denote the positive zeros of J_{m+h} in the ascending order.

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2. LEMMAS

In this section we give several lemmas for latter use.

Lemma 2.1. ([6]) Let $\sigma(dy)$ be the $d-1$ dimensional volume measure on $\partial B_r\ \ (d\geq 2)$, then

$$\int_{\partial B_r} C_m^h(\cos\theta)C_k^h(\cos\theta)\sigma(dy)$$

$$=\begin{cases}\frac{2\pi^{\frac{d}{2}}r^{d-1}h}{(m+h)\Gamma(\frac{d}{2})}C_m^h(1), & m=k,\ d\geq 3,\\ \frac{2\pi r}{m}C_m^0(1), & m=k\neq 0,\ d=2,\\ 2\pi r, & m=k=0,\ d=2,\\ 0, & m\neq k,\ d\geq 2,\end{cases}$$

where $\theta=\angle x0y,\ x\in\mathbb{R}^d$.

Lemma 2.2. ([5]) For $|x|<r,\ \alpha>0$ and $d\geq 2$, then

$$-2\sum_{n=0}^{\infty}\frac{q_{m,n}J_{m+h}(\frac{|x|}{r}q_{m,n})}{(2r^2\alpha+q_{m,n}^2)J'_{m+h}(q_{m,n})}=\frac{I_{m+h}(\sqrt{2\alpha}\,|x|)}{I_{m+h}(\sqrt{2\alpha}r)},$$

where $m\geq 0$ is an integer.

Lemma 2.3. For $|x|>r,\ \ \alpha>0$ and $d\geq 2$, then

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$$\int_0^\infty \frac{\lambda(J_{m+h}(\lambda|x|)N_{m+h}(\lambda r)-J_{m+h}(\lambda r)N_{m+h}(\lambda|x|))}{(\lambda^2+2\alpha)(J_{m+h}^2(\lambda r)+N_{m+h}^2(\lambda r))}d\lambda=-\frac{\pi}{2}\frac{K_{m+h}(\sqrt{2\alpha}|x|)}{K_{m+h}(\sqrt{2\alpha}r)},$$ where $m\geq 0$ is an integer.

Proof. Using the recurrence formulas (see [6]):

$$\tfrac{d}{dx}(x^{\upsilon}K_{\upsilon})=-x^{\upsilon}K_{\upsilon-1},\ \tfrac{d}{dx}(x^{-\upsilon}K_{\upsilon})=-x^{-\upsilon}K_{\upsilon+1},\ \tfrac{d}{dx}(x^{\upsilon}Z_{\upsilon})=-x^{\upsilon}Z_{\upsilon-1},\ \tfrac{d}{dx}(x^{-\upsilon}Z_{\upsilon})=-x^{-\upsilon}K_{\upsilon-1},$$

where $Z_{\upsilon}=J_{\upsilon}$ or N_{υ}, we get

$$\int_0^\infty \frac{\pi}{2}\frac{K_{m+h}(\sqrt{2\alpha}R)}{K_{m+h}(\sqrt{2\alpha}r)}(J_{m+h}(\lambda R)N_{m+h}(\lambda r)-J_{m+h}(\lambda r)N_{m+h}(\lambda R))RdR=-\frac{1}{2\alpha+\lambda^2}.$$

The result follows immediately from the Weber's inversion transform (see [7]). This ends the proof.

Letting $k=0$ in Lemma 2.1, we have

Lemma 2.4. Let $\sigma(dy)$ be the $d-1$ dimensional volume measure on $\partial B_r(d\geq 2)$, then

$$\int_{\partial B_r} C_m^h(\cos\theta)\sigma(dy)=\begin{cases}\frac{2\pi^{\frac{d}{2}}r^{d-1}}{\Gamma(\frac{d}{2})},\ m=0,\\ 0,\ m\geq 1,\end{cases}$$

where $\theta=\angle x0y,\quad x\in\mathbb{R}^d$.

Lemma 2.5. Let $x,\ c\in\mathbb{R}^d(d\geq 2),\ \sigma(dy)$ be the $d-1$ dimensional volume measure on ∂B_r, then

$$\int_{\partial B_r} e^{c\cdot y}C_m^h(\cos\theta)\sigma(dy)=2(r\pi)^{\frac{d}{2}}(\tfrac{|c|}{2})^{-h}I_{m+h}(|c|r)C_m^h(\cos\angle c0x).$$

where $\theta=\angle x0y,\quad x\in\mathbb{R}^d$.

Proof. Using (1.5) in Yin [4] and Lemma 4 in [6, P.245].

3. HITTING SPHERE FOR BROWNIAN MOTION

In this section, we will give the joint density of the first hitting time and the first hitting place of a sphere by Brownian motion is derived based on the Laplace-Gegenbauer transform obtained in Wendel [1]. The result in Theorem 3.1 is due to Hsu [5], which was obtained by solving the heat equation with Dirichlet boundary condition satisfied by the transition density function of the Brownian motion in a ball.

For the interior problem we have

Theorem 3.1. For $x,\ y\in\mathbb{R}^d(d\geq 2),\ |x|<r,\ |y|=r$ and $t>0$, then

(1) for $d\geq 3$, we have

$$P_x(T_r\in dt,\ X(T_r)\in dy)/dt\sigma(dy)=-\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\Gamma(h)(m+h)C_m^h(\cos\theta)q_{m,n}J_{m+h}(\frac{|x|}{r}q_{m,n})}{2\pi^{h+1}r^{h+3}|x|^h\ J'_{m+h}(q_{m,n})}e^{-\frac{q_{m,n}^2}{2r^2}t};\tag{3.1}$$

(2) for $d=2$, we have

$$P_x(T_r\in dt,\ X(T_r)\in dy)/dt\sigma(dy)=-\sum_{n=1}^{\infty}\frac{q_{0,n}J_0(\frac{|x|}{r}q_{0,n})}{2\pi r^3J'_0(q_{0,n})}e^{-\frac{q_{0,n}^2}{2r^2}t}-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{m,n}\cos(m\theta)J_{m+h}(\frac{|x|}{r}q_{m,n})}{\pi r^3J'_m(q_{m,n})}e^{-\frac{q_{m,n}^2}{2r^2}t},\tag{3.2}$$

where $\theta=\angle x0y$, σ is the $d-1$ dimensional volume measure on ∂B_r.

Proof. Let us denote by $H(t,y)$ the right hand side of (3.1). For $\alpha>0$ and integer $k\geq 0$, using Lemmas 2.1 and 2.2 we have

$$\int_0^\infty\int_{\partial B_r} e^{-\alpha t}C_k^h(\cos\theta)H(t,y)dt\sigma(dy) = -\frac{\Gamma(h)}{2\pi^{h+1}r^{h+3}|x|^h}\sum_{m=0}^\infty\sum_{n=1}^\infty\frac{(m+h)q_{m,n}J_{m+h}(\frac{|x|}{r}q_{m,n})}{J'_{m+h}(q_{m,n})}$$
$$\times\int_0^\infty e^{-\alpha t-\frac{q_{m,n}^2}{2r^2}t}dt\times\int_{\partial B_r} C_k^h(\cos\theta)C_m^h(\cos\theta)\sigma(dy) = -\frac{2r^hC_k^h(1)}{|x|^h}\sum_{n=1}^\infty\frac{q_{k,n}J_{k+h}(\frac{|x|}{r}q_{k,n})}{(2r^2\alpha+q_{k,n}^2)J'_{k+h}(q_{k,n})}$$
$$=\left(\frac{r}{|x|}\right)^h C_k^h(1)\frac{I_{k+h}(\sqrt{2\alpha}\,|x|)}{I_{k+h}(\sqrt{2\alpha}r)}.$$

On the other hand, from (3) in Wendel [1] we get

$$\int_0^\infty\int_{\partial B_r} e^{-\alpha t}C_k^h(\cos\theta)P_x(T_r\in dt,\ X(T_r)\in dy) = \frac{r^hC_k^h(1)I_{k+h}(\sqrt{2\alpha}\,|x|)}{|x|^h\ I_{k+h}(\sqrt{2\alpha}r)}. \tag{3.4}$$

It follows from (3.3) and (3.4) and the uniqueness that $P_x(T_r\in dt,\ B(T_r)\in dy) = H(t,y)dt\sigma(dy)$. This proves (3.1). Eq. (3.2) can be proved along the same lines of (3.1) and thus the proof is omitted.

Remark 3.1. When $r=1$, the result (3.1) coincides with (13) in Hsu [5].

Corollary 3.1. For $x\in\mathbb{R}^d\,(d\ge 2)$, $|x|<r$, and $t>0$, then

$$P_x(T_r\in dt)/dt = \sum_{n=1}^\infty\frac{q_{0,n}}{r^2J_{h+1}(q_{0,n})}\left(\frac{|x|}{r}\right)^{-h}J_h(\tfrac{|x|}{r}q_{0,n})e^{-\frac{q_{0,n}^2}{2r^2}t}. \tag{3.5}$$

Proof. Integrating (3.1) or (3.2) with respect to $y\in\partial B_r$, using Lemma 2.4 and $J'_h(q_{0,n}) = -J_{h+1}(q_{0,n})$.

For the exterior problem we have

Theorem 3.2. For $x,\ y\in\mathbb{R}^d\,(d\ge 2)$, $|x|<r$, $|y|=r$ and $t>0$, then

(1) for $d\ge 3$, we have

$$P_x(T_r\in dt,\ X(T_r)\in dy,\ T_r<\infty)/dt\sigma(dy) = -\frac{\Gamma(\frac{d}{2})}{2rh\pi^{\frac{d}{2}+1}(r|x|)^h}\sum_{m=0}^\infty(m+h)C_m^h(\cos\theta)$$
$$\times\int_0^\infty\frac{\lambda(J_{m+h}(\lambda|x|)N_{m+h}(\lambda r)-J_{m+h}(\lambda r)N_{m+h}(\lambda|x|))}{J_{m+h}^2(\lambda r)+N_{m+h}^2(\lambda r)}e^{-\frac{1}{2}\lambda^2 t}d\lambda; \tag{3.6}$$

(1) for $d=2$, we have

$$P_x(T_r\in dt,\ X(T_r)\in dy,\ T_r<\infty)/dt\sigma(dy) = -\sum_{m=0}^\infty\frac{|x|D(m,|x|)}{\pi r}C_m^0(\cos\theta)$$
$$\times\int_0^\infty\frac{\lambda(J_m(\lambda|x|)N_m(\lambda r)-J_m(\lambda r)N_m(\lambda|x|))}{J_m^2(\lambda r)+N_m^2(\lambda r)}e^{-\frac{1}{2}\lambda^2 t}d\lambda, \tag{3.7}$$

Where $\theta=\angle x0y$, σ is the $d-1$ dimensional volume measure on ∂B_r and $D(m,|x|) = \frac{m}{2\pi|x|}$, if $m\ne 0$; $\frac{1}{2\pi|x|}$, if $m=0$.

Proof. Let us denote by $G(t,y)$ the right hand side of (3.6). For $\alpha>0$ and integer $k\ge 0$, using Lemmas 2.1 and 2.3 we have

$$\int_0^\infty\int_{\partial B_r} e^{-\alpha t}C_k^h(\cos\theta)G(t,y)dt\sigma(dy) = -\frac{\Gamma(\frac{d}{2})}{2rh\pi^{\frac{d}{2}+1}(r|x|)^h}\sum_{m=0}^\infty(m+h)\int_{\partial B_r}C_k^h(\cos\theta)C_m^h(\cos\theta)\sigma(dy)$$
$$\times\int_0^\infty\frac{\lambda(J_{m+h}(\lambda|x|)N_{m+h}(\lambda r)-J_{m+h}(\lambda r)N_{m+h}(\lambda|x|))}{J_{m+h}^2(\lambda r)+N_{m+h}^2(\lambda r)}e^{-\frac{1}{2}\lambda^2 t-\alpha t}d\lambda dt = \left(\frac{r}{|x|}\right)^h C_k^h(1)\frac{K_{k+h}(\sqrt{2\alpha}\,|x|)}{K_{k+h}(\sqrt{2\alpha}r)}. \tag{3.8}$$

On the other hand, from (6) in Wendel [1] we get

$$\int_0^\infty \int_{\partial B_r} e^{-\alpha t} C_k^h(\cos\theta) P_x(T_r \in dt,\ X(T_r) \in dy,\ T_r < \infty) = \left(\frac{r}{|x|}\right)^h C_k^h(1) \frac{K_{k+h}(\sqrt{2\alpha}\,|x|)}{K_{k+h}(\sqrt{2\alpha} r)}. \quad (3.9)$$

It follows from (3.8) and (3.9) and the uniqueness that $P_x(T_r \in dt,\ B(T_r) \in dy) = G(t,y)dt\sigma(dy)$. This proves (3.6). (3.7) can be proved along the same lines as the case (3.6) and will be omitted.

The following corollary follows immediately from Theorem 3.2 and Lemma 2.4.

Corollary 3.2. For $x \in \mathbb{R}^d (d \geq 2)$, $|x| < r$ and $t > 0$, then

$$P_x(T_r \in dt,\ T_r < \infty)/dt = -\frac{1}{\pi}\left(\frac{r}{|x|}\right)^h \int_0^\infty \frac{\lambda(J_h(\lambda|x|)N_h(\lambda r) - J_h(\lambda r)N_h(\lambda|x|))}{J_h^2(\lambda r) + N_h^2(\lambda r)} e^{-\frac{1}{2}\lambda^2 t} d\lambda.$$

4. HITTING SPHERE FOR BROWNIAN MOTION WITH DRIFT

In this section, we will give the joint density of the first hitting time and the first hitting place of a sphere by Brownian motion with constant drift. The results can be proved, as in the last section, by inverting the Laplace-Gegenbauer transform for Brownian motion with drift obtained in Yin [4]. Or, using Girsanov's change of measure theorem for Brownian motion. We give the results without proof.

For the interior problem we have

Theorem 4.1. For $c,\ x,\ y \in \mathbb{R}^d (d \geq 2)$, $|x| < r$, $|y| = r$ and $t > 0$, then

(1) for $d \geq 3$, we have

$$P_x^c(T_r \in dt,\ X(T_r) \in dy)/dt\sigma(dy) = -e^{c\bullet(y-x)-\frac{1}{2}|c|^2 t} \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{\Gamma(h)(m+h)C_m^h(\cos\theta) q_{m,n} J_{m+h}(\frac{|x|}{r} q_{m,n})}{2\pi^{h+1} r^{h+3} |x|^h J'_{m+h}(q_{m,n})} e^{-\frac{q_{m,n}^2}{2r^2}t};$$

(2) for $d = 2$, we have

$$P_x^c(T_r \in dt,\ X(T_r) \in dy)/dt\sigma(dy) = -e^{c\bullet(y-x)-\frac{1}{2}|c|^2 t} \left(\sum_{n=1}^\infty \frac{q_{0,n} J_0(\frac{|x|}{r} q_{0,n})}{2\pi r^3 J'_0(q_{0,n})} e^{-\frac{q_{0,n}^2}{2r^2}t} + \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{q_{m,n}\cos(m\theta) J_m(\frac{|x|}{r} q_{m,n})}{\pi r^3 J'_m(q_{m,n})} e^{-\frac{q_{m,n}^2}{2r^2}t} \right),$$

where $\theta = \angle x0y$, σ is the $d-1$ dimensional volume measure on ∂B_r.

Corollary 4.1. For $c, x \in \mathbb{R}^d (d \geq 2)$, $|x| < r$ and $t > 0$, then

(1) for $d \geq 3$, we have

$$P_x^c(T_r \in dt)/dt = -e^{-c\bullet x-\frac{1}{2}|c|^2 t} \frac{2^h \Gamma(h)}{r^2(|c|\cdot|x|)^h} \times \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{(m+h)C_m^h(\cos\angle c0x) q_{m,n} I_{m+h}(r|c|) J_{m+h}(\frac{|x|}{r} q_{m,n})}{J'_{m+h}(q_{m,n})} e^{-\frac{q_{m,n}^2}{2r^2}t};$$

(2) for $d = 2$, we have

$$P_x^c(T_r \in dt)/dt = -e^{-c\bullet x-\frac{1}{2}|c|^2 t} \sum_{n=1}^\infty \frac{q_{0,n} I_0(r|c|) J_0(\frac{|x|}{r} q_{0,n})}{r^2 J_0'(q_{0,n})} e^{-\frac{q_{0,n}^2}{2r^2}t} - e^{-c\bullet x-\frac{1}{2}|c|^2 t} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{2q_{m,n}\cos(m\angle c0x) I_m(r|c|) J_m(\frac{|x|}{r} q_{m,n})}{r^2 J'_m(q_{m,n})} e^{-\frac{q_{m,n}^2}{2r^2}t}$$

Proof. Integrating (3.1) or (3.2) with respect to $y \in \sum^{d-1}(0,r)$ and using Lemma 2.5.

For the exterior problem we have

Theorem 4.2. For $c,\ x,\ y\in\mathbb{R}^d(d\geq 2),\ \ |x|>r,\ \ |y|=r$ and $t>0$, then

(1) for $d\geq 3$, we have

$$P_x^c(T_r\in dt,\ B(T_r)\in dy,\ T_r<\infty)/dt\sigma(dy)=-\frac{\Gamma(\frac{d}{2})e^{c\bullet(y-x)-\frac{1}{2}|c|^2t}}{2rh\pi^{\frac{d}{2}+1}(r|x|)^h}\sum_{m=0}^{\infty}(m+h)C_m^h(\cos\theta)$$
$$\times\int_0^{\infty}\frac{\lambda(J_{m+h}(\lambda|x|)N_{m+h}(\lambda r)-J_{m+h}(\lambda r)N_{m+h}(\lambda|x|))}{J_{m+h}^2(\lambda r)+N_{m+h}^2(\lambda r)}e^{-\frac{1}{2}\lambda^2t}d\lambda;$$

(2) for $d=2$, we have

$$P_x(T_r\in dt,\ B(T_r)\in dy,\ T_r<\infty)/dt\sigma(dy)=-e^{c\cdot(y-x)-\frac{1}{2}|c|^2t}\sum_{m=0}^{\infty}\frac{|x|D(m,|x|)}{\pi r}C_m^0(\cos\theta)$$
$$\times\int_0^{\infty}\frac{\lambda(J_m(\lambda|x|)N_m(\lambda r)-J_m(\lambda r)N_m(\lambda|x|))}{J_m^2(\lambda r)+N_m^2(\lambda r)}e^{-\frac{1}{2}\lambda^2t}d\lambda,$$

where $\theta=\angle x0y$, σ is the $d-1$ dimensional volume measure on ∂B_r and $D(m,|x|)=\frac{m}{2\pi|x|}$, if $m\neq 0$; $\frac{1}{2\pi|x|}$, if $m=0$.

The following corollary follows immediately from Theorem 3.2 and Lemma 2.5.

Corollary 4.2. For $c,\ x\in\mathbb{R}^d(d\geq 2),\ \ |x|>r$ and $t>0$, then

(1) for $d\geq 3$, we have

$$P_x^c(T_r\in dt,\ T_r<\infty)/dt=-e^{-c\bullet x-\frac{1}{2}|c|^2t}\frac{\Gamma(h)2^h}{\pi(|c|\cdot|x|)^h}\sum_{m=0}^{\infty}(m+h)I_{m+h}(r|c|)C_m^h(\cos\angle c0x)$$
$$\times\int_0^{\infty}\frac{\lambda(J_{m+h}(\lambda|x|)N_{m+h}(\lambda r)-J_{m+h}(\lambda r)N_{m+h}(\lambda|x|))}{J_{m+h}^2(\lambda r)+N_{m+h}^2(\lambda r)}e^{-\frac{1}{2}\lambda^2t}d\lambda;$$

(2) for $d=2$, we have

$$P_x^c(T_r\in dt,\ T_r<\infty)/dt=-e^{-c\bullet x-\frac{1}{2}|c|^2t}\left(\frac{1}{\pi}I_0(r|c|)+\sum_{m=1}^{\infty}\frac{m}{\pi}I_m(r|c|)C_m^0(\cos\angle c0x)\right)$$
$$\times\int_0^{\infty}\frac{\lambda(J_m(\lambda|x|)N_m(\lambda r)-J_m(\lambda r)N_m(\lambda|x|))}{J_m^2(\lambda r)+N_m^2(\lambda r)}e^{-\frac{1}{2}\lambda^2t}d\lambda.$$

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